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PRIMES OF THE FORM $[n^c]$

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Dedicated to the memory of Ernst G. Straus

Methods of Vinogradov for estimating exponential sums over primes are modified and made easier to use. Comparisons are made with approaches of Heath-Brown and Vaughan.

1. Introduction. In 1939 I. M. Vinogradov developed a method of estimating exponential sums over primes. His method reduces the estimation of a sum $S = \sum_{p \le N} F(p)$ to the estimation of sums of type 1,

$$\sum_{X \le x \le 2X} a(x) \sum_{\substack{Y < y \le Y_1 \\ xy \le N}} F(xy),$$

where $Y_1 \le 2Y$, Y is large, and sums of type 2,

$$\sum_{X \le x \le 2X} a(x) \sum_{\substack{Y \le y \le 2Y \\ xy \le N}} b(y) F(xy),$$

where X and Y are large.

R. C. Vaughan proved an identity which allows one to express S as the sum of type 1 and type 2 sums:

$$\sum_{V \le n \le X} \Lambda(n) F(n) = S_1 - S_2 - S_3,$$

where

$$S_1 = \sum_{d \le U} \sum_{k \le X/d} \mu(d) \log k F(dk);$$

$$S_2 = \sum_{k \le UV} a(k) \sum_{r \le X/k} F(kr),$$

with

$$a(k) = \sum_{\substack{d \le U,\, n \le V \\ dn = k}} \mu(d) \Lambda(n);$$

and

$$S_3 = \sum_{m > U} \sum_{V \le n \le X/n} \Lambda(n) \left(\sum_{\substack{d \mid m \\ d \le U}} \mu(d) \right) F(mn),$$

where U and V are parameters, to be chosen to our advantage. Here S_3 is a type 2 sum, S_2 is of type 1 and S_1 can easily be reduced to a type 1 sum. D. R. Heath-Brown has proved [**1**] another identity, which allows one to use parameters better. He proved that if $F(x)$ is a function supported in $[N/2, N]$, and U, V, Z are parameters satisfying $3 \leq U < V < Z < N$, $z \geq 4U^2$, $N \geq 64Z^2U$, $V^3 \geq 32N$, then

$$(1) \qquad \left|\sum_n \Lambda(n)F(n)\right| \ll \max|F(n)| + K\log N + L\log^2 N,$$

where

$$K = \sum_m a(m) \sum_{n>Z} F(mn)$$

is a type 1 sum, and

$$L = \sum_m a(m) \sum_{U<n<V} b(n)F(mn)$$

is a type 2 sum. Using the above inequality, he proved that

$$\pi_c(X) = \frac{X}{c\log X} + O(X/\log^2 X)$$

for $c < 755/662$, where $\pi_c(X)$ is the number of $n \leq X$ for which $[n^c]$ is a prime. The above result extends a previously known result for which the above formula for $\pi_c(X)$ holds. The identities of Vaughan and Heath-Brown are easy to use, while the original method of Vinogradov needs some combinatorial arguments. However, using Vinogradov's idea, we can prove the following:

LEMMA 1. *Let α, δ, ε be positive numbers with $\delta \leq 1/2$ and ε small, and let N_1, $N \leq 2N_1$ be large numbers. Let $F(x)$ be a function supported in $[N_1, N]$, $F(x) \ll 1$, and let*

$$L^1 = \max \sum_{\sigma \leq N^\delta} \left| \sum_{X/\sigma \leq x \leq 2X/\sigma} a(x) \sum_{Y/\sigma \leq y \leq 2Y/\sigma} b(y)F(xy\sigma^2)\right|,$$

where the maximum is taken over $|a(x)| \leq 1$, $|b(y)| \leq 1$, $X \in [N^\alpha, N^{\alpha+\delta+\varepsilon}]$, $XY = N_1$. Furthermore, let

$$K^1 = K(\alpha, \delta, F) = \max\left|\sum_{x\in\mathscr{D}} a(x_n)F(x_1 \cdots x_n)\right|,$$

the maximum being taken over all $n \leq [1/\delta] + 1$, all $|a(x)| \leq 1$, and over all subdomains $\mathscr{D}$ of $\{x | X_j \leq x_j \leq 2X_j,\ j = 1,\dots,n\}$ with the following restrictions:

