

## ON THE LOCATION OF THE ZEROS OF CERTAIN COMPOSITE POLYNOMIALS

ABDUL AZIZ

Let  $P(z) = \sum_{j=0}^n C(n, j)A_j z^j$  and  $Q(z) = \sum_{j=0}^m C(m, j)B_j z^j$  be two polynomials of degree  $n$  and  $m$ , respectively,  $m \leq n$  ( $C(n, j) =$  binomial coefficient). In this paper we study the relative location of the zeros of  $P(z)$  and  $Q(z)$  when the coefficients of these polynomials satisfy an apolar type relation and obtain some results. As an application of these results, we present certain generalizations of results of Walsh, Szegő, DeBruijn and Kakeya.

More recently [1] the author has shown that if

$$(1) \quad P(z) = \sum_{j=0}^n C(n, j)A_j z^j, \quad A_0 A_n \neq 0,$$

and

$$(2) \quad Q(z) = \sum_{j=0}^m C(m, j)B_j z^j, \quad B_0 B_m \neq 0,$$

are two polynomials of degree  $n$  and  $m$ , respectively,  $m \leq n$ , such that

$$(3) \quad C(m, 0)B_0 A_n - C(m, 1)B_1 A_{n-1} \\ + \cdots + (-1)^m C(m, m)B_m A_{n-m} = 0,$$

then the following holds.

(a) If  $Q(z)$  has all its zeros in  $|z| \geq r$ , then  $P(z)$  has at least one zero in  $|z| \geq r$ .

(b) If  $P(z)$  has all its zeros in  $|z| \leq r$ , then  $Q(z)$  has at least one zero in  $|z| \leq r$ .

Here we first show that this result equally holds if the circle  $|z| = r$  is replaced by a more general circle  $C$  with center at a point  $c$  and radius  $r$ . In fact, we prove

**THEOREM 1.** *If  $P(z)$  is a polynomial of degree  $n$  defined by (1) and  $Q(z)$  is a polynomial of degree  $m$  defined by (2),  $m \leq n$ , and if their coefficients satisfy the relation (3), then the following holds.*

(i) *If  $Q(z)$  has all its zeros in  $|z - c| \geq r$ , then  $P(z)$  has at least one zero in  $|z - c| \geq r$ .*

(ii) If  $P(z)$  has all its zeros in  $|z - c| \leq r$ , then  $Q(z)$  has at least one zero in  $|z - c| \leq r$ .

For the proof of Theorem 1, we need the following lemma, which is a generalization of a result due to Markovitch [4, p. 64].

LEMMA. Let  $P(z) = \sum_{j=0}^n C(n, j)A_j z^j$  and  $Q(z) = \sum_{j=0}^m C(m, j)B_j z^j$  be two polynomials of degree  $n$  and  $m$ , respectively,  $m \leq n$ . If we form

$$U(z) = \sum_{j=0}^n (-1)^j P^{(n-j)}(z) Q^{(j)}(z),$$

then

$$U(z) = n! \sum_{j=0}^n (-1)^j C(m, j) A_{n-j} B_j.$$

*Proof of the Lemma.* Since  $P(z)$  and  $Q(z)$  are two polynomials of degree  $n$  and  $m$ , respectively, we have

$$(4) \quad P^{(n+1)}(z) = 0 \quad \text{and} \quad Q^{(m+1)}(z) = 0.$$

Now we can write

$$P(z) = \sum_{j=0}^n C(n, j) A_j z^j = \sum_{j=0}^n \frac{P^{(j)}(0)}{j!} z^j$$

and

$$Q(z) = \sum_{j=0}^m C(m, j) B_j z^j = \sum_{j=0}^m \frac{Q^{(j)}(0)}{j!} z^j,$$

