PACIFIC JOURNAL OF MATHEMATICS
Vol. 119, No. 1, 1985

ON SOME INFINITE SERIES OF L. J. MORDELL AND THEIR ANALOGUES

M. V. SUBBARAO AND R. SITARAMACHANDRARAO

In this paper we obtain a general reciprocity relation (Theorem 2.1) for a class of double series, from which we deduce several results including two alternating double series for $\zeta(3)$ (where $\zeta(s)$ is the Riemann zeta function), which complement a result of L. J. Mordell. Later in the paper, we obtain another reciprocity relation for the double series and also extend our investigations to multiple series.

1. Introduction. In 1958 L. J. Mordell (cf. [7], Theorems I and II) considered the multiple series

$$(1.1)\qquad \sum_{l_1,\ldots,l_r=1}^{\infty} 1/l_1l_2\cdots l_r(l_1+\cdots+l_r+a)(l_1+\cdots+l_r+a+1)\cdots(l_1+\cdots+l_r+a+s)$$

where $r \geq 1$, $s \geq 0$ are integers and $a > -r$ is real. In particular, he deduced that

$$(1.2)\qquad \sum_{l_1,l_2=1}^{\infty} 1/l_1l_2(l_1+l_2) = 2\zeta(3)$$

where ζ denotes the Riemann zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for complex s with real part greater than 1 and its analytic continuation. Also he proved (cf. [7], Theorem III) that for positive integral r

$$(1.3)\qquad \sum_{l_1,l_2=1}^{\infty} 1/l_1^{2r}l_2^{2r}(l_1+l_2)^{2r} = \pi^{6r}c_{2r}$$

where c_{2r} is a rational number and that $c_2 = 1/2835$.

In §2 of this paper, we prove a reciprocity relation for a class of infinite series which are closely related to (1.2). We deduce from this, for example that

$$\sum_{l_1,l_2=1}^{\infty} (-1)^{l_1-1}/l_1l_2(l_1+l_2) = \frac{5}{8}\zeta(3),$$

$$\sum_{l_1,l_2=1}^{\infty} (-1)^{l_1+l_2}/l_1l_2(l_1+l_2) = \frac{1}{4}\zeta(3)$$

which are complementary to (1.2). In §3, we evaluate the multiple series in (1.1). Our evaluation is different from that of Mordell and involves Stirling numbers of the first kind. In §4, we prove another reciprocity relation for double series and deduce Mordell's result (1.3) above with an explicit determination of the constant c_{2r} in terms of the Bernoulli numbers.

2. A reciprocity relation for double series. Throughout the paper, $Z^{(0)}$, Z^{+}, R^{+}, R and C respectively denote the sets of all non-negative integers, positive integers, positive reals, reals and complex numbers. For bounded maps $f, g\colon Z^{+} \to C$, $u, v \in Z^{+}$ and $w \in R^{+}$, we write

$$S_{f,g}(u, v, w) = \sum_{r,k=1}^{\infty} \frac{f(r+k)g(k)}{r^{u}k^{v}(r+k)^{w}}. \tag{2.1}$$

We note that the series on the right converges absolutely since f, g are bounded functions, $r + k \geq 2\sqrt{rk}$ and $w \in R^{+}$. We also write

$$C_{f,g}(x, y) = \sum_{r=2}^{\infty} \frac{f(r)}{r^{x}} \sum_{k=1}^{r-1} \frac{g(k) + g(r-k)}{k^{y}} \tag{2.2}$$

for $x > 1$ and $y \in Z^{+}$. Then we have the following reciprocity relation

THEOREM 2.1.

$$\begin{aligned} S_{f,g}(u, v; w) + S_{f,g}(v, u; w) &= \sum_{i=0}^{u-1} \binom{v+i-1}{v-1} C_{f,g}(v+w+i, u-i) \\ &\quad + \sum_{i=0}^{v-1} \binom{u+i-1}{u-1} C_{f,g}(u+w+i, v-i). \end{aligned} \tag{2.3}$$