(i) $X_1 \cdots X_n = N_1$, $X_n \geq N^\alpha$, $X_1 \geq X_2 \geq \cdots \geq X_{n-1} \geq N^\delta$;

(ii) *for any* $\{j_1,\ldots,j_k\} \subset \{1,\ldots,n\}$

$$X_{j_1}\cdots X_{j_k} \notin [N^{\alpha}, N^{\alpha+\delta}] \cup [N^{1-\alpha-\delta}, N^{1-\alpha}];$$

(iii) *if for some* $\{j_1,\ldots,j_l\} \subset \{1,\ldots,n-1\}$

$$X_{j_1}\cdots X_{j_l} \le N^{\alpha+\delta+\varepsilon}, \quad \textit{then } X_{j_1}\cdots X_{j_l}X_n \le N^{\alpha}.$$

Let also

$$M = \max_{N^{2\delta}\le Q_0\le N} \sum_{Q_0\le q\le 2Q_0} \min_{\alpha_1,\delta_1} [K(\alpha_1,\delta_1,F_1) + L(\alpha_1,\delta_1,F_1)]$$

where $F_1(x) = F(qx)$, *the last sum is taken over powerful* q *with* $(q, P) = 1$. *Then*

$$\Sigma_1 \equiv \left|\sum_p F(q)\right| \ll (L + K + M)N^{\varepsilon}.$$

Here K^1 can be treated as a type 1 sum, and in some cases one can take advantage of the sum over all variables; L^1 is the sum of type 2 sums, and, in fact, the main contribution comes from small σ so that L^1 is estimated similarly to L in Heath-Brown's identity. While, as Heath-Brown pointed out, his identity has sometimes an advantage over the identity of Vaughan, his conditions on U, V, Z can be occasionally too restrictive (say, the conditions $U \ll N^{1/5}$, $V \gg N^{1/3}$; note however that in his recent paper. "Prime numbers in short intervals and a generalized Vaughan identity", D. R. Heath-Brown proved a new identity which has no such disadvantages; his new identity is essentially similar to Lemma 1 of this paper).

The lemma has no such restrictions. Also, the type 1 sum K^1 is in fact a multiple sum which can in some cases be estimated better than the type 1 sums in the methods of Vaughan and Heath-Brown described above. This happens, for example, if $F(x) = e(f(x))$, where $f(x)$ grows relatively slowly so that one can apply the Poisson summation formula to the type 1 sum. If we take $\delta \ge \alpha$, $\beta + \delta \ge 1/3$, then L^1 is essentially similar to Heath-Brown's L, and K^1 is "better" (in the sense mentioned above) than the K in Heath-Brown's method. Applied to the Pyatetsky-Shapiro prime number theorem, both Lemma 1 and Heath-Brown's identity (1) lead to the following result:

THEOREM. *Let c be a constant* $< 39/34$. *Then* $\pi_c(X) = X/(c\log X) + O(X/\log^2 X)$.

As Heath-Brown mentioned in his paper, one can write an asymptotic formula for $\pi_c(X)$ which is similar to the known formula for $\pi(X)$. The

Theorem improves slightly the result of D. R. Heath-Brown. The improvement is obtained by using our estimation of multiple sums (Lemma 4 below).

2. Notation. Since the Theorem is proved [1] for $c < 755/662$, we assume that $755/662 \leq c < 39/34$; $\gamma = 1/c$. As usual, $f(x) \lll g(x)$ means that $|f(x)| \ll x^{\varepsilon} g(x)$; $f(x) \cong g(x)$ means that $|f(x)| = O(|g(x)|)$ and $|g(x)| = O(|f(x)|)$; $f(x) \tilde{\Delta} g(x)$ means that

$$f^{(i)}(x) = g^{(i)}(x)(1 + O(\Delta))$$

for all i for which the statement makes sense; p, p_j are primes.