from which it follows that

$$(5) \quad P^{(j)}(0) = C(n, j) j! A_j, \quad j = 0, 1, 2, \dots, n,$$

and

$$(6) \quad Q^{(j)}(0) = C(m, j) j! B_j, \quad j = 0, 1, 2, \dots, m.$$

Now

$$\begin{aligned} U'(z) &= \sum_{j=0}^n (-1)^j P^{(n-j+1)}(z) Q^{(j)}(z) \\ &\quad + \sum_{j=0}^n (-1)^j P^{(n-j)}(z) Q^{(j+1)}(z), \end{aligned}$$

so that by (4) we have

$$\begin{aligned}
 U'(z) &= \sum_{j=1}^m (-1)^j P^{(n-j+1)}(z) Q^{(j)}(z) \\
 &\quad + \sum_{j=0}^{m-1} (-1)^j P^{(n-j)}(z) Q^{(j+1)}(z) \\
 &= \sum_{j=0}^{m-1} (-1)^{j+1} P^{(n-j)}(z) Q^{(j+1)}(z) \\
 &\quad + \sum_{j=0}^{m-1} (-1)^j P^{(n-j)}(z) Q^{(j+1)}(z) \\
 &= 0.
 \end{aligned}$$

Therefore, it follows that  $U(z)$  is constant and thus

$$U(z) = U(0) = \sum_{j=0}^m (-1)^j P^{(n-j)}(0) Q^{(j)}(0).$$

Using (5) and (6) we obtain

$$U(z) = n! \sum_{j=0}^m (-1)^j C(m, j) A_{n-j} B_j.$$

This proves the lemma.

*Proof of Theorem 1.* Consider

$$H(z) = P(z + c) = \sum_{j=0}^n \frac{P^{(j)}(c)}{j!} z^j = \sum_{j=0}^n C(n, j) A_j^* z^j \quad (\text{say})$$

and

$$G(z) = Q(z + c) = \sum_{j=0}^m \frac{Q^{(j)}(c)}{j!} z^j = \sum_{j=0}^m C(m, j) B_j^* z^j \quad (\text{say}).$$

Then  $H(z)$  and  $G(z)$  are two polynomials of degree  $n$  and  $m$ , respectively,  $m \leq n$ . Now we have

$$\begin{aligned}
 &\sum_{j=0}^m (-1)^j C(m, j) A_{n-j}^* B_j^* \\
 &= \sum_{j=0}^m (-1)^j C(m, j) \frac{P^{(n-j)}(c)}{C(n, n-j)(n-j)!} \frac{Q^{(j)}(c)}{C(m, j) j!} \\
 &= \frac{1}{n!} \sum_{j=0}^m (-1)^j P^{(n-j)}(c) Q^{(j)}(c).
 \end{aligned}$$

Using the lemma above, we obtain

$$\begin{aligned} \sum_{j=0}^m (-1)^j C(m, j) A_{n-j}^* B_j^* &= \sum_{j=0}^m (-1)^j C(m, j) A_{n-j} B_j \\ &= 0 \quad (\text{by hypothesis}). \end{aligned}$$

This shows that the coefficients of the polynomials  $H(z)$  and  $G(z)$  satisfy (3). Hence it follows from (a) and (b) that if all the zeros of  $G(z)$  lie in  $|z| \geq r$ , then  $H(z)$  has at least one zero in  $|z| \geq r$  and if all the zeros of  $H(z)$  lie in  $|z| \leq r$ , then  $G(z)$  has at least one zero in  $|z| \leq r$ . Replacing  $z$  by  $z - c$  and noting that  $P(z) = H(z - c)$  and,  $Q(z) = G(z - c)$ , the conclusion of Theorem 1 follows immediately.

As the first application of Theorem 1, we shall prove the following result, which is a generalization of Walsh's Coincidence Theorem [5] for the case when the circular region  $C$  is a circle  $|z - c| = r$ .

