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ON THE MINORANT PROPERTIES IN $C_p(H)$

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We improve in two directions a recent result of B. Simon about the minorant property in $C_p(H)$; the methods also allow us to extend a result of H. Shapiro and to obtain an apparently new result on matrices with positive entries.

Introduction. Let H be a complex Hilbert space, which will always be the space l^2 of square summable sequences or the space l_n^2 of all n-tuples of complex numbers with the hermitian norm, equipped once and for all with an orthogonal basis $(e_i)_{i\in I}$ (I finite or countable). Let $K(H)$ be the set of all compact operators of H; if $C \in K(H)$, put $|C| = \sqrt{C^*C}$ and let $\mu_1(C), \mu_2(C), \dots, \mu_i(C)$ be the eigenvalues of $|C|$, rearranged in decreasing order; if $1 \le p < \infty$, put

$$\|C\|_p = \Big(\sum_{i\in I}(\mu_i(C))^p\Big)^{1/p} = \big(\operatorname{Tr}|C|^p\big)^{1/p} = \big[\operatorname{Tr}(C^*C)^{p/2}\big]^{1/p}$$

(where for $A \in K(H)$, $\operatorname{Tr} A \overset{\text{def}}{=} \sum_{i\in I}\langle Ae_i, e_i\rangle$ is the trace of A whenever it exists).

Let $C_p(H)$ be the set of all $C \in K(H)$ such that $\|C\|_p < \infty$, ($C_\infty(H) = K(H)$ and $\|C\|_\infty = \mu_1(C)$ is the usual operator norm of C). It is well known that $C_p(H)$, with the norm $\|\ \|_p$, is a Banach space (**[11]**).

For $C \in K(H)$, we put

$$c_{ij} = \langle C(e_j), e_i\rangle = \operatorname{Tr}\big(C\cdot(e_i\otimes e_j)\big) = \hat{C}(i, j).$$

In the last inequality, c_{ij} is considered as a Fourier coefficient with respect to the orthonormal (in the Hilbert–Schmidt sense) system $(e_i\otimes e_j)_{(i,j)\in I\times J}$ and this allows us to keep the analogy with the commutative case (**[3]**, **[4]**) in the definitions below (recall that $e_i \otimes e_j$ is the operator of rank one defined by:

$$(e_j\otimes e_j)(x) = \langle x, e_i\rangle e_j).$$

DEFINITION 1. If $A, B \in K(H)$, we say that A is a minorant of B if $|a_{ij}| \le b_{ij}$ for $(i, j) \in I\times J$, that is if $|\hat{A}| \le \hat{B}$. We say that $C_p(H)$ has the minorant property, and we abbreviate this to (m)-property, if

$$A, B \in C_p(H) \quad\text{and}\quad |\hat{A}| \le \hat{B} \Rightarrow \|A\|_p \le \|B\|_p.$$

We introduce a slightly different definition, the role of which appears in the third part of this paper (Theorem 3).

DEFINITION 2. We say that $C_p(H)$ has the positive minorant property, and we abbreviate this to (m^+)-property, if

$$A, B \in C_p(H) \quad \text{and} \quad 0 \le \hat{A} \le \hat{B} \Rightarrow \|A\|_p \le \|B\|_p.$$

We will begin by giving a survey a results already known about the (m)-property. The first questions is: for which p does $C_p(H)$ have (m)? It is well known, and simple to prove, that if $p = 2k$, k an integer, $C_p(H)$ has (m): if A, $B \in C_p(H)$ and $|\hat{A}| \le \hat{B}$, then

$$|\widehat{A^*A}| \le \widehat{B^*B}, \qquad \left|\widehat{(A^*A)^k}\right| \le \widehat{(B^*B)^k}$$

and

$$\|A\|_p^p = \mathrm{Tr}(A^*A)^k \le \mathrm{Tr}(B^*B)^k = \|B\|_p^p.$$

It is also obvious that $C_\infty(H)$ has (m).

In ([**8**]), V. Peller proved that $1 \le p < \infty$, $p \ne 2k$, k an integer, $C_p(l^2)$ does not have (m). The answer for $C_p(H)$ is then analogous to the answer for the commutative case of L^p-spaces that has already been considered and solved by G. H. Hardy–J. Littlewood ([**7**]) and R. P. Boas ([**2**]). It follows from V. Peller's result that there must exist some n for which $C_p(l_n^2)$ does not have (m), if $p \ne 2k$, k an integer; but the proof, which relies on the theory of Hankel operators ([**9**]) and on ([**11**]) does not provide an estimate for n. In ([**12**]), B. Simon gives a simple proof (which relies only on [**2**]) that (m) fails for $C_p(l^2)$ if $p \ne 2k$, and his proof gives an explicit n for which (m) fails for $C_p(l_n^2)$. More prescisely, B. Simon introduces the following definition.

