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EPIDEMIOGRAPHY

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Dedicated to the memory of Ernst G. Straus

Epidemiography designates a class of games played on directed graphs. At stage k of the game, the move made on a graph G is replicated onto k isomorphic copies of G. The player first unable to move is the loser; his opponent the winner. We give conditions on G under which a game terminates, and determine the winner for a number of cases.

1. Introduction. We like to investigate the following form of epidemic chorea (*Dancing Mania*). Let $G_1, G_2, \ldots$ be countably many copies of a given connected directed graph G and let u_1 be a labeled vertex of G_1. At the beginning, u_1 is the only labeled vertex in G_1 and there are no labeled vertices in any of the other copies. Player I now removes the label from u_1, labels in G_1 a vertex u_2 dominated by u_1 and also labels u_2 in G_2. The players alternate turns. At stage k, a player selects any labeled vertex u_i in any of the copies G_j, removes the label from u_i and labels some u_{i+1} dominated by u_i in G_j. He also labels u_{i+1} in k as yet completely unlabeled copies of G.

We shall always assume here that the player first unable to move is the loser. His opponent is the winner, because he manages to kill the epidemic. If G is infinite or contains cycles, the epidemic may rage on and on, in which case the game outcome is declared a draw. However, if G is a finite directed acyclic graph, then the game terminates and one of the players has a winning strategy. This is proved in §2.

Several perverse and maniacal forms of the malady will be examined in §§5 and 6. In §§3 and 4, however, we will prescribe a remedy for the special case where G is a finite directed simple path (one of length 5 is shown in Figure 1). In this case the disease is also known as the Nim epidemic (*Nimania*), which is sometimes observed in post-*pneumonia* patients. As the theory of Nim is a key to the Sprague-Grundy theory for impartial acyclic 2-player games with perfect information and no chance moves, so the proper treatment of Nimania may turn out to be the key to the successful treatment of other forms of Dancing Mania. This is the reason we treat it first.

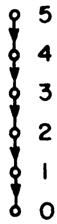


FIGURE 1. Nimania is played on a directed simple path.

Dancing Mania was motivated by the ∞-predecessor Hydra game of Nešetřil [4], which was studied extensively by M. Loebl and P. Savický.

2. There exists a remedy for common mania. Dancing Mania played on a connected finite directed acyclic graph is known as Common Mania (*Coma*). Nimania is of course a special case of Coma.

THEOREM 1. *Coma is a finite game, and one of the two players has a winning strategy.*

Proof. Let $G = (V, E)$ be the connected finite directed acyclic graph on which Coma is played. For $u \in V$, denote by $h(u)$ the length of a longest path emanating from u. Put $H = \max_{u \in V} h(u)$.

With any position P of the game, associate a sequence $(n_1, \ldots, n_H)$ of nonnegative integers, where n_i is the number of copies G_t of G in which there exists a labeled vertex u_j satisfying $h_{G_t}(u_j) = H - i + 1$ $(i \in \{1, \ldots, H+1\})$. With this coding, the set of all coma-positions corresponds to a subset of the set of all sequences $\mathbf{Z}^H$ of nonnegative integers.

Consider the lexicographic ordering $<$ of $\mathbf{Z}^H$:

$$(m_1, \ldots, m_H) < (n_1, \ldots, n_H)$$

if there exists j such that $m_i = n_i$ for $i < j$ and $m_j < n_j$.

