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BINOMIAL COEFFICIENTS WHOSE PRODUCTS ARE PERFECT kTH POWERS

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In memory of Ernst Straus

A P_k-set is a finite set of positions in Pascal's triangle which, when translated anywhere within the triangle, covers entries whose product is a perfect kth power. A characterization of such sets is obtained, and the minimum cardinality $f(k)$ of all P_k-sets is determined.

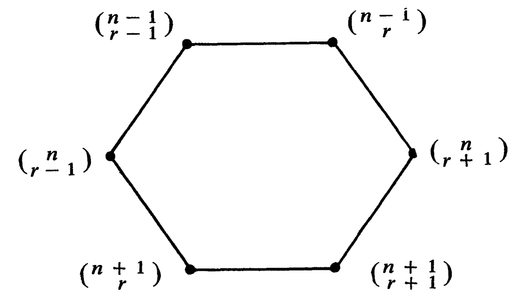
1. Introduction. In 1971, V. Hoggatt and W. Hansell [4] proved that the product of the 6 neighbors of any interior entry of the Pascal triangle is a perfect square. A corollary of this appeared as a problem on the Putnam Examination [9]. In Figure 1, for example, the product of the 6 entries enclosed in circles or squares is $360{,}000 = 600^2$.

The usual proof of this theorem consists in showing that the product of the circled entries is equal to the product of the entries enclosed in squares, i.e.

$$(1)\qquad \binom{n-1}{r-1}\binom{n}{r+1}\binom{n+1}{r}=\binom{n-1}{r}\binom{n}{r-1}\binom{n+1}{r+1}.$$

Because of the positions in Figure 1 of the factors on the 2 sides of (1), this identity has been called the *Star of David property* of Pascal's triangle.

To reformulate the star of David Theorem, we consider the following hexagon:



The theorem says that if this hexagon is translated in such a way that its vertices lie on entries of the Pascal triangle, the product of these entries is

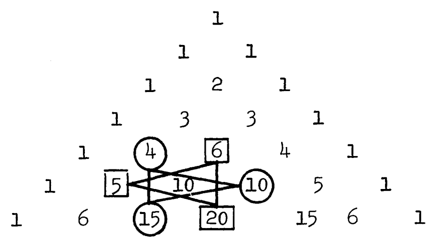


FIGURE 1

always a perfect square. Subsequently, many other configurations with this so-called "translatable perfect-square property" were found [**2**], [**5**], [**6**], [**7**], [**8**]. These examples lead naturally to the following problems.

Problem 1. Given an integer $k \geq 2$, are there configurations of points which, when translated so that they lie on entries of the Pascal triangle, always make the product of these entries a perfect kth power?

Problem 2. Characterize all sets S with this translatable perfect kth power property. (We call them P_k-sets.)

Problem 3. Determine the smallest possible cardinality $f(k)$ of a P_k-set.

Problem 2 was raised by Hoggatt [**6**] and Uisiskin [**8**] for the special case $k = 2$.

In this paper we will present solutions to these problems.

2. Characterization of P_k-sets. In this paper the symbol $\binom{n}{r}$ denotes both the value $n!/r!(n-r)!$ of the binomial coefficient, and its position in the Pascal triangle. Hopefully this abuse of notation will cause no confusion. The usual inequalities $0 \leq r \leq n$ are assumed throughout.

In order to determine all P_k-sets, we first consider the entries of Pascal's triangle modulo a prime p. We denote by A_p the set of entries $\binom{n}{r}$ with $n < p$, by B_p the set of entries with $p \leq n \leq 2p-1$, $0 \leq r \leq n-p$, by C_p the set of entries with $p \leq n \leq 2p-1$, $p \leq r \leq n$, and by T_p the set of entries with $p \leq n \leq 2p-1$, $n-p+1 \leq r \leq p-1$. The following figure shows these sets in the case $p = 5$.