DEFINITION 3. *If* $1 \le p < \infty$, $p \ne 2k$, $N(p)$ denotes the smallest integer n such that $C_p(l_n^2)$ does not have (m).

B. Simon proved in ([**12**])

(i) $N(p) \le 2[p/2] + 5$

(ii) $N(p) \ge 3$ if $p \ge 2$

(where $[x]$ is the greatest integer v such that $v \le x$) and he remarked that it would be interesting to know the precise value of $N(p)$.

This paper will be divided into three parts. In the first part, we improve (i) to show that

$$N(p) \le [p/2] + 2. \qquad \text{(I)}$$

(Equivalently, if $p < 2(n-1)$, $C_p(l_n^2)$ does not have (m)). We give a simple and explicit counter example (Theorem 1, α) whereas in ([**2**]) and ([**12**]) the methods are variational. Our counter example, in its variational version (Theorem 1, β) extends to the non commutative case a result of H. Shapiro ([**10**]). It is also possible to formulate β in terms of the minorant property for some Banach space of operators $C_\varphi(H)$, where φ is an Orlicz function: we refer to ([**4**]) for this.

In the second part, we improve (ii) to show that

$$\text{(II)} \qquad N(p) \geq 4 \quad \text{if } p > 4$$

(equivalently, $C_p(l_3^2)$ has (m) if $p > 4$; because of (I), it has not (m) if $1 \leq p < 4, p \neq 2$). It follows from (I) and (II) that

$$N(p) = [p/2] + 2 \quad \text{if } 1 < p < 6, p \neq 2, 4.$$

We conjecture that this is the correct value of $N(p)$ for all p, $p \neq 2k$. Equivalently, we conjecture that $C_p(l_n^2)$ has (m) if $p \geq 2(n-1)$.

In the third part, using the Gâteaux-differentiability of the C_p-norm, we prove that the (m^+) property is equivalent to the following:

$$\text{if } B \in C_p(H) \quad \text{and} \quad \hat{B} \geq 0, \quad \text{then} \widehat{(B^*B)^{p/2-1}B^*} \geq 0 \qquad \text{(Theorem 3)}.$$

In particular, from this result and from (II) we derive the following fact: if B is a (3×3) matrix with positive entries, for $\alpha \geq 2$ the matrix $(B^*B)^\alpha$ has positive entries. An analogous result, with a more direct approach, was obtained by B. Virot (Theorem 5).

In view of these results, it would be interesting to know if the (m^+) property is actually weaker than (m). We know no case in which (m^+) holds and not (m). What we know about (m^+) is collected in Theorem 4.

Part I. We shall prove the following theorem.

THEOREM 1.

(α) *Let* $1 \leq p < \infty, p \neq 2k$. *Then* $N(p) \leq [p/2] + 2$ (*equivalently, if* $p < 2(n-1)$, $C_p(l_n^2)$ *does not have* (m)).

(β) *More generally, let* φ *be a strictly increasing* C^∞ *convex function on* R^+, *vanishing at zero and* $\psi(t) = \varphi(\sqrt{t})$. *Suppose that some derivative of* ψ *is negative at some point of* $]0, \infty[$, *and let n be the smallest integer such that this happens*; *then, there exists* A *and* B *in* $K(l_n^2)$ *such that* $|\hat{A}| \leq \hat{B}$, *but* $\mathrm{Tr}[\varphi(|A|)] > \mathrm{Tr}[\varphi(|B|)]$.

Before giving the proof, we shall make some comments: another way to formulate (β) is the following: if for all $A, B \in K(l^2)$ such that

$|\hat{A}| \leq \hat{B}$, we have: $\mathrm{Tr}[\varphi(|A|)] \leq \mathrm{Tr}[\varphi(|B|)]$ (provided both members are finite) then all the derivatives of ψ must be positive on $]0, \infty[$ and so, by the classical result of Bernstein ([**13**]), φ must be of the form:

$$\varphi(t) = \sum_{n \geq 0} a_n t^{2n}, \quad \text{with } a_n \geq 0 \text{ for all } n \geq 0.$$

So, (β) is the extension in the operator case of the result of Shapiro ([**10**]).