**THEOREM 2.** *Let  $G(z_1, z_2, \dots, z_n)$  be a symmetric  $n$ -linear form of total degree  $m$ ,  $m \leq n$ , in  $z_1, z_2, \dots, z_n$  and let  $C: |z - c| \leq r$  be a circle containing the  $n$  points  $w_1, w_2, \dots, w_n$ . Then in  $C$  there exists a least one point  $w$  such that*

$$G(w, w, \dots, w) = G(w_1, w_2, \dots, w_n).$$

*Proof of Theorem 2.* We write

$$P(z) = \prod_{j=1}^n (z - z_j) = \sum_{j=0}^n C(n, j) A_j z^j \quad (\text{say}),$$

so that

$$(7) \quad C(n, j) A_{n-j} = (-1)^j S(n, j) A_n, \quad j = 1, 2, \dots, n,$$

where  $S(n, j)$  are the symmetric functions consisting of the sum of all possible products of  $z_1, z_2, \dots, z_n$  taken  $j$  at a time.

Now if  $G(w_1, w_2, \dots, w_n) = G^*$ , then the difference  $G(z_1, z_2, \dots, z_n) - G^*$  is linear, symmetric and of total degree  $m \leq n$ , in the variables  $z_1, z_2, \dots, z_n$ . So that by the well-known theorem of algebra,  $G(z_1, z_2, \dots, z_n) - G^*$  can be expressed as a linear combination of the elementary symmetric functions  $S(n, j)$ ,  $j = 0, 1, \dots, m$ . That is, there

exist constants  $B_j$  so that

$$\begin{aligned} G(z_1, z_2, \dots, z_n) - G^* &= B_0 + S(n, 1)B_1 + S(n, 2)B_2 + \dots + S(n, m)B_m \\ &= \frac{1}{A_n} \{ B_0 A_n - C(n, 1)B_1 A_{n-1} + \dots + (-1)^m C(n, m)B_m A_{n-m} \} \end{aligned}$$

by using (7).

If we define the polynomial  $Q(z)$  by

$$Q(z) = \sum_{j=0}^m C(m, j) \frac{C(n, j)}{C(m, j)} B_j z^j = G(z, z, \dots, z) - G^*,$$

then the relation

$$G(w_1, w_2, \dots, w_n) - G^* = 0$$

shows that the polynomials  $P(z)$  and  $Q(z)$  satisfy the condition of Theorem 1. Since all the zeros of  $P(z)$  lie in  $|z - c| \leq r$ , we conclude from the 2nd part of Theorem 1 that  $Q(z) = G(z, z, \dots, z) - G^*$  has at least one zero in the circle  $|z - c| \leq r$ . This is equivalent to the desired result.

We next apply Theorem 1 to deduce the following partial generalization of a result due to Szegö [4, p. 65].

**THEOREM 3.** *If all the zeros of a polynomial  $Q(z) = \sum_{j=0}^m C(m, j) B_j z^j$  of degree  $m$  lie in  $|z - c| \geq r$  and if  $\alpha$  is a zero of the polynomial  $P(z) = \sum_{j=0}^n C(n, j) A_j z^j$ ,  $A_0 A_n \neq 0$ , of degree  $n$ ,  $m \leq n$ , then every zero  $w$  of the polynomial  $R(z) = \sum_{j=0}^m C(m, j) A_j B_j z^j$  of degree  $m$ , has the form  $w = -\alpha\beta$ , where  $\beta$  is a suitably chosen point in  $|z - c| \geq r$ .*

*Proof of Theorem 3.* If  $w$  is a zero of  $R(z)$ , then

$$(8) \quad R(w) = \sum_{j=0}^m C(m, j) A_j B_j w^j = 0.$$

Equation (8) shows that the polynomials

$$\begin{aligned} z^n P(-w/z) &= C(n, 0)(-1)^n A_n w^n + \dots + C(n, m)(-1)^m A_m w^m z^{n-m} \\ &\quad + \dots + C(n, n) A_0 z^n \end{aligned}$$

and

$$Q(z) = C(m, 0)B_0 + C(m, 1)B_1 z + \dots + C(m, m)B_m z^m$$

satisfy the conditions of Theorem 1. Since all the zeros of  $Q(z)$  lie in  $|z - c| \geq r$ , it follows from the first part of Theorem 1 that  $z^n P(-w/z)$

has at least one zero in  $|z - c| \geq r$ . If  $\alpha_1, \alpha_2, \dots, \alpha_n$  are the zeros of  $P(z)$ , then the zeros of  $z^n P(-w/z)$  are  $-w/\alpha_1, -w/\alpha_2, \dots, -w/\alpha_n$ . One of these zeros must be  $\beta$ , where  $|\beta - c| \geq r$ . Hence we must have  $w = -\alpha_j \beta$  for some  $j = 1, 2, \dots, n$ . This completes the proof of Theorem 3.