3. The main results. To prove Lemma 1, we use the ideas of I. M. Vinogradov [3]. We can obviously assume that $\alpha + \delta < 1$, otherwise $\Sigma_1 \ll L$. Let

$$P = \prod_{p \leq N^{\delta}} p; \qquad Q = \prod_{N^{\delta} \leq p \leq N} p;$$

$$\Sigma_k = \frac{1}{k!} \sum_{p_1 \cdots p_k | Q} F(p_1 \cdots p_k), \quad W_k = \sum_{y_1 \cdots y_k | Q} F(y_1 \cdots y_k),$$

where p_j, y_j of the above sums Σ_k, W_k range independently over the interval $[1, N]$; $W_{k,1}(q) = \Sigma_{y_1, \ldots, y_k} F(y_1 \cdots y_k q)$, where q is powerful and the sum is taken over y_j such that $p | y_j$ implies $p | Q$, $p | q$; $F(x) = 0$ for $x \notin [N/2, N]$; $r_0 = [1/\delta]$ if $\{1/\delta\} \neq 0$ and $r_0 = 1/\delta - 1$ otherwise. As in Theorem 3 of [3], page 156, we use the identities $W_r = r\Sigma_1 + r^2 \Sigma_2 + \cdots + r^{r_0} \Sigma_{r_0}$ $(r = 1, 2, \ldots, r_0)$ to express Σ_1 as a linear combination of $W_1, \ldots, W_{r_0}$ so that

$$|\Sigma_1| \ll |W_1| + \cdots + |W_{r_0}| \ll \max_k |W_k|.$$

Using induction on r_0, one can show that

$$\begin{aligned} W_k = {} & W_{k,1}(1) + \sum_{p | Q} \sum_{j=2}^{r_0} C(j, k) W_{k,1}(p^j) \\ & + \sum_{p_1 p_2 | Q} \sum_{j_1, j_2 = 2}^{r} C(j_1, j_2, k) W_{k,1}(p_1^{j} p_2^{j}) + \cdots \\ & + \sum_{p_1 \cdots p_r | Q} \sum_{j_1, \ldots, j_{r_0} = 2}^{r_0} C(j_1, \ldots, j_{r_0}, k) W_{k,1}(p_1^{j_1} \cdots p_{r_0}^{j_{r_0}}) \end{aligned}$$

so that

$$\Sigma_1 \lll \max_k \sum_q |W_{k,1}(q)|.$$

Here for $P(q) = P\prod_{p|q} p$ we have

$$\sum_q |W_{k,1}(q)|$$

$$= \sum_q \left| \sum_y F(y_1 \cdots y_k q) \sum_{d_1|(P(q),\, y_1)} \mu(d_1) \cdots \sum_{d_k|(P(q),\, y_k)} \mu(d_k) \right|$$

$$= \sum_q \left| \sum_{d_1, \dots, d_k | P(q)} \mu(d_1) \cdots \mu(d_k) \sum_m F(qd_1 \cdots d_k m_1 \cdots m_k) \right|$$

$$\ll \sum_q \left| \sum_{d_1, \dots, d_k | P} \mu(d_1) \cdots \mu(d_k) \sum_m F(qd_1 \cdots d_k m_1 \cdots m_k) \right|,$$

and

$$\Sigma_1 \lll \max_k \Bigg(\max_D \sum_{(d,\, m) \in D} \mu(d_1) \cdots \mu(d_k) F(d_1 \cdots d_k m_1 \cdots m_k)$$

$$+ \max_{Q_0} \sum_{Q_0 \le q \le 2Q_0} \max_D \Bigg| \sum_{(d,\, m) \in D} \mu(d_1) \cdots \mu(d_k)$$

$$\cdot F(qd_1 \cdots d_k m_1 \cdots m_k) \Bigg| \Bigg),$$

where the maxima are taken over $Q_0 \in [N^{2\delta}, N]$ and D of the form

$$D = \left\{ (d, m) | X_1 \le m_1 \le 2X_1, \dots, X_k \le m_k \le 2X_k, d_1|P, \dots, d_k|P \right\}.$$

We will show that the first sum is $\lll K^1 + L^1$; the proof that the second sum is $\lll M$ can be obtained similarly. We consider the following cases

1. $X_1 \cdots X_k \ge N^{1-\alpha-\delta}$.