Proof of Theorem 1 (α). Let $n = [p/2] + 2$, U be the unitary permutation operator of l_n^2 defined by.

$$U(e_1) = e_2, \ldots, U(e_{n-1}) = e_n,\ U(e_n) = e_1.$$

Let S be the symmetry operator defined by $S(e_1) = -e_1$, $S(e_i) = e_i$ *if* $2 \leq i \leq n$. (Any operator S such that $S(e_i) = \varepsilon_i e_i$, $\varepsilon_i = \pm 1$, $\varepsilon_i \cdots \varepsilon_n = -1$ would also work.)

Put $A = I + SU$ and $B = I + U$. It is clear that $|\hat{A}| \leq \hat{B}$ (in fact $|\hat{A}| = \hat{B}$) and we claim that

$$(1) \qquad \mathrm{Tr}(A^*A)^q > \mathrm{Tr}(B^*B)^9.$$

It is easy to compute explicitly the proper values of A and B, ad therefore those of A^*A and B^*B. (Observe that A and B are normal matrices). In fact, if one puts $\alpha = e^{i\pi/n}$, $\omega = \alpha^2$, $v_k = \sum_{j=1}^n \omega^{jk} e_j$, $w_k = \sum_{j=1}^n (\omega^k \alpha)^j e_j$ for $1 \leq k \leq n$, one has

$$U(v_k) = \bar{\omega}^k v_k, \quad \text{and} \quad SU(w_k) = \bar{\omega}^k \bar{\alpha} w_k,$$

so that the eigenvalues of A^*A and B^*B are respectively $|1 + e^{(2k+1)i\pi/n}|^2$ and $|1 + e^{2ki\pi/n}|^2$, the corresponding eigenvectors being, respectively, w_k and v_k $1 \leq k \leq n$. But we shall use a different presentation.

In order to prove (1), observe that

$$(2) \qquad A^*A = 2(I + V) \quad \text{and} \quad B^*B = 2(I + W)$$

$$\text{with } V = \frac{SU + (SU)^*}{2},\ W = \frac{U + U^*}{2}.$$

It is easy to check the following relations:

$$(3) \qquad \mathrm{Tr}\, U^k = \begin{cases} 0 & \text{if } k \not\equiv 0(n), \\ n & \text{if } k = 0(n), \end{cases} \qquad \text{and}$$

$$\mathrm{Tr}(SU)^k = \begin{cases} 0 & \text{if } k \not\equiv 0(n), \\ (-1)^\rho n & \text{if } k = \rho n. \end{cases}$$

Moreover, by the binomial formula, we have:

(4) If $l \geq 1$,

$$V^l = \sum_{\substack{|k| \leq l \\ k \equiv l(2)}} a_{kl} (SU)^k \quad \text{and} \quad W^l = \sum_{\substack{|k| \leq l \\ k \equiv l(2)}} a_{kl} U^k$$

where a_{kl} are strictly positive coefficients. Put, for $l \geq 1$,

$$C_l = (1/l!)q(q-1)\cdots(q-l+1)$$

and note that:

$$\sum |C_l| < \infty \tag{5}$$

($|C_{l+1}/C_l| = 1 - (q+1)/l + O(1/l^2)$, so that $|C_l| = O(l^{-q-1})$ when l tends to infinity),

$$C_1 > 0, \dots, C_{n-1} > 0,\ C_n < 0, \quad \text{and} \quad \operatorname{sign} C_{n+\Gamma} = (-1)^{r+1} \quad \text{if } r \geq 0. \tag{6}$$

Using (5) and the fact that $\operatorname{Max}(|V\|_\infty, \|W\|_\infty) \leq 1$, we can write:

$$(I+V)^q - (I+W)^q = \sum_{l=1}^{\infty} C_l\left[V^l - W^l\right]. \tag{7}$$

Taking the traces of both members, and taking account of (3) and (4), we get:

$$\begin{aligned}\operatorname{Tr}(I+V)^q - \operatorname{Tr}(I+W)^q &= \sum_{l\geq 1} C_l \sum_{\substack{|k|\leq l \\ k \equiv l(2)}} a_{kl}\left[\operatorname{Tr}(SU)^k - \operatorname{Tr} U^k\right] \\ &= \sum_{\substack{r\in Z \\ k=(2r+1)h}} \ \sum_{\substack{l\geq |k| \\ l\equiv k(2)}} (-2n)a_{kl}C_l.\end{aligned} \tag{8}$$