Let P' be the position after a move of a player from position P at stage k. Let P' be coded by $(m_1, \ldots, m_H)$. Since $(u, v) \in E$ implies $h(u) > h(v)$, the sequence $(m_1, \ldots, m_H)$ has either the form

$$(m_1, \ldots, m_H) = \left(n_1, \ldots, n_i - 1, n_{i+1}, \ldots, n_j + k + 1, \ldots, n_H\right)$$

for some $i < H$ and some $j > i$, or the form

$$(m_1, \ldots, m_H) = (n_1, \ldots, n_{H-1}, n_H - 1).$$

In either case, $(m_1, \ldots, m_H) < (n_1, \ldots, n_H)$. Since this lexicographic ordering is a well-ordering of the set $\mathbf{Z}^H$, the game will terminate after a finite number of moves. For games with this latter property, it is known that one of the two players has a winning strategy. See e.g. Berlekamp, Conway and Guy [**2**, the extras of Chapters 1 and 2]. □

3. Nimania. To find a remedy for Nimania, it is convenient to forget about graphs, reverting to the language of nonnegative integers instead. In Nimania, two players, I and II, alternate in making moves. Given a positive integer n. In his first move, player I subtracts 1 from n. If $n = 1$, the result is the empty set and the game ends with player I winning. If $n > 1$, one additional copy of the resulting number $n - 1$ is adjoined, so at the end of the first move there are two (indistinguishable) copies of $n - 1$ (denoted by $(n - 1)^2$). At the k-th stage, a player selects a copy of a positive integer m of the present position, and subtracts 1 from it. If $m = 1$, the copy is deleted. If $m > 1$, k copies of $m - 1$ are adjoined to the resulting $m - 1$ ($k \geq 1$). The player first unable to move loses and his opponent wins.

Since the numbers in successive positions decrease, it is clear that the game terminates. This also follows directly from Theorem 1. The basic question we address ourselves to is whether player I or II wins for any given n.

EXAMPLE. (i) $n = 1$. As we saw above, player I wins. (ii) $n = 2$. Player I moves to 1^2, player II to 1, hence player I again wins. (iii) $n = 3$. The following self-explanatory diagram (Figure 2) shows that again player I can win. Unlike the above two cases, however, not all moves of player I are winning: Player I has to select his moves carefully to win, following the lower branches of the diagram.

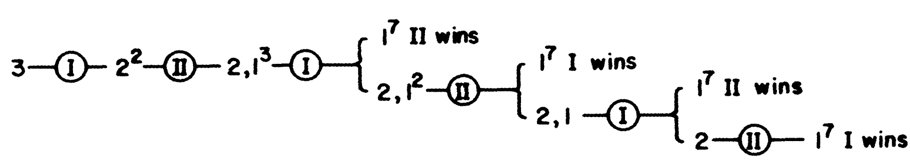


FIGURE 2. A proof that player I wins $n = 3$ of Nimania. The numbers in circles indicate the player making the move.

An attempt to resolve the problem for $n = 4$ by a similar diagram construction is rather frustrating, because of the size and the many branches of such a diagram.

The purpose of the next section is to prove the rather surprising fact that player I has a winning strategy for every $n \geq 1$.

4. Player I conquers Nimania recursively.

LEMMA 1. (i) *Starting from a position* 1^a *at stage* k, *player* I *can win iff* a *is odd* ($a \geq 0$, $k \geq 1$).

(ii) *Starting from the position* $2, 1^a$ *at stage* k, *player* I *can win in Nimania iff either* k *is odd or* a *is odd* ($a \geq 0$, $k \geq 1$).

Proof. (i) Clear. (ii) The options of the given position are depicted in Figure 3. On the upper path, II wins iff $a + k + 1$ is odd, iff $a + k$ is even. So I wins on the upper path iff $a + k$ is odd. Thus player I will opt the lower path if $a + k$ is even.

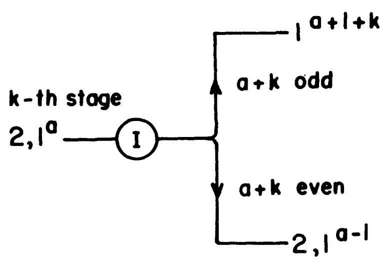


FIGURE 3. The options of $2, 1^a$ at stage k of Nimania.