Here $d_1 \cdots d_k \in [X_{k+1}, 2^{k+1} X_{k+1}]$, where

$$X_{k+1} = N\left(2^{k+1} X_1 \cdots X_k\right)^{-1}.$$

If for some $\{j_1, \dots, j_t\} \subset \{1, \dots, k+1\}$ we have $X_{j_1} \cdots X_{j_t} \in [N^{\alpha}, N^{\alpha+\delta}] \cup [N^{1-\alpha-\delta}, N^{1-\alpha}]$, then we write $x = m_{j_1} \cdots m_{j_t}$, $y = d_1 \cdots d_k m_1 \cdots m_k / x$ and get $\Sigma_1 \lll L^1 + M$; otherwise we denote by l the number such that $X_1 \ge X_2 \ge \cdots \ge X_l \ge N^{\delta} > X_{l+1} \ge \cdots \ge X_k$. Here $X_1 \cdots X_l > N^{1-\alpha}$, because otherwise we would have, for some $j \ge l$, $X_1 \cdots X_j \in [N^{1-\alpha-\delta}, N^{1-\alpha}]$. If for some $\{j_1, \dots, j_t\} \subset \{1, \dots, l\}$ we have $X_{j_1} \cdots X_{j_t} \le N^{\alpha+\delta}$, then $X_{j_1} \cdots X_{j_t} X_{l+1} \cdots X_k \le N^{\alpha}$ (otherwise either $X_{j_1} \cdots X_{j_t} \in [N^{\alpha}, N^{\alpha+\delta}]$ or $X_{j_1} \cdots X_{j_t} X_{l+1} \cdots X_n \in [N^{\alpha}, N^{\alpha+\delta}]$ for some $n \in [l+1, k]$) and, similarly, $X_{j_1} \cdots X_{j_t} X_{l+1} \cdots X_{k+1} > N^{\alpha+\delta}$. Using the argument similar to case 2 below, we obtain $\Sigma_1 \lll L^1 + M$.

2. $X_1 \cdots X_k < N^{1-\alpha-\delta}$.

Using Lemma 5 of [3], page 144, we divide the set of all integers $d_j | P$ $(j = 1,\dots,k)$ into $\ll N^{\varepsilon}$ subsets such that for any subset there exist numbers $\varphi_1,\dots,\varphi_k$ such that $\varphi_j \le d_j \le \varphi_j^{1+h}$, where h is a small positive number. Also, since every subset consists of some squarefree numbers having the same number of prime divisors (see [3]), for every subset $\mu(d_1) \cdots \mu(d_k) = \text{const}$. This divides the sum

$$S = \left| \sum_{(d,m) \in \mathscr{D}} \mu(d_1) \cdots \mu(d_k) F(d_1 \cdots d_k m_1 \cdots m_k) \right|$$

into $\ll N^{\varepsilon}$ subsums. Taking the largest subsum S_0 which corresponds to the subset $\mathscr{D}_0$, we get

$$S \lll \left| \sum_{m} \sum_{d \in \mathscr{D}_0} F(d_1 \cdots d_k m_1 \cdots m_k) \right| = S_0,$$

where $m = (m_1,\dots,m_k)$, $m_j \in [\,X_j, 2X_j)$ $(j = 1,\dots,k)$, $d = (d_1,\dots,d_k)$, $d_j \in [\varphi_j, \varphi_j^{1+h}]$ $(j = 1,\dots,k)$. We assume $S_0 \ne 0$. Here $\varphi_1 \cdots \varphi_k \ge N^{\alpha}$, since otherwise

$$d_1 \cdots d_k m_1 \cdots m_k \le N^{\alpha} \cdot 2^k N^{1-\alpha-\delta} < N/2 \le N_1.$$