REMARK 1. We may use the 2nd part of Theorem 1 to establish Theorem 3 of [1] exactly in the same way as above for the case when the disk  $|z| \leq r$  is replaced by more general disk  $|z - c| \leq r$ .

We shall now prove the following theorem, which is a generalization of a result due to DeBruijn [2].

THEOREM 4. *From the two given polynomials*

$$P(z) = \sum_{j=0}^n C(n, j) A_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^m C(m, j) B_j z^j$$

*of degree  $n$  and  $m$ , respectively,  $m \leq n$ , let us form the third polynomial*

$$R(z) = \sum_{j=0}^m C(m, j) A_j B_j z^j$$

*of degree  $m$ . Given a subset  $S$  of the  $w$  plane, let  $P(z) \in S$  for  $|z| \leq r$  and  $Q(z) \neq 0$  for  $|z| < 1$ . Then  $R(z) \in B_0 S$  for  $|z| \leq r$  where*

$$B_0 S = \{ B_0 s; s \in S \}.$$

*Proof of Theorem 4.* Let  $\delta$  be a real or a complex number. We replace the polynomial  $P(z)$  by the polynomial  $F(z) = P(z) - \delta$  and hence  $R(z)$  by  $H(z) = R(z) - \delta B_0$ . If  $\delta \notin S$ , then  $F(z)$  does not vanish in  $|z| \leq r$ . So that all the zeros of  $F(z)$  lie in  $|z| > r$ . Also, by hypothesis, the zeros of  $Q(z)$  lie in  $|z| \geq 1$ . Now if  $w$  is a zero of  $H(z)$ , then by Theorem 2 of [1],  $w$  has the form  $w = -\alpha\beta$  where  $\alpha$  is a suitably chosen point in  $|z| > r$  and  $\beta$  is a zero of  $Q(z)$ . Hence  $|w| = |\alpha||\beta| \geq |\alpha| > r$ . This shows  $H(z)$  does not vanish in  $|z| \leq r$ . If therefore,  $\delta B_0$  is a value assumed by  $R(z)$  in  $|z| \leq r$ , then  $\delta$  is a value assumed by  $P(z)$  in  $|z| \leq r$ . Since  $P(z) \in S$  for  $|z| \leq r$ , it follows that  $R(z) \in B_0 S$  for  $|z| \leq r$  and this completes the proof.

REMARK 2. Let  $P(z)$ ,  $Q(z)$  and  $R(z)$  be as in Theorem 4, and  $S$  be a subset of the  $w$  plane. Let  $Q(z) \in S$  for  $|z| \leq r$  and  $P(z) \neq 0$  for  $|z| < 1$ . Then by using Theorem 3 with  $c = 0$ , it can be shown, similarly as above, that  $R(z) \in A_0 S$  for  $|z| \leq r$ , where  $A_0 S = \{ A_0 s; s \in S \}$ .

As an application of Theorem 4 and Remark 2, we deduce the following result which is also a generalization of another result of De-Bruijn [2].