Note that if $a + k$ is even, we may assume that every player at his turn will reduce the multiplicity of 1 until it becomes 0, because otherwise he leaves his opponent in position 1^{k+a+1}, from which the opponent can win. At stage $k + a$ the position is thus 2, and the next player moves to 1^{k+a+1} and loses. Therefore the player who moved to position 2 wins. This player is I iff a is odd, iff k is odd. □

The following theorem deals with a general position of the form n^{α_n}, $(n-1)^{\alpha_{n-1}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}$ with $\alpha_i \geq 0$ $(1 \leq i \leq n)$. From it our result follows readily.

THEOREM 2. *Starting from a position* $A_n = n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}$ *at stage* k, *with the proviso that either* $\alpha_2 > 1$ *or* $\alpha_i > 0$ *for some* $3 \leq i \leq n$, *player* I *can win in Nimania iff* k *is odd* ($k \geq 1$).

Proof. We shall use induction on the size of the position, where positions are ordered lexicographically. That is, we assume the truth of the assertion for a subset of the set of positions satisfying the proviso which are lexicographically less than A_n, and show that it implies the truth of the assertion for A_n.

The set of options of A_n is

$$\begin{aligned} S = \Big\{ B_n &= n^{\alpha_n - 1}, (n-1)^{\alpha_{n-1}+1+k}, (n-2)^{\alpha_{n-2}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}, \\ B_{n-1} &= n^{\alpha_n}, (n-1)^{\alpha_{n-1}-1}, (n-2)^{\alpha_{n-2}+1+k}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}, \\ &\vdots \\ B_2 &= n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, (n-2)^{\alpha_{n-2}}, \ldots, 2^{\alpha_2 - 1}, 1^{\alpha_1+1+k}, \\ B_1 &= n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, (n-2)^{\alpha_{n-2}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1 - 1} \Big\}, \end{aligned}$$

where the option B_i is in the set S iff $\alpha_i > 0$ $(1 \le i \le n)$.

If $a_i > 0$ for some $i \ge 3$, then every option B_j of A_n again satisfies the proviso of the theorem. This is clear for $j \ne i$. In B_i, α_i transforms into $\alpha_i - 1$ and α_{i-1} into $\alpha'_{i-1} = \alpha_{i-1} + k + 1$. If $i > 3$, then the proviso is satisfied, since $\alpha'_{i-1} > 0$ and $i - 1 \ge 3$. If $i = 3$, the proviso is satisfied since $\alpha'_2 = \alpha_2 + k + 1 \ge 2$. Therefore the induction hypothesis implies that player II can win starting from B_j iff $k + 1$ is odd, iff k is even, for every option $B_j \in S$ $(1 \le j \le n)$. Equivalently, player I can win iff k is odd.

We may thus assume $\alpha_i = 0$ $(i \ge 3)$. Then $A_n = A_2 = 2^{\alpha_2}, 1^{\alpha_1}$, $\alpha_2 \ge 2$, so $B_2 = 2^{\alpha_2 - 1}, 1^{\alpha_1+1+k}$. If $\alpha_1 > 0$, then $B_1 = 2^{\alpha_2}, 1^{\alpha_1 - 1}$ exists, and the induction hypothesis implies that if player II moves from B_1, then player I can win iff k is odd. If $\alpha_2 > 2$, then the induction hypothesis applied to B_2 implies again that player I can win iff k is odd. If $\alpha_2 = 2$, then $B_2 = 2$, 1^{α_1+1+k}. By Lemma 1(ii), player II can win iff $k + 1$ is odd or $\alpha_1 + 1 + k$ is odd, iff k even or $k + \alpha_1$ even. So player I can win in B_2 iff k is odd and α_1 is even. It follows (considering also the case $\alpha_1 = 0$) that player I can win iff k is odd. □

COROLLARY 1. *Starting from any position n in Nimania (at stage* 1), *player* II *wins trivially if $n = 0$, and player* I *has a winning strategy for every* $n \ge 1$.