Let q be the smallest integer such that $\varphi_1 \cdots \varphi_q \ge N^{\alpha}$. If $\varphi_1 \cdots \varphi_q \le N^{\alpha+\delta}$, then we write $x = d_1 \cdots d_q$, $y = d_{q+1} \cdots d_k m_1 \cdots m_k$ and, since

$$N^{\alpha} \le \varphi_1 \cdots \varphi_q \le d_1 \cdots d_q \le (\varphi_1 \cdots \varphi_q)^{1+h} \le N^{(\alpha+\delta)(1+h)} \le N^{\alpha+\delta+h},$$

we get $\Sigma_1 \lll S \lll S_0 \lll L^1$. Now we assume that $\varphi_1 \cdots \varphi_q > N^{\alpha+\delta}$. We take γ defined by $N^{\alpha-\gamma} = \varphi_1 \cdots \varphi_{q-1}$ and use part (ii) of Lemma 5, [3], page 144 to show that there exist two sequences (u) and (v) such that $N^{\gamma} \le u \le N^{\gamma+\delta+h}$ for all $u \in (u)$, and such that the products uv with $(u, v) = 1$, $uv \le N$ comprise precisely the numbers d_q of the subset, each repeated the same number of times. We obtain:

$$S_0 \ll \left| \sum_{d_1,\dots,d_{q-1}} \sum_{v \in (v)} \sum_{\substack{u \in (u) \\ (u,v)=1}} \sum_{d_{q+1},\dots,d_k m} \sum_{m} F\big(d_1 \cdots d_{q-1} uv d_{q+1} \cdots d_k \cdot m_1 \cdots m_k\big) \right|$$

$$= \left| \sum_{d,m} \sum_{u,v} \sum_{\sigma | (u,v)} \mu(\sigma) F\big(d_1,\dots,d_{q-1} uv d_{q+1} \cdots d_k m_1 \cdots m_k\big) \right|$$

$$\le \sum_{\sigma} \left| \sum_{d,m} \sum_{\substack{\sigma u \in (u) \\ \sigma v \in (v)}} F\big(d_1 \cdots d_{q-1} uv\sigma^2 d_{q+1} \cdots d_k m_1 \cdots m_k\big) \right|.$$

Writing $d_1 \cdots d_{q-1}u = x$, $vd_{q+1} \cdots d_k m_1 \cdots m_k = y$, we get $\sigma x \geq \varphi_1 \cdots \varphi_{q-1} N^\gamma = N^\alpha$,

$$\sigma x \leq \left(\varphi_1 \cdots \varphi_{q-1}\right)^{1+h} N^{\gamma+\delta+h} \leq N^{(\alpha-\gamma)(1+h)+\gamma+\delta+h} \leq N^{\alpha+\delta+\varepsilon}$$

if $h \leq \varepsilon/2$, so that

$$\Sigma_1 \lll S_0 \lll \sum_\sigma \left| \sum_{x,y} a(x)b(y)F(xy\sigma^2) \right| \lll L^1 + M.$$

This proves Lemma 1.

We also need five more Lemmas. Lemma 2 is the Poisson summation formula (see, for example, Lemma 6 of [**1**], or Lemma 1 of [**2**]); Lemma 3 is Weyl's inequality (see Lemma 5 of [**1**] or Lemma 2 of [**2**]); Lemma 4 is Theorem 2 of [**2**]; Lemma 5 can be proved similarly to Lemma 4 of [**1**] (in fact, $5\gamma - 4$ can be replaced with $5\gamma - 4 + (1-\gamma)/48$, which, however, does not lead to any improvement for $\pi_c(X)$ because our Lemma 6 is not good enough). Improving Lemma 6, one might hope to extend the boundary for c to $c < 239/207$ (instead of $c < 15/13$, which is the best one can obtain by using Heath-Brown's identity).