The indices l which appear on the right-side of (8) are all of the form:

$$\begin{aligned} l &= |k| + 2l' = |2r+1|n + 2l' = (2r'+1)n + 2l' \\ &= n + 2l'' \quad \text{with } l'' \geq 0. \end{aligned}$$

By (6), $C_l < 0$, so the right-hand side of (8) is a sum of positive terms and

$$\operatorname{Tr}(I+V)^q > \operatorname{Tr}(I+W)^q.$$

In view of (2), this implies (1), and (α) is proved.

(β) We shall need the following relations, which are obvious consequences of (3) and (4)

$$\begin{cases} \operatorname{Tr} V^l = \operatorname{Tr} W^l = 0 & \text{if } 1 \leq l \leq n-1,\ l \text{ odd}, \\ \operatorname{Tr} V^l = \operatorname{Tr} W^l = a_{0l} & \text{if } 1 \leq l \leq n-1,\ l \text{ even}, \\ \operatorname{Tr} V^n - \operatorname{Tr} W^n = -4n2^{-n} < 0. \end{cases} \tag{9}$$

Let n be as in the hypotheses of (β), ξ be a positive number such that $\psi^{(n)}(\xi) < 0$, U, S, I as before, $a \geq 0$ and $b \geq 0$ such that $a^2 + b^2 = \xi$ with

$b \downarrow 0$ and $a \uparrow \xi$, and put:

$$A = aI + bSU, \qquad B = aI + bU,$$

(so that A and B belong to a neighborhood of ξI). It is clear that $|\hat{A}| \leq \hat{B}$, (in fact, $|\hat{A}| = \hat{B}$) and we claim that

(10) $\quad \mathrm{Tr}[\psi(A^*A)] > \mathrm{Tr}[\psi(B^*B)]$ for b small enough.

In order to prove (10), observe that

(11) $\quad A^*A = \xi I + 2abV$ and $B^*B = \xi I + 2abW.$

V and W, being normal operators, can be diagonalized so that the following symbolic Taylor formulas are valid:

$$\psi(A^*A) = \sum_{l=0}^{n-1} \frac{(2ab)^l}{l!}\psi^{(l)}(\xi)V^n + O(b^{n+1}),$$

$$\psi(B^*B) = \sum_{l=0}^{n-1} \frac{(2ab)^l}{l!}\psi^{(l)}(\xi)W^l + \frac{(2ab)^n}{n!}\psi^{(n)}(\xi)W^n + O(b^{n+1}).$$

Subtracting and taking traces, we get in view of (9):

(12) $\quad \mathrm{Tr}[\psi(A^*A)] - \mathrm{Tr}[\psi(B^*B)^{-}] = -4n2^{-n}\frac{(2ab)^n}{n!}\psi^{(n)}(\xi) + O(b^{n+1}).$

Since $\psi^{(n)}(\xi) < 0$, (12) proves (10) for b small enough.

Part 2. We shall prove the following theorem:

THEOREM 2. *Let* $p \geq 4$, A *and* B *two* (3×3) *matrices such that* $|\hat{A}| \leq \hat{B}$. *Then* $\|A\|_p \leq \|B\|_p$.
Equivalently, $N(p) \geq 4$ *if* $p \geq 4, p \neq 2k$.

We shall need the two following lemmas, the first of which plays a fundamental role in the theory of C_p-spaces.

LEMMA 1 (**[6]**, **[11]**). *Let* $a_1 \geq \cdots \geq a_N \geq 0$ *and* $b_1 \geq \cdots \geq b_N \geq 0$. *Let* φ *be an increasing convex function on* $[0, \infty[$. *Suppose that*

$$\sum_1^k a_j \leq \sum_1^k b_j \quad \textit{for } k = 1,\ldots,N.$$

Then, $\sum_1^N \varphi(a_j) \leq \sum_1^N \varphi(b_j)$.

LEMMA 2. *Let* $0 < \alpha_1 < \cdots < \alpha_n$ *be positive exponents, and let* $P(x) = a_0 + a_1 x^{\alpha_1} + \cdots a_n x^{\alpha_n}$ *a "polynomial" with real coefficients* a_i, *not all zero. Put*

$$V = \#\{i/a_i a_{i+1} < 0\}, \qquad Z = \#\{x > 0/P(x) = 0\}.$$

Then $Z \leq V$.