Proof. This is the special case $k = 1$, $\alpha_n = 1$, $\alpha_i = 0$ $(1 \le i < n)$ of Theorem 1 when $n \ge 3$. For $n = 1$ and 2 the result was considered in the above examples. □

Translated back into graph-terminology, Corollary 1 states: Given a finite directed simple path G with a vertex u initially labeled. Then player II wins trivially in Nimania on G (at stage 1) if $h(u) = 0$, and player I can win for every u with $h(u) \geq 1$.

5. Raving epidemy and other variations of Nimania. The general idea of raving epidemics is that instead of labeling an immediately dominated node of a previously labeled node u_i, we label at stage k a node u_{i+1} such that the directed path $(u_i, \ldots, u_{i+1})$ contains p edges. As before we also label u_{i+1} in k as yet unlabeled copies of G. This epidemic comes in a number of variations: (i) The node u_{i+1} must be at *distance* p from u_i. (ii) The node u_{i+1} must be on a path emanating from u_i, where the *path length* between u_i and u_{i+1} is p (but the distance may be less than p). (iii) Label *all* nodes at distance p from u_i. (iv) Label all nodes at the end of paths of length p from u_i. (v) Label *some* vertices at distance p from u_i. (vi) Label some vertices at the end of paths of length p from u_i.

In order to relate some of these variations to the class of games played on directed graphs as introduced in §1, the following definition is useful.

DEFINITION. Two games are called *equivalent* if they have the same value.

In the present context of impartial games, the value of a game is the Sprague-Grundy function value of the game-graph of the game. See e.g. Conway **[3]** or **[2]**.

Let $G = (V, E)$ be a connected directed graph with a vertex u initially labeled. Construct $G_1 = (V_1, E_1)$, where V_1 and E_1 are defined recursively as follows: $u \in V_1$. For every $v \in V_1$, put w into V_1 and (v, w) into E_1 if the distance from v to w in G is p. For every $x \in V_1$ put (x, w) into E_1 if the distance from x to w in G is p. Then Dancing Mania on G_1 is equivalent to variation (i) on G. (A formal proof of this equivalence can be given along the lines of the second part of the proof of Theorem 3 below.) Now define $G_2 = (V_2, E_2)$ recursively: $u \in V_2$. For every $v \in V_2$, put w into V_2 and (v, w) into E_2 if there exists a path from v to w of length p in G. For every $x \in V_2$ put (x, w) into E_2 if there exists a path from x to w of length p in G. Then Dancing Mania on G_2 is equivalent to variation (ii) on G.

REMARK. The variations (iii)–(vi) do not fit easily into the general framework of directed graphs set up in Section 1 but an argument similar to Theorem 1 shows that if G is an acyclic directed graph and every vertex of G has finitely many successors, then each variation constitutes a finite game. (A vertex v is a *successor* of u if there is a directed path from u to v.)

Note that if G is a finite directed simple path such as in Figure 1, then all these six variations coincide. We call this game *Nimania*$_p$ (where Nimania$_1$ = Nimania). We show below how to cure Nimania$_p$ for any $p \geq 1$ by reducing the symptoms to those of Nimania.

THEOREM 3. *Let $G = (V, E)$ be a connected finite directed acyclic graph with the following property*:

(P) *For every pair of vertices x and y, all the paths from x to y have the same length.*

Then Coma on G is equivalent to Nimania played on some finite simple path. In particular, player I *can win Coma on G iff the initially labeled vertex u satisfies $h(u) \geq 1$.*