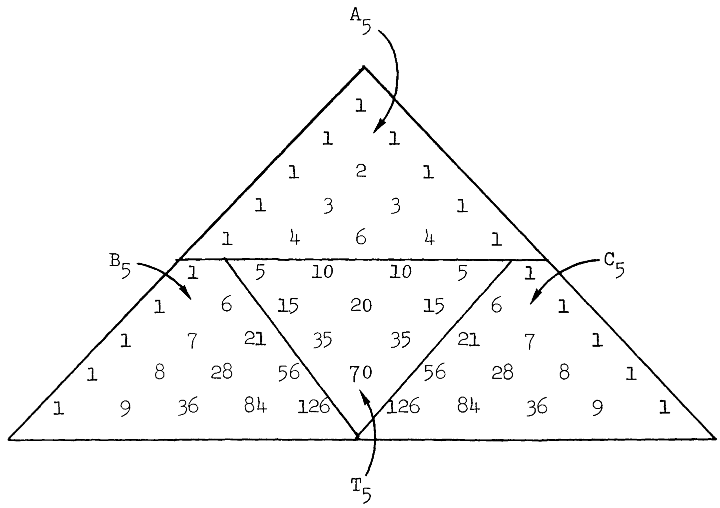


FIGURE 2

LEMMA 1. *Let p be a prime. The binomial coefficient* $\binom{n}{r}$ *is not divisible by p if* $\binom{n}{r}$ *is in* A_p, B_p *or* C_p.

Proof. Clearly $\binom{n}{r}$ is not divisible by p if $n < p$, i.e. if $\binom{n}{r} \in A_p$.

Next suppose that $\binom{n}{r} \in B_p$. Since $p \le n \le 2p - 1$, the numerator of $\binom{n}{r} = n!/r!(n-r)!$ is divisible by p but not by p^2. If $0 \le r \le n - p$, then $(n-r)!$ is divisible by p, while if $p \le r \le n$, then $r!$ is divisible by p. In either case $\binom{n}{r}$ is not divisible by p. Finally the symmetry

$$\binom{n}{r} = \binom{n}{n-r}$$

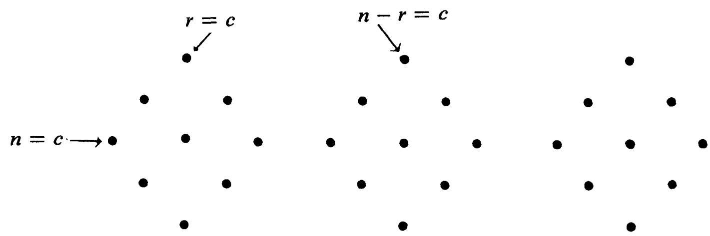
implies that $\binom{n}{r}$ is not divisible by p if it lies in C_p.

LEMMA 2. *If* $\binom{n}{r}$ *is in* T_p, *it is divisible by p but not by* p^2.

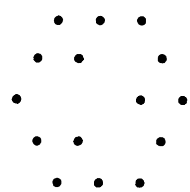
Proof. The numerator of $\binom{n}{r} = n!/r!(n-r)!$ is divisible by p but not by p^2, since $p \le n \le 2p - 1$. The denominator is not divisible by p, since $r \le p - 1$ and $n - r \le p - 1$.

Using these lemmas, we can obtain the following characterization of P_k-sets.

THEOREM 1. *S is a* P_k*-set if and only if the number of points of S on each line* $n = c$, $r = c$, *and* $n - r = c$ (*c any constant*) *is a multiple of k*.



(a)



(b)

FIGURE 3

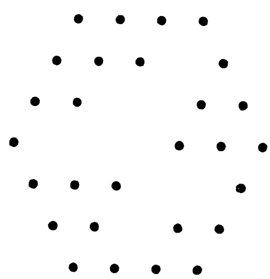


FIGURE 4

For example, the sets in Figure 3 have the perfect cube property, while the set in Figure 4 has the perfect 4th power property

Proof of Theorem 1. (i) Suppose first that the number of points of S on each line $n = c$, $r = c$ and $n - r = c$ is divisible by k. Then the product of the binomial coefficients covered by any translate of S is of the form

$$\prod_{i=1}^{|S|} \frac{n_i!}{r_i!(n_i - r_i)!}, \tag{2}$$

where for each integer N, the numbers of factors $n_i!$, $r_i!$, $(n_i - r_i)!$ equal to N are multiples of k. Therefore (2) is a perfect kth power.