Proof. It is known (cf. [**8**], p. 48, eqn. (9)) that for $u, v \in Z^{+}$

$$\begin{aligned} \frac{1}{k^{v}(k-r)^{u}} &= (-1)^{u} \sum_{i=0}^{v-1} \binom{u+i-1}{u-1} \frac{1}{k^{v-i}r^{u+i}} \\ &\quad + \sum_{i=0}^{u-1} \binom{v+i-1}{v-1} \frac{(-1)^{i}}{r^{v+i}(k-r)^{u-i}}. \end{aligned} \tag{2.4}$$

Hence by (2.1)

$$S_{f,g}(u,v;w) = \sum_{r,k=1}^{\infty} \frac{f(r+k)g(k)}{r^u k^v (r+k)^w} = (-1)^u \sum_{r=2}^{\infty} \frac{f(r)}{r^w} \sum_{k=1}^{r-1} \frac{g(k)}{k^v (k-r)^u}$$
$$= \sum_{i=0}^{v-1} \binom{u+i-1}{u-1} \sum_{r=2}^{\infty} \frac{f(r)}{r^{u+w+i}} \sum_{k=1}^{r-1} \frac{g(k)}{k^{v-i}}$$
$$+ \sum_{i=0}^{u-1} \binom{v+i-1}{v-1} \sum_{r=2}^{\infty} \frac{f(r)}{r^{v+w+i}} \sum_{k=1}^{r-1} \frac{g(r-k)}{k^{u-i}}.$$

Now the theorem follows in view of (2.2).

To illustrate the reciprocity relation, we define the functions ε and i_s (for real s) by

$$\varepsilon(n) = (-1)^n, \qquad i_s(n) = n^s,$$

for $n \in Z^+$. Then we have

COROLLARY 2.1. *For $u \in Z^+$ and $w \in R^+$*

$$\sum_{r,k=1}^{\infty} \frac{1}{r^u k^u (r+k)^w} = \sum_{i=0}^{u-1} \binom{u+i-1}{u-1} C(u+w+i, u-i), \tag{2.5}$$

$$\sum_{r,k=1}^{\infty} \frac{(-1)^{r+k}}{r^u k^u (r+k)^w} = \sum_{i=0}^{u-1} \binom{u+i-1}{u-1} D(u+w+i, u-i) \tag{2.6}$$

and

$$\sum_{r,k=1}^{\infty} \frac{(-1)^{k}}{r^u k^u (r+k)^w} = \sum_{i=0}^{u-1} \binom{u+i-1}{u-1} E(u+w+i, u-i), \tag{2.7}$$

where $C(\ ,\ )$, $D(\ ,\ )$ and $E(\ ,\ )$ respectively denote $C_{i_0 i_0}(\ ,\ )$, $C_{\varepsilon, i_0}(\ ,\ )$ and $C_{i_0,\varepsilon}(\ ,\ )$. Further for $w \in Z^+$

$$\sum_{r,k=1}^{\infty} \frac{1}{rk(r+k)^w} = (w+1)\zeta(w+2) - \sum_{i=2}^{w} \zeta(i)\zeta(w+2-i), \tag{2.8}$$

$$\sum_{r,k=1}^{\infty} \frac{(-1)^{r+k}}{rk(r+k)} = \frac{1}{4}\zeta(3) \tag{2.9}$$

and

$$\sum_{r,k=1}^{\infty} \frac{(-1)^{k-1}}{rk(r+k)} = \frac{5}{8}\zeta(3). \tag{2.10}$$

Proof. On taking $u = v$ and (i) $f = g = i_0$, (ii) $f = \varepsilon$, $g = i_0$ and (iii) $f = i_0$, $g = \varepsilon$ in turn in Theorem 2.1, we obtain (2.5), (2.6) and (2.7).

To prove (2.8) we take $u = 1$ in(2.5) and compare it with the well known result (cf. [8], p. 49, eqn. (3))

$$(2.11)\quad \sum_{i=2}^{w} \zeta(i)\zeta(w+2-i) = (w+1)\zeta(w+2) - 2\sum_{r=2}^{\infty} \frac{1}{r^{w+1}} \sum_{k=1}^{r-1} \frac{1}{k}$$

valid for $w \in Z^+$.