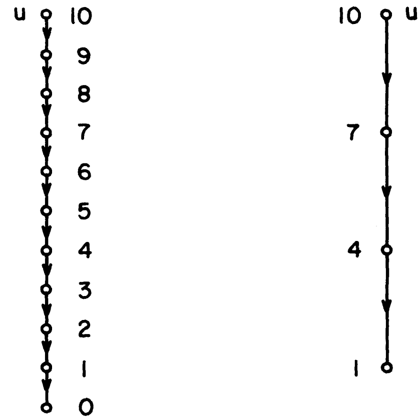
Proof. Let h be the function defined in §2. For $(u, v) \in E$ we have $h(u) = h(v) + l$ for some $l \geq 1$. Since there is a path of length 1 between u and v, property (P) implies that $l = 1$. Therefore the graph on which the actual Coma game is started is a simple path of length $h(u)$, if u is the initially labeled vertex. This graph is isomorphic to that of Nimania played on a finite directed simple path of length $h(u)$.

Consider the sum (disjunctive compound) game (Dancing Mania (G), Nimania $(h(u))$, where u is the initially labeled vertex in G). For the notion of sum of games see e.g. [3] or [2]. Since the two graphs are isomorphic, player II can win by playing moves *corresponding* to those of player I in the set of graphs isomorphic to the set in which player I moves. Hence the Sprague-Grundy function value of the game-graph of Coma on G is the same as that of the game-graph of Nimania played on a path of length $h(u)$. In particular, player I can win Coma played on G iff $h(u) \geq 1$. $\square$

COROLLARY 2. *Let m be a nonnegative integer. Starting from position m of Nimania$_p$ (in stage 1), player* II *wins trivially if $m < p$, and player* I *can win for every $m \geq p$.*

Proof. We revert to graph-theoretic language. If G is a directed simple path, then the graph G_1 induced by G clearly satisfies property (P). It follows that player I can win for every $m \geq p$.

To illustrate Corollary 2, Figure 4(a) shows a directed simple path of length 10. The induced graph G_1 for $p = 3$ is shown in (b), which is again a directed simple path.



(a) The graph G. (b) The induced graph G_1.

FIGURE 4. Nimania$_3$ played on a directed simple path G of length 10.

Let S be a partially ordered set. For $a, b \in S$, we say that *a covers b* if $b < a$ and there is no $x \in S$ satisfying $b < x < a$. *The covering relation graph G* induced by S is the directed graph $G = (S, E)$, where $(a, b) \in E$ if a covers b.

COROLLARY 3. *Let G be the covering relation graph induced by a finite semi-modular lattice. Then player* I *wins Dancing Mania on G.*

Proof. Any semi-modular lattice satisfies the Jordan-Dedekind chain condition, which states that all maximal chains between fixed points have the same length. See e.g. Abbott [**1**, Ch. 5]. This condition is the same as property (P). □

REMARK. Property (P) is not a necessary condition for the existence of a winning strategy for player I. Thus the digraph depicted in Figure 5 violates property (P), but Corollary 1 implies that player I can win if u is initially labeled, since the first move decides which of the two directed simple paths the game will unravel on.

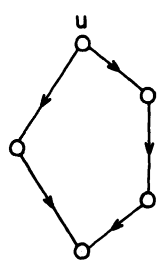


FIGURE 5. Player-I can win Dancing Mania on this graph which violates property (P).

6. Sweeping epidemy. Sweeping epidemy is a further complication of raving epidemy. Instead of labeling a node u_{i+1} on a directed path emanating from u_i removed from u_i by exactly p edges as in the latter, we may label, in the former, any node lying on this path *up to* p edges away from u_i. Everything else is as in raving epidemy. Analogously to the latter, sweeping epidemy has a number of variations. For the case of a directed simple path, for which we denote the game by Nimania$_{\leq p}$, these variations are again identical.

In this section we describe a recursive cure for the case $p = 2$, that is, for Nimania$_{\leq 2}$. Note that Nimania$_{\leq 2}$ on a finite directed simple path $G = (V, E)$ is equivalent to Dancing Mania on $G_3 = (V, E_3)$ where $E_3 = E \cup E_4$ and $(x, y) \in E_4$ if the distance from x to y in G is 2.