LEMMA 2. *Let $q < a < b < 2a$. Let $f(x)$ be a real function such that $f''(x) \asymp M^{-1}$ for $x \in [a,b]$; $f(z)$ be analytic for $z \in \{z \mid \forall x \in [a,b], |z - x| \leq \sqrt{c_0 M \log b}\}$. Let $f'(a) = \alpha$, $f'(b) = \beta$, and define x_n for $n \in [\alpha, \beta]$ by $f'(x_n) = n$. Then*

$$\sum_{a \leq x \leq b} e(f(x)) = \sum_{\alpha \leq n \leq \beta} [f''(x_n)]^{-1/2} e\left(\frac{1}{8} + f(x_n) - nx_n\right)$$
$$+ O\left(M^{-1/2} + \log(2 + Mb)\right).$$

LEMMA 3. *Let I be a subinterval of $(X, 2X]$, and let Q be a positive number. Let z_n be complex numbers. Then*

$$\left| \sum_{n \in I} z_n \right|^2 \leq \left(1 + XQ^{-1}\right) \sum_{|q| \leq Q} \left(1 - |q|Q^{-1}\right) \sum_{n, n+q \in I} \bar{z}_n z_{n+1}.$$

LEMMA 4. *Let α, β, γ be real numbers, $\alpha\beta\gamma(\alpha - 1)(\beta - 1)(\gamma - 1) \cdot (\alpha + \beta + \gamma - 1)(\alpha + \beta - \gamma - 2) \neq 0$, and let $X \geq Y \geq Z$, $XYZ = N$. Let $\mathcal{D}$ be a subdomain of $\{(x, y, z) \mid X \leq x \leq 2X,\ Y \leq y \leq 2Y,\ Z \leq z \leq 2Z\}$, bounded by $O(1)$ algebraic curves, and let $f(x, y, z)$ be a real C^∞*

function such that $f(x, y, z) \tilde{\Delta} FX^{-\alpha}Y^{-\beta}Z^{-\gamma}x^{\alpha}y^{\beta}z^{\gamma}$ *throughout* $\mathscr{D}$. *Then*

$$\left|\sum_{(x,y,z)\in\mathscr{D}} e(f(x,y,z))\right|$$
$$\lll N\left\{FN^{-1} + Z^{-1} + N^{-1/4} + Y^{-2/5}Z^{-2/5} + \left(FY^2Z^2\right)^{-1/8} + \left(F^6X^6N^{-8}\Delta^3\right)^{1/8}\right\}^{1/2}.$$

LEMMA 5. *Let L be the type* 2 *sum from Lemma* 1 *with* $F(x) = e(hx^{\gamma})$ *and* $X \in [N^{1-\gamma+\eta}, N^{5\gamma-4-\eta}]$.

Let $\eta > 0$ be a sufficiently small constant. Then $L \ll N^{1-\eta/7}$.

Proof. To prove the Lemma, we use Heath-Brown's estimate for a fixed σ from his Lemma 4 and, summing over σ, obtain the needed result.

LEMMA 6. *Let* α_i, β_j, A_i, $B_j > 0$, $B \geq A > 0$, *and let* $f(x) = \sum_{i=1}^{m} A_i x^{-\alpha_i} + \sum_{j=1}^{n} B_j x^{\beta_j}$. *Then*

$$M = \max_{A \leq x \leq B} f(x)$$
$$\leq 2^{m+n}\left(\sum_{i=1}^{m} A_i B^{-\alpha_i} + \sum_{j=1}^{n} B_j A^{\beta_j} + \sum_{i=1}^{m}\sum_{j=1}^{n} A_i^{\beta_j/\alpha_i+\beta_j} B_j^{\alpha_i/\alpha_i+\beta_j}\right).$$

This Lemma can be easily proved by induction on (m, n).

LEMMA 7. *Let* $H = N^{1-\gamma}\log^3 N$, $N/2 \leq N_1 \leq N$,

$$S = \sum_{1 \leq h \leq H}\ \sum_{X \leq x \leq 2X}\left|\sum_{\substack{Y \leq y \leq 2Y\\ N_1 \leq xy \leq N}} e(hx^{\gamma}y^{\gamma})\right|.$$

$$S \lll NH\left(Y^{-3/8} + X^{-1/2} + \left(XY^4\right)^{-1/18} + N^{-47/360}\right).$$