To prove (2.9) we take $u = w = 1$ in (2.6) and compare it with (cf. [**8**], lines 5 and 16, p $= 2$)

$$(2.12)\quad \sum_{r=2}^{\infty} \frac{(-1)^r}{r^2} \sum_{k=1}^{r-1} \frac{1}{k} = \frac{1}{8}\zeta(3).$$

To prove (2.10), first we recall the following result due to the second author and A. Sivaramasarma (cf. [**10**], eqn. (1.3))

$$(2.13)\quad \sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1}{r^2(k+r)} = \frac{3}{4}\zeta(3).$$

Since

$$\sum_{k=1}^{2r-1} \frac{(-1)^k}{k} = -\sum_{k=1}^{r} \frac{1}{k+r} - \frac{1}{2r},$$

we have by (2.7) for $w \in R^+$

$$\begin{aligned}(2.14)\quad \sum_{r,k=1}^{\infty} (-1)^k/rk(r+k)^w &= E(w+1,1)\\ &= \sum_{r=2}^{\infty} \frac{1}{r^{1+w}} \sum_{k=1}^{r-1} \frac{(-1)^k + (-1)^{r-k}}{k}\\ &= \frac{1}{2^w} \sum_{r=1}^{\infty} \frac{1}{r^{1+w}} \sum_{k=1}^{2r-1} \frac{(-1)^k}{k} = \frac{1}{2^w} \sum_{r=1}^{\infty} \frac{1}{r^{1+w}} \left(-\sum_{k=1}^{r} \frac{1}{r+k} - \frac{1}{2r} \right)\\ &= -\frac{1}{2^w} \sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1}{r^{1+w}(r+k)} - \frac{\zeta(2+w)}{2^{1+w}}.\end{aligned}$$

Now (2.10) follows from (2.13) and (2.14) with $w = 1$. This completes the proof of Corollary 2.1.

REMARK 2.1. As mentioned in the introduction, L. J. Mordell (cf. [**6**] eqn. (10)), using different arguments, proved (2.8) in case $w = 1$. Since

$$\begin{aligned}\sum_{r,k=1}^{\infty} \frac{1}{rk(r+k)} &= \left(\sum_{\substack{r,k=1\\ r\le k}}^{\infty} + \sum_{\substack{r,k=1\\ r\ge k}}^{\infty} - \sum_{\substack{r,k=1\\ r=k}}^{\infty} \right) \frac{1}{rk(r+k)}\\ &= 2\sum_{r=1}^{\infty} \sum_{k\le r} \frac{1}{kr(k+r)} - \frac{1}{2}\zeta(3),\end{aligned}$$

Mordell's result given in (1.2) above is equivalent to

$$\sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1}{rk(r+k)} = \frac{5}{4}\zeta(3) \tag{2.15}$$

which is due to the second author and A. Sivaramasarma (cf. [10], eqn. (1.2)).

REMARK 2.2. The result given in (2.11) is often rediscovered and dates back to Euler (cf. [8], Footnotes on p. 47). Some recent authors ascribe it to G. T. Williams (cf. [13], Theorem III). The special case $a = 2$ of (2.11) was also proved by W. E. Briggs, S. Chowla, A. J. Kempner and W. E. Kientka [**2**]. For a recent proof of (2.11) we refer to the second author and A. Sivaramasarma [**10**]. Their arguments are based on a generalisation of a transformation formula due to J. Lehner and M. Newman and also discuss equivalent forms of (2.11), (2.13) and (2.15).

REMARK 2.3. We note that the following result is implicit in the proof of (2.10)

$$\sum_{r=1}^{\infty} \sum_{k=1}^{2r-1} \frac{(-1)^{k-1}}{r^2 k} = -2E(2,1) = \frac{5}{4}\zeta(3). \tag{2.16}$$

This may be compared with the following result stated by S. Ramanujan (cf. [**9**], p. 108) and proved recently by the second author and A. Sivaramasarma [**11**]

$$\sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1}{r^2(2k-1)} = \frac{7}{4}\zeta(3).$$

