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UNIVERSAL OBSERVABILITY AND CODIMENSION ONE SUBGROUPS OF BOREL SUBGROUPS

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A subgroup H of an affine algebraic group G is *observable in* G if the quotient variety G/H is quasi-affine (equivalently, if each character on H is the character of a one-dimensional H-submodule of an irreducible G-module). The question is how to characterize the universally observable groups, i.e., those which are observable in every group in which they can be embedded. We remark that the relation that G/H should be *affine* for every embedding of H is equivalent to H being reductive, by work of Cline, Parshall, Scott. A sufficient condition for the universal observability of a solvable group is that a certain monoid of characters for the inner operation of H on its hyperalgebra should be a group. Here, we give a two-dimensional example (a codimension one subgroup of a Borel subgroup of GL_2) to show that this sufficient condition is not necessary. Secondly, we give a method for testing a group for the *failure* of universal observability, which we use to show the *non*-universal observability of a famly of codimension one subgroups of Borel subgroups of GL_n $(n \geq 3)$. We remark that the universal observability of an affine algebraic group is equivalent to the universal observability of its solvable radical. Consequently, we only need to sort the solvable groups for those that are universally observable.

Introduction. An affine algebraic group H is called *universally observable* if every embedding of H in an affine algebraic group G gives a quasi-affine quotient G/H (cf. [1]). Here we study aspects of universal observability for solvable groups which are codimension one subgroups of Borel subgroups of Gl_n or Sl_n.

We use two methods to establish the universal observability of certain groups. (1) Sweedler's method: The inner operation of H on its hyperalgebra generates a monoid of characters (see §1). That this monoid should be a group is a sufficient condition for universal observability of a solvable group. (2) Direct calculation for two-dimensonal groups for which the first method fails (see §2). This work began as a study of whether the sufficient condition (1) for universal observability was also necessary. Our principal results are as follows. (1) Using the first method, we show that $\mathrm{Ker}\,\chi$ is universally observable when $\chi\colon B \to G_m$ is a character in the dominant chamber of a Borel subgroup B of Sl_n. (2) In characteristic zero, we show the universal observability of the semidirect product $G_a \times_w G_m$, where the multiplicative group operates on the additive group by the wth power character ($w > 2$). These groups provide a negative answer to the question of whether Sweedler's sufficient condition is also necessary.

There is something singular about the appearance of the universally observable groups $G_a \times_w G_m$. In fact, $G_a \times_w G_m$ is a codimension one subgroup of a Borel subgroup of Gl_2, but when we look for codimension one subgroup of a Borel subgroup of Gl_2, but when we look for codimension one subgroups of Borel subgroups of Gl_n ($n > 2$) which are analogously situated and universally observable, we find none (see §3). Our method for looking for the failure of universal observability for solvable subgroups H of Gl_n (considered as an n-dimensional representation of H) is to reposition H in Gl_n by tensoring the representation with a character on H. We detect a change in the geometry of the quotient by using the following theorem from **[4]**. For a character χ on a Borel subgroup B of Sl_n, the quotient $\mathrm{Sl}_n/\mathrm{Ker}\,\chi$ is quasi-affine exactly when $\pm\chi$ lies in the dominant chamber of B. Repositioning attempts to move a character from one chamber into a different chamber.

This work also suggests the possible modification of the definition of universal observability to be that a group H is u.o.* if, for every morphism $H \to G$ with finite kernel, the quotient of G by the image of H is quasi-affine. $G_a \times_w G_m$ is not u.o.*, since it is isogeneous to $G_a \times_1 G_m$, a Borel subgroup of PGl_2, which is not universally observable. This would open up the question of Sweedler to further work.

I offer the following question here. Let H be a subgroup of $\mathrm{Gl}(V)$. Suppose that, for every character χ on H, $H \to \mathrm{Gl}(V \otimes \chi)$ gives a quasi-affine quotient of $\mathrm{Gl}(V \otimes \chi)$ by the image of H. Is H then universally observable?