Proof. We apply Lemma 3 to get

$$(1)\qquad |S|^2 \ll HX\sum_{h,x}\left|\sum_{y} e(hx^{\gamma}y^{\gamma})\right|^2$$
$$\ll H^2N^2/Q + HNQ^{-1}\left|\sum_{q=1}^{Q}\sum_{h}\sum_{x,y} e(f(x,y,q))\right|$$
$$\lll H^2N^2/Q + HNQ^{-1}\sum_{h}\left|\sum_{H_1 \leq q \leq 2H_1}\ \sum_{x,y} e(f(x,y,q))\right|,$$

where $1 \le H_1 \le Q/2$; $Q \ll Y$ is a parameter to be defined later; $f(x, y, q)\tilde{\Delta}\gamma hqx^{\gamma}y^{\gamma-1} \cong hH_1N^{\gamma}/Y \equiv F$ with $\Delta = H_1/Y$. Next we apply Lemma 2 to the sum over x and y successively and Abel's summation formula:

$$(3)\qquad \begin{aligned} S_1 &\equiv \sum_q \sum_{x,y} e(f(x, y, q)) \\ &= \sum_{q,y} \left\{ \sum_n [f_{x^2}(x_n, y, q)]^{-1/2} e\left(\frac{1}{8} + f(x_n, y, q) - nx_n\right) \right. \\ &\qquad\qquad \left. + O(XF^{-1/2} + \log N) \right\} \\ &\lll X(hH_1N^{\gamma}/Y)^{-1/2} \left| \sum_{q,n} \sum_y e(f_1(n, q, y)) \right| \\ &\quad + NH_1(hH_1N^{\gamma}/Y)^{-1/2} + H_1Y \\ &\ll X(hH_1N^{\gamma}/Y)^{-1/2} \\ &\quad \cdot \left\{ \left| \sum_{q,n} \sum_m \left[\frac{\partial^2 f_1(n, q, y_m)}{\partial y^2} \right]^{-1/2} e(f_1(n, q, y_m) - my_m) \right| \right. \\ &\qquad\qquad \left. + Y \cdot (hH_1N^{\gamma}/Y)^{-1/2} + \log N \right\} \\ &\quad + NH_1(hH_1N^{\gamma}/Y)^{-1/2} + H_1Y \\ &\lll NY(hH_1N^{\gamma})^{-1} \left| \left(\sum_{q,m,n} e(g(m, n, q)) \right) \right| \\ &\quad + YH_1 + NY\left(hH_1N^{\gamma}Y^{-1}\right)^{-1/2}, \end{aligned}$$

where $f_1(n, q, y) = f(x_n, y, q) - nx_n$,

$$g(m, n, q) = f_1(n, q, y_m) - my_m\tilde{\Delta}c_0\sqrt{n}\, m^{\alpha_n}(qh)^{\beta_1} \cong F,$$

c_0 is a constant, $\alpha_1 = -\gamma/(2 - 2\gamma)$, $\beta_1 = 1/2 - \alpha_1$, $m \cong hH_1N^{\gamma-1} \equiv M_1$, $n \cong hH_1N^{\gamma}Y^{-2} \equiv M_2$. Now we apply Lemma 3 with an appropriate Q_1 and Lemma 2 to the sum over n to get

$$(4)\qquad \begin{aligned} |S_2|^2 &\equiv \left| \sum_{q,m,n} e(g(m, n, q)) \right|^2 \ll (H_1M_1M_2)^2 Q_1^{-1} \\ &\quad + H_1M_1M_2Q_1^{-1} \sum_{q_1=1}^{Q_1} \left| \sum_{q,m} \sum_h e(g(m, n + q_1, q) - g(m, n, q)) \right| \end{aligned}$$