Lemma 2 is a generalization of the well-known theorem of Descartes for polynomials; a sketch of the proof can be found in ([**1**]), but the slavish imitation of Descartes's proof is quicker.

Proof of Theorem 2. Let A and B be two (3×3) matrices such that $|\hat{A}| \leq \hat{B}$. Let us denote by $\lambda_1 \geq \lambda_2 \geq \lambda_3$ (resp. $\mu_1 \geq \mu_2 \geq \mu_3$) the eigenvalues of A^*A (resp. B^*B) rearranged in decreasing order. Let $q = p/2 > 2$ we claim that

$$\sum_1^3 \lambda_i^q \leq \sum_1^3 \mu_i^q. \tag{13}$$

To prove (13), we may as well assume that

$$\sum_1^3 \lambda_i = \sum_1^3 \mu_i. \tag{14}$$

In fact, the extreme points of the parallelotope of R^9 defined by $|x_{ij}| \leq b_{ij}$ are the points (x_{ij}) such that $|x_{ij}| = b_{ij}$, $1 \leq i, j \leq 3$; so that one has a convex combination $a_{ij} = \sum_k \lambda_k a_{ij}^{(k)}$, with $|a_{ij}^{(k)}| = b_{ij}$ for all i, j, k. If $A^{(k)}$ is the element of $K(l_3^2)$ defined by $\langle A^{(k)}(e_j), e_i \rangle = a_{ij}^{(k)}$, we then have $A = \sum \lambda_k A^{(k)}$. For the operators $A^{(k)}$, (14) holds because, for every k,

$$\sum_i \lambda_i^{(k)} = \sum_{i,j} \left| a_{ij}^{(k)} \right|^2 = \sum_{i,j} b_{ij}^2 = \sum \mu_i.$$

(If $\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \lambda_3^{(k)}$ are eigenvalues of $A^{(k)*}A^{(k)}$.) If we are able to deduce the result when (14) holds, we have: $\|A^{(k)}\|_p \leq \|B\|_p$ for all k, and then

$$\|A_p\|_p \leq \sum \lambda_k \|A^{(k)}\|_p \leq \sum \lambda_k \|B\|_p = \|B\|_p.$$

So, in the following, we shall assume that (14) holds.

Suppose that (13) is false and consider the continuous function

$$g(r) = \sum \lambda_i^r - \sum \mu_i^r, \quad \text{so that } g(q) > 0.$$

Let ν be an integer such that $\nu > q$; since $C_4(l_3^2)$ and $C_{2\nu}(l_3^2)$ have (m), we have:

$$g(2) \leq 0 \quad \text{and} \quad g(\nu) \leq 0. \tag{15}$$

By the intermediate value theorem, there exists $q_1 \in [2, q[$ and $q_2 \in]q, v]$ such that

(16) $$g(q_1) = g(q_2) = 0 \qquad (q_1 < q_2).$$

Notice that

(7) $$\lambda_1 \leq \mu_1.$$

In fact, $\lambda_1 = \|A\|_\infty^2$, $\mu_1 = \|B\|_\infty^2$, and $C_\infty(l_3^2)$ trivially has (m). We shall prove that (14) and (16) contradict Lemma 2 for some property chosen polynomial P. Let us distinguish two cases:

Case 1. $\lambda_3 \geq 0$.

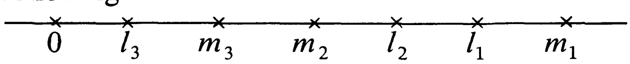
If $\lambda_1 + \lambda_2 \leq \mu_1 + \mu_2$, (14), (17) and the application of Lemma 1 to $\varphi(t) = t^q$ gives us (13). So, we may assume $\lambda_1 + \lambda_2 > \mu_1 + \mu_2$. Because of (17) we then have $\lambda_2 > \mu_2$ and because of (14)

$$\lambda_3 = \mu_3 + (\mu_1 + \mu_2 - \lambda_1 - \lambda_2) < \mu_3.$$

Multiplying all λ_i's, μ_i's by the same constant, we may assume that $\lambda_3 \geq 1$; so that if

$$l_i = \operatorname{Log} \lambda_i \quad \text{and} \quad m_i = \operatorname{Log} \mu_i$$

we have the following situation



We can write

$$g(r) = P(x) = \sum x^{l_i} - \sum x^{m_i} \quad \text{with } x = e^r.$$

In view of the preceding picture, let us rewrite:

$$P(x) = x^{l_3} - x^{m_3} - x^{m_2} + x^{l_2} + x^{l_1} - x^{m_1}$$

with the notations of Lemma 2, we have $V = 3$ and $Z \geq 4$, in fact, due to (14) and (16), P vanishes at $1, e, e^{q_1}, e^{q_2}$. This proves (13) by contradiction.