**COROLLARY 1.** *If the polynomials  $P(z)$  and  $Q(z)$  defined in Theorem 4 have the properties*

$$|P(z)| \leq 1 \quad \text{and} \quad |Q(z)| \leq 1 \quad \text{for } |z| \leq 1,$$

*then the polynomial  $R(z)$  of Theorem 4 has the property*

$$|R(z)| \leq 1 - \left| |A_0| - |B_0| \right| \quad \text{for } |z| \leq 1.$$

*Proof of Corollary 1.* Let  $\alpha$  be a complex number such that  $|\alpha| > 1$ . Then the polynomial  $G(z) = (Q(z) - \alpha)/(B_0 - \alpha)$  does not vanish in  $|z| \leq 1$ . Since  $|P(z)| \leq 1$  for  $|z| \leq 1$ , we use Theorem 4 with  $S: |z| \leq 1$ ,  $r = 1$ ,  $Q(z)$  replaced by  $G(z)$ , and  $R(z)$  by  $(R(z) - \alpha A_0)/(B_0 - \alpha)$ . Hence it follows that

$$|R(z) - \alpha A_0| \leq |B_0 - \alpha| \quad \text{for } |z| \leq 1.$$

This gives

$$(9) \quad |R(z)| \leq |\alpha A_0| + |B_0 - \alpha| \quad \text{for } |z| \leq 1$$

and for all  $\alpha$  with  $|\alpha| > 1$ .

Since  $|B_0| = |Q(0)| \leq 1$ , we can choose an argument of  $\alpha$  such that  $|B_0 - \alpha| = |\alpha| - |B_0|$ . Using this in (9) and letting  $|\alpha| \rightarrow 1$  we get

$$(10) \quad |R(z)| \leq 1 + |A_0| - |B_0| \quad \text{for } |z| \leq 1.$$

If we take  $F(z) = (P(z) - \alpha)/(A_0 - \alpha)$ ,  $|\alpha| > 1$  and use the above remark, we can similarly show that

$$(11) \quad |R(z)| \leq 1 + |B_0| - |A_0| \quad \text{for } |z| \leq 1.$$

The required result now follows by combining (10) and (11) and the proof is complete.

As another application of Theorem 1, we shall next prove the following generalization of a result due to Walsh [5].

**THEOREM 5.** *From the two given polynomials*

$$P(z) = \sum_{j=0}^n a_j z^j = a_n \prod_{j=1}^n (z - \alpha_j)$$

*and*

$$Q(z) = \sum_{j=0}^m b_j z^j = b_m \prod_{j=1}^m (z - \beta_j)$$

*of degree  $n$  and  $m$ , respectively,  $m \leq n$ , let us form the thir polynomial*

$$R(z) = \sum_{j=0}^m (n-j)! a_{n-j} Q^{(j)}(z)$$

of degree  $m$ . Then the following holds:

(i) If all the zeros of  $P(z)$  lie in a circle  $C: |z - c| \leq r$ , then all the zeros of  $R(z)$  lie in the point set  $S$  consisting of the  $m$  circular regions obtained by translating  $C$  in the magnitude and direction of the vectors  $\beta_j$ .

(ii) If all the zeros of  $Q(z)$  lie in the circular region  $A: |z - c| \geq r$ , then every zero  $w$  of  $R(z)$  has the form  $w = \beta + \alpha$  where  $\beta$  is a suitably chosen point in  $A$  and  $\alpha$  is a zero of  $P(z)$ .

*Proof of Theorem 5.* If  $w$  is any zero of  $R(z)$ , then

$$(12) \quad R(w) = \sum_{j=0}^m (n-j)! a_{n-j} Q^{(j)}(w) = 0.$$

Equation (12) shows that the polynomials

$$P(z) = \sum_{j=0}^n a_j z^j \quad \text{and} \quad Q(w-z) = \sum_{j=0}^m (-1)^j \frac{Q^{(j)}(w)}{j!} z^j$$

of degree  $n$  and  $m$ , respectively,  $m \leq n$ , satisfy the conditions of Theorem 1. Since all the zeros of the polynomial  $P(z)$  lie in  $|z - c| \leq r$ , it follows from the 2nd part of Theorem 1 that the polynomial  $Q(w-z)$  has at least one zero in  $|z - c| \leq r$ . But the zeros of  $Q(w-z)$  are  $w - \beta_1, w - \beta_2, \dots, w - \beta_m$ . One of these zeros must be  $\alpha$ , where  $|\alpha - c| \leq r$ . That is, we must have  $w = \alpha + \beta_j$  for some  $j = 1, 2, \dots, m$ , where  $\alpha$  is a point of  $C$ . This completes the proof of the first part of the Theorem.