3. Evaluation of multiple series. For $r \in Z^+$, $s \in Z^{(0)}$ and $a > -r$, we write

$$T_{r,s}(a) = \sum_{(l,r)} 1/l_1, l_2, \ldots, l_r(l_1 + \cdots + l_r + a + 1) \cdots (l_1 + \cdots + l_r + a + s)$$

where the symbol (l, r) under the summation sign indicates that the sum extends over all r-tuples $(l_1, \ldots, l_r)$ of positive integers. L. J. Mordell (cf. [7], Theorems I and II) proved that

$$T_{r,s}(a) = \frac{r!}{s!}\left(\frac{1}{(s+1)^{r+1}} + \sum_{i=1}^{\infty} \frac{(-1)^i(a-1)(a-2)\cdots(a-i)}{i!(s+i+1)^{r+1}}\right). \tag{3.1}$$

In this section, we give a different evaluation of $T_{r,s}(a)$ involving $S_n^{(r)}$, the Stirling numbers of the first kind, which are defined by

$$x(x-1)\cdots(x-n+1)=\sum_{r=1}^{n} S_n^{(r)}x^r.$$

THEOREM 3.1.

$$T_{r,s}(a)=(-1)^r r!\sum_{n=r}^{\infty}\frac{(-1)^n S_n^{(r)}}{n!(a+n)\cdots(a+n+s)}. \tag{3.2}$$

Further for $w \in Z^+$

$$\sum_{(l,r)} 1/l_1 l_2 \cdots l_r(l_1+\cdots+l_r+a)=(-1)^r r!\sum_{n=r}^{\infty}\frac{(-1)^n S_n^{(r)}}{n!(a+n)^w}. \tag{3.3}$$

Proof. It is well known (cf. [**4**], p. 146, eqn. (3)) that

$$(\log(1-z))^r=r!\sum_{n=r}^{\infty}(-1)^n S_n^{(r)}z^n/n!.$$

Hence

$$\begin{aligned}
T_{r,1}(a) &= \sum_{(l,r)} 1/l_1\cdots l_r(l_1+\cdots+l_r+a) \\
&= \sum_{(l,r)}\frac{1}{l_1\cdots l_r}\int_0^1 x^{l_1+\cdots+l_r+a-1}\,dx \\
&= \int_0^1 x^{a-1}\left(\sum_{l=1}^{\infty}\frac{x^l}{l}\right)^r dx=\int_0^1 x^{a-1}(\log(1-x))^r\,dx \\
&= (-1)^r\int_0^{\infty} e^{-ay}(\log(1-e^{-y}))^r\,dy \\
&= (-1)^r\int_0^{\infty} e^{-ay}r!\sum_{n=r}^{\infty}\frac{(-1)^n S_n^{(r)}e^{-ny}}{n!}\,dys \\
&= (-1)^r r!\sum_{n=r}^{\infty}\frac{(-1)^n S_n^{(r)}}{n!}\int_0^{\infty} e^{-(a+n)y}\,dy \\
&= (-1)^r r!\sum_{n=r}^{\infty}\frac{(-1)^n S_n^{(r)}}{n!(n+a)}.
\end{aligned} \tag{3.4}$$

Since for fixed $r \in Z^+$, $S_n^{(r)} = O((n-1)!(\log n)^{r+1})$ (cf. [4], p. 161, line 3), the interchange of summation and integration could be easily justified. Also since

$$\text{(3.5)} \qquad \frac{s!}{A(A+1)\cdots(A+s)} = \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^i}{A+i},$$

we have

$$T_{r,s}(a) = \frac{1}{s!} \sum_{i=0}^{s} (-1)^i \binom{s}{i} T_{r,1}(a+i)$$

and consequently (3.2) follows from (3.4) and (3.5).

To prove (3.3), we differentiate term wise $(w-1)$ times the series appearing in (3.4). The term wise differentiations are easily justified. This completes the proof of Theorem 3.1.

REMARK 3.1. Our evaluation of $T_{r,s}(a)$ is different from what of Mordell and has the advantage of yielding (3.3) readily. However, Mordell's evaluation as given in (3.1) can be utilized to obtain the result

$$\text{(3.6)} \qquad \sum_{(l,r)} 1/l_1 \cdots l_r (l_1 + \cdots + l_r)^2$$
$$= \frac{r!}{2}\left((r+1)\zeta(r+2) - \sum_{i=2}^{r} \zeta(i)\zeta(r+2-i)\right).$$

In fact, term wise differentiations of the series defining $T_{r,s}(a)$ and the series appearing on the right of (3.1) yield in case $s = 0$

$$\sum_{(l,r)} 1/l_1 \cdots l_r (l_1 + \cdots + l_r + a)^2$$
$$= r! \sum_{i=1}^{\infty} \frac{(-1)^{i-1}(a-1)\cdots(a-i)}{i!k(i+1)^{r+1}} \left(\frac{1}{a-1} + \cdots + \frac{1}{a-i}\right)$$

so that (3.6) follows from (2.11).