Case 2. $\lambda_3 = 0$.

By (18) $\mu_3 > 0$ and as before we may assume $\mu_3 > 1$, we then consider $\mathbf{Q}(x) = -x^{m_3} - x^{m_2} + x^{l_2} + x^{l_1} - x^{m_1}$, and have $V \leq 2$, $Z \geq 3$, which again proves (13) by contradiction.

Part 3. First, recall the following lemma on the Gâteaux-differentiability of the norm in normed spaces E with strictly convex dual E'.

LEMMA 3 ([**5**]). *Let E be a complex normed space, and $b \in E$, $b \neq 0$; assume that the norm is smooth at b, that is to say: there exists a unique $\tilde{b} \in E'$, $\|\tilde{b}\| = 1$, $\langle \tilde{b}, b\rangle = \|b\|$. Then, the norm in E is Gâteaux-differentiable at b; more precisely,*

$$\forall c \in E, \quad \lim_{\substack{t \in R \\ t \to 0}} \frac{\|b + tc\| - \|b\|}{t} = R\left[\langle \tilde{b}, c\rangle\right].$$

Observe that the lemma is applicable when the norm of E' is strictly convex, a fortiori when it is uniformly convex; if $E = C_p(H)$, $1 < p < \infty$, $E' = C_{p'}(H)$ is uniformly convex by the Clarkson-McCarthy inequalities ([**11**]), so that Lemma 3 may be applied to $C_p(H)$ $(1 < p < \infty)$. If $B \in C_p(H)$, and $\|B\|_p = 1$, we easily compute:

$$\tilde{B} = (B^*B)^{p/2-1}B^*. \tag{20}$$

(This has a meaning even when $1 \leq p < 2$, and B is not injective by putting $\tilde{B}(x) = 0$ if $B(x) = 0$.) It is clear that $\langle \tilde{B}, B\rangle = \operatorname{Tr} \tilde{B}B = 1\|B\|_p$ and to check that $\|\tilde{B}\|_{p'} = 1$, use a polar decomposition of B and the invariance of the $C_{p'}$-norm under multiplication by a unitary operator; by extension, in the following $\tilde{B}$ will always be given by (20), even when $\|B\|p \neq 1$.

THEOREM 3. *Let* $1 < p < \infty$

(1) *the following are equivalent*:

(a) $C_p(H)$ *has* (m^+)

(b) $B \in C_p(H)$ *and* $\hat{B} \geq 0 \Rightarrow \hat{\tilde{B}} \geq 0$. (*In particular* $\overline{(B^*B)^{p/2}} \geq 0$).

(2) *In particular, if B is a* (3×3) *matrix such that* $\hat{B} \geq 0$, *then*

$$(B^*B)^{\alpha} \geq 0 \quad \text{for } \alpha \geq 2.$$

Proof. (a) $\Rightarrow$ (b). Let $B \in C_p(H)$, with $\hat{B} \geq 0$; we may assume that $\|B\|_p = 1$; let $C \in C_p(H)$, with $\hat{C} \geq 0$; first, it is clear that $\tilde{B}$ has real entries (B^*B can be diagonalized by the real orthogonal group); besides, if $t \geq 0$, we get from (a):

$$\|B\|_p \leq \|B + tC\|_p,$$

so that by Lemma 3

$$\lim_{\substack{> \\ t \to 0}} \frac{\|B + tC\|_p - \|B\|_p}{t} = \mathscr{R}[\operatorname{Tr} \tilde{B}C] = \operatorname{Tr} \tilde{B} \geq 0.$$

Testing this with the operators of rank one $C = e_i \otimes e_j$, we get (b).

(b) $\Rightarrow$ (a). Let $B, C \in C_p(H)$ with $\hat{B} \geq 0$, $\hat{C} \geq 0$, let us prove that $\|B\|_p \leq \|B + C\|_p$. Observe that, if $0 \leq t_0 \leq 1$

$$\left[\frac{d}{dt}\|B + tC\|_p\right]_{t=t_0} = \operatorname{Tr}\left[(\widetilde{B + t_0 C})C\right] \geq 0 \text{ by (b)},$$

so that the function $t \to \|B + tC\|_p$ is increasing.