(ii) Conversely, suppose that S is a P_k-set. Let p be a prime so large that S can be translated into each of the sets A_p, B_p, C_p and T_p. When S is translated into T_p, the product of the binomial coefficients covered by it is exactly divisible by $p^{|S|}$. When S is then translated vertically so that its topmost elements go into the set A_p, this power of p decreases to $p^{|S|-\nu}$, where ν is the number of elements of S on its top row. Therefore both $|S|$ and $|S| - \nu$ are divisible by k, from which we conclude that $\nu \equiv 0 \pmod{k}$. We now continue to translate S upwards; the same reasoning shows that the number of points on each horizontal row of S is divisible by k. Similarly, by translating S in the direction of B_p or C_p, we find that the number of points of S on each line $r = c$ and $n - r = c$ is a multiple of k.

COROLLARY. *If S is a P_k-set, then $|S|$ is divisible by k.*

3. P_k-sets of minimal cardinality. Let $f(k)$ be the smallest cardinality of all P_k-sets. In this section we will prove that $f(k) = k(2k - 1)$. In particular $f(2) = 6$, which shows that the original Hoggatt-Hansell example of a perfect square set [4] is minimal. Moreover $f(3) = 15$ and $f(4) = 28$, so Figures 3b and 4 are respectively minimal P_3- and P_4-sets.

THEOREM 2. $f(k) = k(2k - 1)$.

Proof. (i) We first show that if S is any P_k-set, then $|S| \geq k(2k - 1)$. For this purpose, we translate S within the Pascal triangle so that at least one of its elements lies on the "left side" $r = 0$ of the triangle (cf. Figure 5, where $k = 3$).

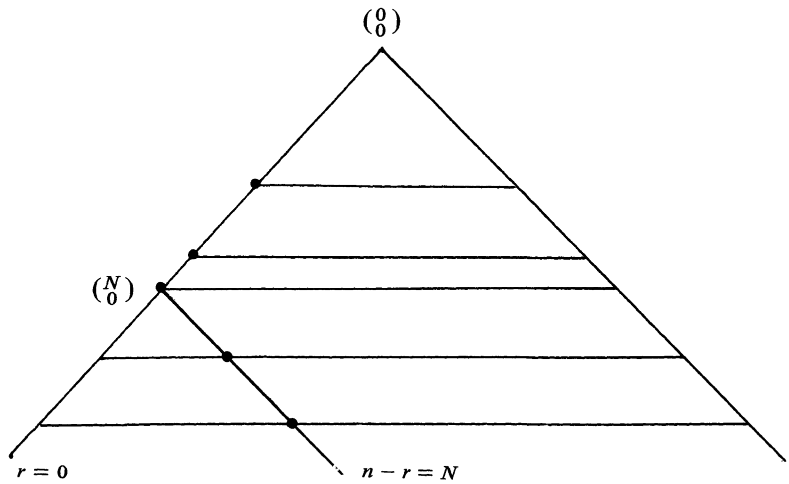


FIGURE 5

In this position, let $\binom{N}{0}$ be the element $\binom{n}{0}$ of S with the greatest value of n. By the characterization given in Theorem 1, S contains at least k points on the line $r = 0$, and at least k points on the line $n - r = N$. This gives a total of $2k - 1$ points $\binom{n}{r}$ of S, all having different values of n. The horizontal lines through these points are therefore distinct, and each such line contains at least k points of S. Hence $|S| \geq k(2k - 1)$.

(ii) To complete the proof of Theorem 2, we must construct a P_k-set S_k with cardinality $k(2k - 1)$. Many such constructions are possible; the one presented here is motivated by a consideration of the continuous analogue ($k \to \infty$) of the problem. In this analogue, we seek a subset S_∞ of a regular hexagon H of unit side such that every straight line parallel to a side of H is either disjoint from S_∞, or intersects S_∞ in a set of linear Lebesgue measure 1.

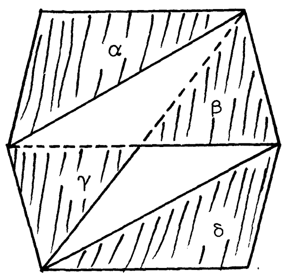


FIGURE 6

The shaded region S_∞ in Figure 6 (where solid lines are part of S_∞, but dotted lines are not) is easily seen to have this property. The sets S_k described below can be regarded as suitably normalized discrete approximations to S_∞. Examples for $k = 3$ and $k = 4$ are provided by Figures 3b and 4 respectively.

The technical details are as follows.