I benefited from discussions with John Ballard in the course of the work on §2.

The last remark in the abstract is included at the thoughtful suggestion of the referee. Please see Theorems 9 and 10 of **[1]** to establish the fact stated.

1. The hyperalgebra and universal observability.[1]

1.1. Let H be an affine algebraic group defined over a field K. Let $X(H)$ be the character group of H. The inner operation of H on itself induces an operation of H on the coordinate ring A_H and dually, on the hyperalgbra $\mathrm{hy}(H)$. See **[3]** for the definition of the hyperalgebra.

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[1]The material in this section is adapted from unpublished notes of M. Sweedler **[5]**.

Definition. $C(H)$ is the smallest family of finite-dimensional H-modules which contains all finite-dimensional submodules of hy(H) and which is closed under the operations of taking submodules, direct sums, and exterior, symmetric and tensor powers.

Definition. S is the monoid of those characters on H which are characters of one-dimensional modules in $C(H)$.

1.2. Lemma. *$C(H)$ is closed under quotients when S is a group.*

Proof. Let U be a submodule (of some dimension m) of a module V in $C(H)$. U, $\Lambda^m U$, $\Lambda^m V$ are modules in $C(H)$, and the character χ of the one-dimensional module $\Lambda^m U$ is an element of S. Since the kernel of the canonical map $V \otimes \Lambda^m U \to \Lambda^{m+1} V$ is $U \otimes \Lambda^m U$, $(V/U) \otimes \Lambda^m U$ is isomorphic to the image of $V \otimes \Lambda^m U$ in $\Lambda^{m+1} V$, which is an element of $C(H)$.

Let W be a one-dimensional module in $C(H)$ with character χ^{-1}. Then V/U is isomorphic to $(V/U) \otimes \Lambda^m U \otimes W$, which in turn is isomorphic to an element of $C(H)$. This completes the proof.

1.3. Lemma. *Let H be a subgroup of a group G. Each module in $C(H)$ is an H-submodule of a module in $C(G)$.*

Proof. The inclusion $H \hookrightarrow G$ induces the inclusive morphism of H-modules i: hy(H) $\to$ hy(G). If M is a finite-dimensional submodule of hy(H), then $i(M)$ is an H-submodule of a finite-dimensional G-submodule of hy(G) since hy(G) is locally finite. This property of hy(H) relative to hy(G) evidently extends to the property of $C(H)$ relative to $C(G)$ stated in the Lemma.

1.4. Proposition. *Let H be solvable and connected. H is universally observable if S is a group.*

Proof. H is the semidirect product of its unipotent radical U and any maximal torus T. The character group $X(T)$ is a direct product $S_1 \times S_2$, where S_1 contains the restriction of S to T as a subgroup of finite index. Correspondingly, T is the direct product $T_1 \times T_2$ where S_1 is trivial on T_2 and S_2 is trivial on T_1.

First, we show that T_2 centralizes H. The following conditions are equivalent: (1) T_2 centralizes H. (2) T_2 operates trivially on A_H. (3) T_2

operates trivially on hy(H). (4) T_2 operates trivially on all 1-dimensional H-module quotients of finite-dimensional H-submodules of hy(H). The equivalences are all immediate. The equivalence of (3) and (4) uses the facts that H is solvable and that T operates linear reductively.

By Lemma 1.2, $C(H)$ contains the one-dimensional quotients of finite-dimensional submodules of hy(H). Hence, the characters of these modules lie in S and are trivial on T_2. By the equivalence of (1) and (4), T_2 centralizes H, and so, $H = (UT_1) \times T_2$ and $X(H) = X(UT_1) \times X(T_2)$.

Let $H \to G$ be any embedding. For any character $\chi_2 \in X(T_2)$, $K\chi_2$ is a T_2-submodule of some finite-dimensional G-module V. Let W be the H-submodule of V generated by $K\chi_2$. The central subgroup T_2 of H operates on W by the character χ_2.