(continues)

$$\ll (H_1M_1M_2)^2Q_1^{-1} + H_1M_1M_2Q_1^{-1}$$

$$\times \sum_{q_1=1}^{Q_1} \left| \sum_{q,m} \left\{ \sum_{v} [g_{n^2}(m, n_v + q_1, q) - g_{n^2}(m, n_v, q)]^{-1/2} \right. \right.$$

$$\cdot e\left(\frac{1}{8} + g(m, n_v + q_1, q) - g(m, n_v, q) - vn_v\right)$$

$$\left. \left. + O\left(F^{-1/2}M_2 + \log N\right) \right\} \right|$$

$$\ll (H_1M_1M_2)^2Q_1^{-1} + (H_1M_1M_2)^2(F)^{-1/2}$$

$$+ H_1M_1M_2^2Q_1^{-1} \cdot (Q_1Y)^{-1/2} \sum_{q_1} \left| \sum_{q,m} \sum_{v} e(\varphi(q, m, v)) \right|$$

where $v \cong V \equiv q_1Y/M_2$, $Q_1 \ll M_2$,

$$\varphi(q, m, v)\tilde{\Delta}_1 c_1 v^{1/3} q_1^{2/3} m^{2\alpha_1/3}(qh)^{2\beta_1/3} \cong Fq_1/M_2,$$

$\Delta_1 = H_1/Y + q_1/M_2$. Using Lemma 4 to estimate the last sum over q, m, v, we get

$$S_3 \equiv \left| \sum_{q,m,v} e(\varphi(q, m, v)) \right|$$

$$\lll H_1M_1V\left\{ Fq_1(M_2M_1H_1V)^{-1} + H_1^{-1} + M_1^{-1} + V^{-1} + (H_1M_1V)^{-1/4} \right.$$

$$+ (H_1 + M_1 + V)^{2/5}(H_1M_1V)^{-2/5}$$

$$+ \left[(H_1 + M_1 + V)^2 M_2 (Fq_1)^{-1} (H_1M_1V)^{-2}\right]^{1/8}$$

$$+ \left[(Fq_1/M_2)^6 (H_1 + M_1 + V)^6 (H_1M_1V)^{-8}\right.$$

$$\left.\left. \times (M_1/Y + q_1/M_2)^3\right]^{1/8}\right\}^{1/2}.$$

Substituting this into (3), we choose Q_1 (using Lemma 6) to minimize the obtained expression; then we substitute the obtained estimate into (2) and (1) and, choose (using Lemma 6) Q to our advantage, we complete the proof of the lemma.

Now we can prove the Theorem. As in **[1]**, it suffices to prove (using Abel's formula) that

$$S_0 = \sum_{1 \le h \le H} \left| \sum_{N_1 \le p \le N} e(hp^\gamma) \right| \ll N(\log N)^{-2}, \tag{5}$$

where

$$H = N^{1-\gamma} \log^3 N.$$

While the formulation of Lemma 1 is slightly longer than (1), the application is not more difficult, and we will use it to prove (5). We take $\alpha = 1 - \gamma + \eta$, $\alpha + \delta = 5\gamma - 4 - \eta$, where η is sufficiently small, and use Lemma 1:

$$S_0 \lll (L^1 + K^1) + N^{1-\eta}.$$

According to Lemma 5, $L \lll N^{1-\eta/7}$, and we need to estimate K. Here $\alpha + \delta > 14/39 - \eta$ so that n from Lemma 1 is equal to 2 or 3. If $n = 3$, then $X_1 \geq N^{(\gamma-\eta)/2}$, $X_2 \geq N^{5\gamma-4-\eta}$, and, denoting $y = x_1$, $x = x_2x_3$, we use Lemma 7 to get

$$\sum_h K \lll N^{2-\gamma}\left(X_1^{-3/8} + (X_2X_3)^{-1/2}l + \left(X_1^4X_2X_3\right)^{-1/18} + N^{-47/360}\right)$$
$$\ll N^{1-\eta}$$

if η is small. If $n = 2$, then we use van der Corput's estimate (or Lemma 2 and a trivial estimate of the right-hand side sum) and obtain

$$\begin{aligned}\sum_h K &\ll \sum_h \sum_{x_2 \cong X_2} \left| \sum_{x_1 \cong X_1} e\left(hx_1^\gamma x_2^\gamma\right)\right| \\ &\lll \sum_h X_2\left[X_1(hN^\gamma)^{-1/2} + (hN^\gamma)^{1/2}\right] \\ &\ll N^{1-\gamma+1/2}X_2 \lll N^{2-2\gamma+1/2+\eta} \leq N^{1-\eta}.\end{aligned}$$

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