(2) follows trivially from (1) and from Theorem 2.

In view of Theorem 3, the property (m^+) is of interest: it is quite simple to verify that the properties (m) and (m^+) are equivalent in $C_p(l^2)$ (Theorem 4(c) below), but in the finite-dimensional case and if $p > 2$, we do not know if (m^+) is weaker than (m). We shall prove the following results about (m^+):

THEOREM 4.

(a) *If* $1 \leq p < 2$, $C_p(l_2^2)$ *does not have* (m^+).

(b) *If n is even*, $n \geq 4$, $C_p(l_n^2)$ *does not have* (m^+) *if* $1 \leq p < n - 2$, $p \neq 2k$.

If n is odd, $n \geq 5$, $C_p(l_n^2)$ *does not have* (m^+) *if* $1 \leq p < n - 3$, $p \neq 2k$.

(d) *If* $p \geq 2$ *and* $C_p(l_n^2)$ *has* (m^+), *then* $A, B \in K(l_n^2)$ *and* $|\hat{A}\exists \leq \hat{B}$ *implies the following*

$$\begin{cases} \|A\|_p \leq (2^p - 1)^{1/p}\|B\|_p & \text{if } \hat{A} \text{ is real,} \\ \|A\|_p \leq 2(2^p - 1)^{1/p}\|B\|_p & \text{if } \hat{A} \text{ is complex.} \end{cases}$$

Proof. (a) Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & d \end{bmatrix} \quad \text{for } d > 0.$$

We shall prove that $\|A\|_p > \|B\|_p$ for d small enough (depending on p); the eigenvalues of B are:

$$\lambda_1 = \frac{1}{2}\left(d + 1 + \sqrt{d^2 - 2d + 5}\right) \quad \text{and}$$

$$\lambda_2 = \frac{1}{2}\left(d + 1 - \sqrt{d^2 - 2d + 5}\right).$$

Since B is symmetric, we have $\mu_1(B) = |\lambda_1| = \lambda_1$ and $\mu_2(B) = \|\lambda_2\| = -\lambda_2$ for d small enough, so that:

$$2^p\|B\|_p^p = \left(d + 1 + \sqrt{d^2 - 2d + 5}\right)^p + \left(-d - 1 + \sqrt{d^2 - 2d + 5}\right)^p$$
$$= f(d)$$

and we claim that $f'(0) < 0$ if $p < 2$. In fact:

$$\frac{1}{p}f'(0) = \left(1 - \frac{1}{\sqrt{5}}\right)(1 + \sqrt{5})^{p-1} - \left(1 + \frac{1}{\sqrt{5}}\right)(\sqrt{5} - 1)^{p-1},$$

so that $f'(0) < 0$ iff $(\sqrt{5}+1)^{p-2} < (\sqrt{5}-1)^{p-2}$, which is equivalent to $p < 2$.

(b) the proof is similar to a proof of ([**10**]).

We shall first examine the case n even; put $m = n/2$, $q = p/2$. We then have $q < m-1$ and we may clearly assume $q > m-2$. By (b) of Theorem 3, it suffices to find an $(n \times n)$ matrix B such that $\hat{B} \geq 0$, and such that $(B^*B)^q$ has some negative entry: this will be the case if $\mathrm{Tr}[(B^*B)^q C] < 0$ for some C with positive entries; we shall take:

$B = \sqrt{1-r^2}\, I + rU$, where U is as in Theorem 1 and $0 < r < 1$,

$C = U^m$,

$B^*B = I + \rho W$ where $\rho = 2r\sqrt{1-r^2} \to 0$ when $r \to 0$, and where W is as in (2). Observe that if $C_l = (1/l!)q(q-1)\cdots(q-l+1)$ for $l \geq 1$, we have $C_m < 0$ and

$$(21) \qquad (B^*B)^q = I + \sum_{l=1}^{m-1} C_l \rho^l W^l + C_m \rho^m W^m + O(\rho^{m+1}).$$