EXAMPLES. (i) $n = 1$. Clearly player I wins. (ii) $n = 2$. Player I can move to 0 winning swiftly, or he can move to 1^2, in which case he also wins, though only after having prolonged the malady. (iii) $n = 3$. Player I can move to 1^2 and again win.

Note the striking similarity to the examples at the end of §3. When the interns at Radixal University Hospital were confronted with this evidence, they naturally concluded that the prognosis of Nimania$_{\leq 2}$ is the same as for Nimania. They were in for a shock when Dr. Manny Plaeg, the Department Head, pointed out to them, many victims later, that the above examples are the *only* cases where player I can win: For any $n > 3$, player II can in fact win! We now proceed to prove this.

LEMMA 2. *Starting from the position* $2, 1^a$ *at stage* k, *player* I *can win in Nimania*$_{\leq 2}$ *iff either* k *is even or* a *is even* $(a \geq 0, k \geq 1)$.

Proof. From $2, 1^{2a}$ player I moves to 1^{2a} and wins. If k is even, then player I moves from $2, 1^{2a+1}$ to 1^{k+2a+2}, winning. If k is odd, however, then this move is losing, as are also the remaining two moves: to 1^{2a+1} and to $2, 1^{2a}$. □

LEMMA 3. (i) *Starting from position* 2^2 *at stage* k, *player* I *can win in Nimania*$_{\leq 2}$ *iff* k *is even* $(k \geq 1)$.

(ii) *Starting from position* $2^b, 1^a$ *at stage* k, *player* I *can win in Nimania*$_{\leq 2}$ *iff* k *is even* $(b \geq 2, a \geq 0, k \geq 1)$.

Proof. (i) The options of 2^2 are shown in Figure 6.

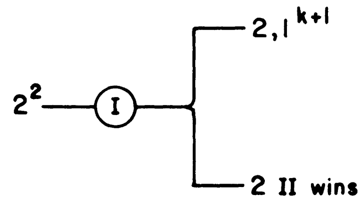


FIGURE 6. The options of 2^2 at stage k of Nimania$_{\leq 2}$.

On the upper path, Lemma 2 implies that player II can win iff $k+1$ is even, iff k is odd. Hence player I can win iff k is even.

(ii) We use induction on $a+b$. The options of the position are shown in Figure 7.

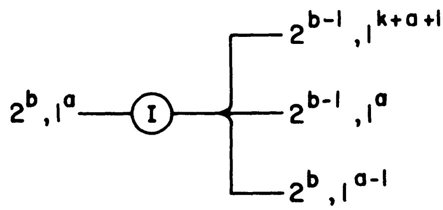


FIGURE 7. The options of $2^b, 1^a$ at stage k of Nimania$_{\leq 2}$.

Suppose first $b = 2$. By Lemma 2, player II may lose on the top path iff $k+1$ is odd and a is even. So player I can win on the top path iff both k and a are even. On the middle path, player II may lose iff $k+1$ and a are both odd. Thus player I can win on the middle path iff k is even and a is odd. By part (i) we may assume $a > 0$. Hence the bottom path exists. On it, the induction hypothesis and part (i) imply that player II can win iff $k+1$ is even, iff k is odd. So player I can win on the bottom path iff k is even. A simple best strategy for player I is thus to use the bottom path.

Now suppose $b > 2$. Then on each of the paths the induction hypothesis implies that player II can win iff $k+1$ is even, iff k is odd. Thus player I can win iff k is even. □

LEMMA 4 (i) *Starting from position* 3 *at stage* k, *player* I *can win in Nimania*$_{\leq 2}$ $(k \geq 1)$.

(ii) *Starting from the position* $3, 1^a$ *at stage* k, *player* I *can win in Nimania*$_{\leq 2}$ *iff either* k *is even or* a *is even* $(a \geq 0,\ k \geq 1)$.