We also evaluate a related multiple series

THEOREM 3.2. *For $r, w \in Z^+$ and $a \in R^+$, let*

$$\text{(3.7)} \qquad V_{r,w}(a) = \sum_{(l,r)} 1/l_1! l_2! \cdots l_r! (l_1 + \cdots + l_r + a)^w.$$

Then

$$\text{(3.8)} \qquad V_{r,w}(a) = \sum_{s=1}^{r} (-1)^{r-s} \binom{r}{s} \left\{ \sum_{n=1}^{\infty} \frac{s^n}{n!(n+a)^w} \right\}.$$

Proof. We have

$$\begin{aligned}
V_{r,1}(a) &= \sum_{l,r} 1/l_1! \cdots l_r!(l_1 + \cdots + l_r + a) \\
&= \sum_{l,r} \frac{1}{l_1! \cdots l_r!} \int_0^1 x^{l_1 + \cdots + l_r + a - 1}\, dx \\
&= \int_0^1 x^{a-1} \left(\sum_{(l,r)} \frac{x^{l_1 + \cdots + l_r}}{l_1! \cdots l_r!} \right) dx = \int_0^1 x^{a-1}(e^x - 1)^r\, dx \\
&= \int_0^1 x^{a-1} \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} e^{sx}\, dx \\
&= \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \int_0^1 x^{a-1} e^{sx}\, dx \\
&= \frac{(-1)^r}{a} + \sum_{s=1}^{r} \frac{(-1)^{r-s}\binom{r}{s}}{s^a} \int_0^s e^x x^{a-1}\, dx.
\end{aligned}$$

Now on substituting the infinite series representation of e^x and integrating term wise, the theorem follows.

4. Mordell's series. In this section, we prove a reciprocity relation for a class of harmonic double series related to Mordell's series given in (1.3).

For $m, n, p \in Z^+$, we write

$$M(m, n, p) = \sum_{r,k=1}^{\infty} \frac{1}{r^m k^n (r + k)^p}.$$

Then we have the following reciprocity relation

THEOREM 4.1.

(4.1) $$M(2m,2n,2p) + M(2n,2p,2m) + M(2p,2m,2n) = \frac{4}{(2m)!(2n)!} \sum_{0 \le r \le \min(m,\, n)} \left\{ m\binom{2n}{r} + n\binom{2m}{r} \right\} \cdot (2m + 2n - 2r - 1)!(2r)!\zeta(2r)\zeta(2m + 2n + 2p - 2r)$$

(4.2) $$= \frac{(2\pi)^{2m+2n+2p}(-1)^{m+n+p}}{(2m)!(2n)!} \sum_{0 \le r \le \min(m,\, n)} \left[\left\{ n\binom{2m}{r} + m\binom{2n}{r} \right\} \cdot \frac{(2m + 2n - 2r)!}{(2m + 2n + 2p - 2r)!} B_{2r} B_{2m+2n+2p-2r} \right].$$

Proof. If $B_n(x)$ denotes the Bernoulli polynomial of order n defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!}$$

for $|t| < 2\pi$, it is well known (cf. [1], §12.11) that

$$B_{2n}(x) = \alpha_n \sum_{r=1}^{\infty} \frac{\cos 2\pi rx}{r^{2n}} \tag{4.3}$$

for $0 \leq x \leq 1$. Here

$$\alpha_n = (-1)^{n+1} 2(2n)!/(2\pi)^{2n}. \tag{4.4}$$

Thus if B_n denotes $B_n(0)$, then it is known due to Euler (cf. [1], Theorem 12.17) the for $n \in Z^+$

$$B_{2n} = (-1)^{n+1} 2(2n)! \zeta(2n)/(2\pi)^{2n}. \tag{4.5}$$

Since $B_0 = 1$ and $\zeta(0) = -1/2$, we note that (4.5) is true for $n = 0$ also. Now by (4.3), we have for $m, n, p \in Z^+$

$$B_{2m}(x) b_{2n}(x) B_{2p}(x) = \alpha_m \alpha_n \alpha_p \sum_{r,k,l=1}^{\infty} \frac{\cos 2\pi rx \cos 2\pi kx \cos 2\pi lx}{r^{2m} k^{2n} l^{2p}}.$$