When one computes $W^l C$ for $0 \leq l \leq m-1$, only the following powers of U appear:

$$m, \quad m \pm 1, \quad m \pm 2, \ldots, m \pm (m-1).$$

In view of (3), we then have:

$$(22) \qquad \mathrm{Tr}\, W^l C = 0 \quad \text{if } 0 \leq l \leq m-1.$$

On the other hand, we have

$$(23) \qquad \mathrm{Tr}\, W^m C = 2^{-m+1}\, \mathrm{Tr}\, I = n 2^{-m+1} > 0.$$

(21), (22), (23) give:

$$(24) \qquad \mathrm{Tr}\big[(B^*B)^q C\big] = n 2^{-m+1} \rho^m c_m + O(\rho^{m+1})$$

and the right-hand side of (24) is negative for r small enough. The case n odd is treated in a similar way: just put $m = (n-1)/2$, B as before, and $C = U^{m+1}$.

(c) A close examination of ([**12**]) shows that in fact $C_p)(l^2)$ has (m^+) iff $p = 2k$ or $p = \infty$ so that $C_p(l^2)$ has (m^+) iff $C_p(l^2)$ has (m). One can also argue directly as follows: if $C_p(l^2)$ has (m^+), then trivially: $|\hat{A}| \leq \hat{B}$ implies $\|A\|_p \leq 4\|B\|_p$. But it follows from the tensor product argument of ([**12**]) that if, for a certain constant M

$$|\hat{A}| \leq \hat{B} \quad \text{implies } \|A\|_p \leq M\|B\|_p.$$

then $C_p(l^2)$ has (m); so we conclude that (m^+) is equivalent to $m)$ for $C_p(l^2)$.

(d) It is enough to deal with the case $\hat{A}$ real; put $A^+ = (a_{ij}^+)$, $A^- = (a_{ij}^-)$, we have $A = A^+ - A^-$ and $0 \leq A^+ + A^- \leq \hat{B}$. By the Clarkson-MacCarthy inequality ([**1**]) for $p \geq 2$

$$\|A^+ - A^-\|_p^p + \|A^+ + A^-\|_p^p \leq 2^{p-1}\left|\|A^+\|_p^p + \|A^-\|_p^p\right|.$$

So that using (m^+) twice we get:

$$\|A^+ - A^-\|_p^p \leq (2^p - 1)\|A^+ + A^-\|_p^p \leq (2^p - 1)\|B\|_p^p.$$

Let us now explain Virot's result, which is as follows:

THEOREM 5 (*B. Virot*). *Let E be a positive-definite* (3×3) *matrix with positive entries, f a real positive function on* $[0, \infty[$ *such that* $f(t)$ *is convex and* $f(\sqrt{t})$ *concave; then* $f(E)$ *has posiitve entries; in particular,* E^α *has positive entries if* $\alpha \geq 1$.

Proof. By a standard perturbation argument, we may assume that the eigenvalues of E are distinct, let them be $\lambda_1 < \lambda_2 < \lambda_3$ one easily computes real numbers a_0, a_1, a_2 such that:

$$f(E) = a_0 I + a_1 E + a_2 E^2. \tag{25}$$

If $\Delta = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$, one finds in particular:

$$\Delta a_1 = (f(\lambda_2) - f(\lambda_1))(\lambda_3^2 - \lambda_1^2) - (f(\lambda_2) - f(\lambda_1))(\lambda_2^2 - \lambda_1^2),$$

$$\Delta a_2 = (f(\lambda_3) - f(\lambda_1))(\lambda_2 - \lambda_1) - (f(\lambda_1))(\lambda_3 - \lambda_1).$$

So that $a_1 > 0$ iff

$$\frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2^2 - \lambda_1^2} > \frac{f(\lambda_3) - f(\lambda_1)}{\lambda_3^2 - \lambda_1^2}.$$

This will be true if $f(\sqrt{t})$ is concave. In the same way $a_2 > 0$ iff

$$\frac{f(\lambda_3) - f(\lambda_1)}{\lambda_3 - \lambda_1} > \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}.$$

This will be true if f is convex; so that under the hypotheses of Theorem 5, (25) shows that $(f(E))_{ij} \geq 0$ if $i \neq j$; we know nothing about the sign of a_0, but it is a priori clear, that $(f(E))_{ii} \geq 0$ since $f(E)$ is a positive-definite operator; this can be applied to the function $f(t) = t^\alpha$ for $1 \leq \alpha \leq 2$; but, then, E^α has positive entries for all $\alpha \geq 1$ because $E^\alpha = E^\mu E^\beta$ with μ a positive integer and β a real number such that $1 \leq \beta \leq 2$.

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