Proof. (i) If k is even, player I moves to 2^{k+1} to win (Lemma 3 (ii)), otherwise he moves to 1^{k+1} and wins.

(ii) The options are shown in Figure 8. By Lemma 3 (ii), player I can win on the top path iff k is even. On the middle path, player I can win iff $k+a$ is odd. By part (i) we may assume $a > 0$. Therefore the bottom path

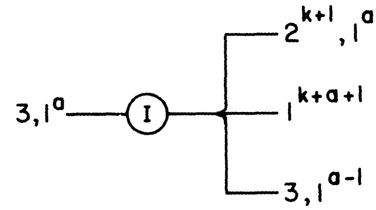


FIGURE 8. The options of $3, 1^a$ at stage k of Nimania$_{\leq 2}$.

exists, and by induction player II can win iff $a - 1$ is even or $k + 1$ is even. So player I can win on this path iff a and k are both even.

It follows that the only case where player I may lose is when both a and k are odd. □

An analog of Theorem 2 for Nimania$_{\leq 2}$ will now be proved. Our result (Corollary 4) follows immediately from it.

THEOREM 4. *Starting from a position* $A_n = n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}$ *at stage* k, *with the proviso that either* $\alpha_2 + \alpha_3 > 1$ *or* $\alpha_i > 0$ *for some* $4 \leq i \leq n$, *player* I *can win in Nimania*$_{\leq 2}$ *iff* k *is even* ($k \geq 1$).

Proof. As in the proof of Theorem 2, we shall again use induction on the size of the position.

The set of options of A_n is

$$
\begin{aligned}
S = \Big\{ B_n &= n^{\alpha_n - 1}, (n-1)^{\alpha_{n-1}+k+1}, (n-2)^{\alpha_{n-2}}, (n-3)^{\alpha_{n-3}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}, \\
C_n &= n^{\alpha_n - 1}, (n-1)^{\alpha_{n-1}}, (n-2)^{\alpha_{n-2}+k+1}, (n-3)^{\alpha_{n-3}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}, \\
B_{n-1} &= n^{\alpha_n}, (n-1)^{\alpha_{n-1}-1}, (n-2)^{\alpha_{n-2}+k+1}, (n-3)^{\alpha_{n-3}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}, \\
C_{n-1} &= n^{\alpha_n}, (n-1)^{\alpha_{n-1}-1}, (n-2)^{\alpha_{n-2}}, (n-3)^{\alpha_{n-3}+k+1}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}, \\
&\vdots \\
B_2 &= n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, (n-2)^{\alpha_{n-2}}, (n-3)^{\alpha_{n-3}}, \ldots, 2^{\alpha_2 - 1}, 1^{\alpha_1+k+1}, \\
C_2 &= n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, (n-2)^{\alpha_{n-2}}, (n-3)^{\alpha_{n-3}}, \ldots, 2^{\alpha_2 - 1}, 1^{\alpha_1}, \\
B_1 = C_1 &= n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, (n-2)^{\alpha_{n-2}}, (n-3)^{\alpha_{n-3}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1 - 1} \Big\},
\end{aligned}
$$

where the options B_i and C_i are in S iff $\alpha_i > 0$ $(1 \leq i \leq n)$.

If $\alpha_i > 0$ for some $i \geq 4$, then every option B_j and C_j of A_n again satisfies the proviso of the theorem. This is clear for every option B_j and

C_j with $j \neq i$. In B_i, α_i transforms into $\alpha_i - 1$ and α_{i-1} into $\alpha'_{i-1} = \alpha_{i-1} + k + 1$. If $i > 4$, then the proviso is satisfied, since $\alpha'_{i-1} > 0$ and $i - 1 \geq 4$. If $i = 4$, then the proviso is satisfied since $\alpha'_3 = \alpha_3 + k + 1 \geq 2$. In C_i, α_i transforms into $\alpha_i - 1$ and α_{i-2} into $\alpha'_{i-2} = \alpha_{i-2} + k + 1$. If $i > 5$, then the proviso is satisfied since $\alpha'_{i-2} > 0$ and $i - 2 \geq 4$. If $i = 4$ or 5, then the proviso is satisfied since then $\alpha'_{i-2} = \alpha_{i-2} + k + 1 \geq 2$, hence the sum of the second and third exponents is at least 2.