The series on the right converges uniformly for $x \in [0, 1]$ and hence can be integrated term wise. We note that the integral from 0 to 1 of the terms of the series vanishes except when either $r = k + l$ or $k = l + r$ or $l = r + k$ since

$$\begin{aligned} 4 \cos A \cos B \cos C = {} & \cos(-A + B + C) + \cos(A - B + C) \\ & + \cos(A + B - C) + \cos(A + B + C). \end{aligned}$$

Thus

$$\begin{aligned} & \int_0^1 B_{2m}(x) B_{2n}(x) B_{2p}(x)\, dx \\ & = \frac{\alpha_m \alpha_n \alpha_p}{4} \{ M(2m, 2n, 2p) + M(2n, 2p, 2m) + M(2p, 2m, 2n) \}. \end{aligned} \tag{4.6}$$

However, it is known due to L. Carlitz [3] that for $m, n, p \in Z^+$

$$\begin{aligned} & \int_0^1 B_m(x) B_n(x) B_p(x)\, dx \\ & = (-1)^{p+1} p! \sum_{0 \leq r \leq \min(m/2,\, n/2)} \left\{ n\binom{m}{2r} + m\binom{n}{2r} \right\} \\ & \quad \cdot \frac{(m + n - 2r - 1)!}{(m + n + p - 2r)!} B_{2r} B_{m+n+p-2r}. \end{aligned} \tag{4.7}$$

Now (4.2) follows from (4.6) and (4.7) while (4.1) follows from (4.5) and (4.2).

REMARK 3.1. L. J. Mordell (cf. [7], Theorem III) noted that for $m \in Z^+$, $M(2m, 2m, 2m)$ is a rational multiple of π^{6m}. However, from Theorem 4.1 we have the more explicit evaluation

$$M(2m,2m,2m) = \frac{(-1)^m 2^{6m+1}\pi^{6m} m}{3((2m)!)^2} \cdot \sum_{0 \le r \le m} \binom{2m}{2r} \frac{(4m-2r-1)!}{(6m-2r)!} B_{2r} B_{6m-2r}$$

$$= \frac{8m}{3((2m)!)^2} \sum_{0 \le r \le m} \binom{2m}{2r} (2m)!(4m-2r-1)!\zeta(2r)\zeta(6m-2r).$$

5. Some open problems. At the end of his paper, L. J. Mordell [7] remarked that it would be of interest to find the corresponding result for (1.3) wherein the occuring exponents $2r$ inside the summation are replaced by $2r+1$. However, we note that, much earlier to Mordell's work [7], L. Tornheim [**12**] essentially solved this problem by proving that $M(2r+1, 2r+1, 2r+1)$ may be explicitly determined as a polynomial in $\zeta(2), \zeta(3), \dots, \zeta(6r+3)$ with rational coefficients. In addition to proving (1.2), Tornheim makes a systematic and thorough investigation of harmonic double series.

Our attempts to evaluate the series

$$\sum_{m,n=1}^{\infty} \frac{(-1)^{m-1}}{m^r n^r (m+n)^r}, \qquad \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{m^r n^r (m+n)^r},$$

in particular when r is even and

$$\sum_{m,n=1}^{\infty} \frac{(-1)^d}{m^a} \left(\frac{1}{1^b} + \frac{1}{2^b} + \cdots + \frac{1}{m^b} \right)^c$$

where $a > 1$ and $b, c, d \in Z^+$ were not successful.

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Received July 14, 1982 and in revised form May 2, 1984. Research by the first author is partially supported by a Canadian National Research Council Grant.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA
CANADA

Current address: R. Sitaramachandrarao
Department of Mathematics
The University of Toledo
Toledo, Ohio