To prove the 2nd part of the theorem, we observe that the polynomial  $R(z)$  can also be written in the form

$$R(z) = \sum_{j=0}^m (n-j)! a_{n-j} Q^{(j)}(z) = \sum_{j=0}^m (m-j)! b_{m-j} P^{(n-m+j)}(z),$$

so that if  $w$  is a zero of  $R(z)$ , then we have

$$R(w) = \sum_{j=0}^m (m-j)! b_{m-j} P^{(n-m+j)}(w) = 0.$$

This equation shows that the polynomials

$$P(w-z) = \sum_{j=0}^n (-1)^j \frac{P^{(j)}(w)}{j!} z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^m b_j z^j$$

also satisfy the conditions of Theorem 1. Since all the zeros of  $Q(z)$  lie in  $|z - c| \geq r$ , it follows from the first part of Theorem 1 that  $P(w-z)$  has at least one zero in  $|z - c| \geq r$ . But the zeros of  $P(w-z)$  are  $w - \alpha_1, w - \alpha_2, \dots, w - \alpha_n$ , therefore, it follows that for at least one  $j = 1, 2, \dots, n$ , we have  $w = \alpha_j + \beta$ , where  $\beta$  is a suitably chosen point in  $A$ . This is equivalent to the desired result.



Exactly in the same way as Theorem 5, we may deduce the following result from Theorem 1 of [1].

**THEOREM 6.** *From the two given polynomials*

$$P(z) = \sum_{j=0}^n a_j z^j = a_n \prod_{j=1}^n (z - \alpha_j)$$

and

$$Q(z) = \sum_{j=0}^m b_j z^j = b_m \prod_{j=1}^m (z - \beta_j)$$

of degree  $n$  and  $m$ , respectively,  $m \leq n$ , let us form the polynomials

$$R_1(z) = \sum_{j=0}^m \frac{(m-j)!(n-m+j)!}{j!} a_{m-j} Q^{(j)}(z)$$

of degree  $m$  and

$$R_2(z) = \sum_{j=0}^m (n-j)! b_{m-j} P^{(j)}(z)$$

of degree  $n$ . Then we have the following:

(i) *If all the zeros of  $P(z)$  lie in  $|z| \geq r$ , then every zero  $w$  of  $R_1(z)$  has the form  $w = \alpha + \beta$ , where  $\alpha$  is a suitably chosen point in  $|z| \geq r$  and  $\beta$  is a zero of  $Q(z)$ .*

(ii) *If all the zeros of  $Q(z)$  lie in  $|z| \leq r$ , then every zero  $w$  of  $R_2(z)$  has the form  $w = \alpha + \beta$ , where  $\beta$  is a suitably chosen point in  $|z| \leq r$  and  $\alpha$  is a zero of  $P(z)$ .*

As an immediate consequence of Theorems 5 and 6 we have the following corollary, which presents a generalization of a result of Takeya [3].

**COROLLARY 2.** *If all the zeros of a polynomial  $P(z) = \sum_{j=0}^n a_j z^j$  of degree  $n$  lie in  $|z| \leq r_1$  and all the zeros of a polynomial  $Q(z) = \sum_{j=0}^m b_j z^j$  of degree  $m$  lie in  $|z| \leq r_2$ , then all the zeros of the polynomials*

$$H(z) = \sum_{j=0}^m (n-j)! a_{n-j} Q^{(j)}(z)$$

of degree  $m$  and

$$R(z) = \sum_{j=0}^m (n-j)! b_{m-j} P^{(j)}(z)$$

of degree  $n$  lie in  $|z| \leq r_1 + r_2$ .

This follows from the fact that  $|\alpha| \leq r_1$  and  $|\beta| \leq r_2$  imply  $|w| = |\alpha + \beta| \leq r_1 + r_2$ .

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UNIVERSITY OF KASHMIR  
HAZRATBAL SRINAGAR - 190006  
KASHMIR, INDIA