We may thus assume $\alpha_i = 0$ ($i \geq 4$). Then $A_n = A_3 = 3^{\alpha_3}, 2^{\alpha_2}, 1^{\alpha_1}$, with $\alpha_2 + \alpha_3 \geq 2$. Suppose first $\alpha_2 + \alpha_3 = 2$. There are three subcases:

(a) $A_3 = 3, 2, 1^\alpha$ ($\alpha \geq 0$). The options of A_3 are shown in Figure 9.

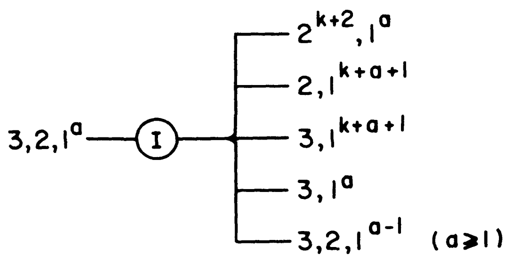


FIGURE 9. The options of $3, 2, 1^a$ at stage k of Nimania$_{\leq 2}$.

On the top path, Lemma 3 (ii) implies that player I can win iff k is even. For the second path we employ Lemma 2 to conclude that player II may lose iff both $k + 1$ and $k + a + 1$ are odd, iff both k and $k + a$ are even. Thus player I can win on the second path iff both k and a are even. Exactly the same conclusion is reached for the third path, using Lemma 4 (ii). Using the same lemma, player II may lose on the fourth path iff $k + 1$ is odd and a is odd. So player I can win on this path iff k is even and a is odd. Finally, if $a \geq 1$ the bottom path exists, and player I can win on it iff k is even by induction.

The upshot of all five options is that player I can win iff k is even.

(b) $A_3 = 3^2$, 1^a ($a \geq 0$). The options of the position are depicted in

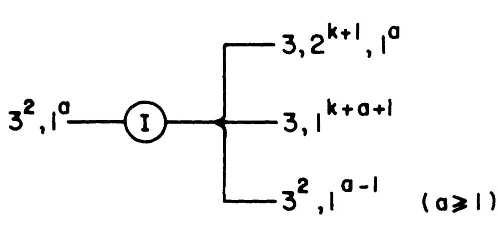


FIGURE 10. The options of $3^2, 1^a$ at stage k of Nimania$_{\leq 2}$.

Figure 10. Player I can win on the top path iff k is even by induction. On the middle path, using Lemma 4 (ii), player II may lose iff $k + 1$ and $k + a + 1$ are both odd iff k and $k + a$ are both even. Thus player I can win on this path iff k and a are both even. If $a \geq 1$ the bottom path exists, and player I can win on it iff k is even by induction.

We see again that the best strategy permits player I to win iff k is even.

(c) $A_2 = 2^2, 1^a$ $(a \geq 0)$. By Lemma 3 player I can win iff k is even.

We may thus assume $a_2 + \alpha_3 > 2$. Then the result follows by examining the options and using induction. □

COROLLARY 4. *Starting from any position* $n > 3$ (*at stage* 1), *player* II *has a winning strategy in Nimania*$_{\leq 2}$. *For* $n = 1, 2$ *and* 3, *player* I *can win*.

Proof. The case $k = 1$ and $\alpha_n = 1$ for $n \geq 4$ of Theorem 4 shows that player II can win for $n \geq 4$. The cases $n = 1, 2$ and 3 were considered in the examples above. □

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