Let S_k consist of all entries $\binom{n}{r}$ of the Pascal triangle satisfying one of the 4 following conditions:

$$
\begin{array}{lll}
(\alpha) & k - 1 \leq n \leq 2k - 2, & 0 \leq r \leq 2k - 2 - n \\
(\beta) & k - 1 \leq n \leq 2k - 2, & k \leq r \leq n \\
(\gamma) & 2k - 1 \leq n \leq 3k - 3, & n - 2k + 2 \leq r \leq k - 1 \\
(\delta) & 2k - 1 \leq n \leq 3k - 3, & 4k - 3 - n \leq r \leq 2k - 2
\end{array}
$$

The Greek letters here have been chosen to correspond to the four shaded triangles in Figure 6.

We assert that for any integer c, the number of points of S_k on the line $n = c$ is k if $k - 1 \leq c \leq 3k - 3$, and is 0 otherwise. The second assertion is clear. To prove the first one, we suppose first that $k-1 \leq c \leq 2k-2$. From (α) and (β) we find that the number of points of S_k on the line $n = c$ is

$$(2k-2-c+1)+(c-k+1)=k.$$

On the other hand, if $2k-1 \leq c \leq 3k-3$, we find from (γ) and (δ) that the number of points of S_k on the line $n = c$ is

$$[(k-1)-(c-2k+2)+1]+[(2k-2)-(4k-3-c)+1]=k.$$

It follows in particular that

$$|S_k| = [(3k-3)-(k-1)+1]k = (2k-1)k.$$

Next, we note that the inequalities defining S_k can be rewritten in the form

$$\begin{array}{lll} (\alpha) & 0 \leq r \leq k-1, & k-1 \leq n \leq 2k-2-r \\ (\beta) & k \leq r \leq 2k-2, & r \leq n \leq 2k-2 \\ (\gamma) & 0 \leq r \leq k-1, & 2k-1 \leq n \leq 2k-2+r \\ (\delta) & k \leq r \leq 2k-2, & 4k-3-r \leq n \leq 3k-3. \end{array}$$

From this we can compute the number of points of S_k on each line $r = c$, where c is an integer. This number is clearly 0 unless $0 \leq c \leq 2k-2$. If $0 \leq c \leq k-1$, then (α) and (γ) show that the number of points of S_k on the line $r = c$ is

$$[(2k-2-c)-(k-1)+1]+[(2k-2+c)-(2k-1)+1]=k.$$

On the other hand, if $k \leq c \leq 2k-2$, then (β) and (δ) show that this number is

$$[(2k-2)-c+1]+[(3k-3)-(4k-3-c)+1]=k.$$

In the same way it can be shown that the number of points of S_k on the line $n - r = c$ is k if $0 \leq c \leq 2k-2$, and 0 otherwise. The verification of this is rather tedious, since the portions of S_k defined by (α) and (δ) must be divided into two parts. We therefore omit the details.

4. Concluding remarks. In the preceding section, we determined P_k-sets of minimal cardinality. There are, of course, other ways of measuring minimality of a set. For example, suppose that the location of the Pascal triangle in the plane is normalized so that the distance between nearest neighbors of the triangle is 1. We can then ask for a P_k-set of minimal diameter, or for one whose convex hull has minimal area. The set S_k constructed above has diameter $2(k-1)$, and its convex hull has area $3(k-1)^2\sqrt{3}/2$. These values are minimal, since the convex hull of any

P_k-set S contains a regular hexagon of side $k - 1$. This is easily seen by allowing lines parallel to the sides of a regular hexagon (in the obvious orientation) to approach S from outside its convex hull until they first intersect S. The convex hulls of these intersections have length $\geq k - 1$, from which the desired property follows.

Among many generalizations and analogues of the above results, we mention those where binomial coefficients are replaced by multinomial coefficients, and where (especially in the Star of David property) products of binomial coefficients are replaced by g.c.d.'s or l.c.m.'s. These extensions lead to some very curious arithmetic and geometric theorems which, for reasons of space, cannot be dealt with here.

Finally, in connection with the continuous analogue discussed in §3, we pose the following problem:

Does there exist a non-empty plane set whose intersection with each line in any of 4 given directions is either empty or of linear Lebesgue measure 1? This problem also has natural extensions to sets in Euclidean space $\mathbf{R}^n$.

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