Let n be the index of S in S_1 above, and let m be the dimension of W. The character for the operation of H on $\Lambda^m W$ ($\hookrightarrow \Lambda^m V$) has the form $(\chi_1, m\chi_2)$, where χ_1 is a charcter on UT_1. $-n\chi_1$ lies in S, and $K(-n\chi_1)$ is an H-submodule of some G-module, by Lemma 1.3. Since $K(n(\chi_1, m\chi_2))$ is an H-submodule of some G-module $[K(\chi_1, m\chi_2)$ already has this property], so is $K(-n\chi_1) \otimes K(n(\chi_1, m\chi_2)) = K(1, nm\chi_2)$. Therefore, for any $Y_1 \in X(UT_1)$ and $\chi_2 \in X(T_2)$, $K(mn(Y_i, \chi_2))$ is a H-submodule of some G-module (where m depends on χ_2). By the observation [**1**, Th. 9], H is observable in G.

2. We present a class of solvable connected universally observable groups H whose monoid S of characters arising in $C(H)$ is *not* a group.

Let K be an algebraically closed field of characteristic zero and w an integer greater than 2. Let H be the semidirect product $G_a \times_w G_m$, where the multiplicative group G_m operates on the additive group G_a via the wth-power character.

THEOREM. *$H = G_a \times_w G_m$ is universally observable, but the monoid of characters of H in $C(H)$ is not a group.*

Proof. Universal observability is shown in §2.3. Here we prove the part dealing with the monoid S of characters. Evidently, S is contained in the monoid generated by the characters appearing in the operation of G_m on hy(H). Since G_m operates trivially on hy(G_m), the latter monoid coincides with the monoid of characters of G_m on hy(G_a), which equals $\{nw \mid n \in \mathbf{Z}_{\geq 0}\}$. This monoid contains only the trivial group as a subgroup.

REMARK. In order to show that $H = G_a \times_w G_m$ ($w > 2$) is universally observable, we need only consider embeddings $H \to G$ where G is a general linear group Gl_n, $n \in \mathbf{Z}_{>0}$. In fact, if $H \to G$ is any embedding, then we may embed G in some Gl_n. If Gl_n/H is quasi-affine, so is G/H.

2.1. Let Gl_n be the general linear group over an algebraically closed field K of characteristic zero. B, U, and T are the upper triangular, upper unipotent, and diagonal subgroups, and gl_n is the Lie algebra of $n \times n$ matrices over K. We work with the adjoint representation of Gl_n on gl_n.

Let $A = (A_{ij})$ be a non-zero nilpotent $n \times n$ matrix in Jordan form

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{pmatrix}, \quad \text{where } A_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ & & & 0 \end{pmatrix} \text{ is } n_i \times n_i.$$

Let C_A be the subgroup of Gl_n of elements that act trivially on A (i.e., the centralizer of A), and let D_A be the subgroup of diagonal elements of C_A.

LEMMA. *D_A is a maximal torus of C_A.*

Proof. Let

$$I = \{1 < i < n | A_{i,i+1} = 0\} = \{i_1 < i_2 < \cdots < i_{m-1}\}.$$

C_A is the group

$$\begin{aligned} &\{(a_{ij}) \in \mathrm{Gl}_n | a_{i,k-1}A_{k-1,k} = A_{i,i+1}a_{i+1,k}; 1 \le i, k \le n\} \\ &\quad = \{(a_{ij}) \in \mathrm{Gl}_n | a_{ik}A_{k,k+1} = A_{i,i+1}a_{i+1,k+1}; \\ &\qquad\qquad 1 \le i \le n, 0 \le k \le n-1\}. \end{aligned}$$

Here and below, a term a_{ik} or A_{ik} is understood to be zero if i or k equals 0 or $n + 1$.

C_A is the set of all (a_{ij}) satisfying c1, c2, c3 for $1 \le i \le n$, and $0 \le k \le n - 1$.

c1. $a_{ik} = a_{i+1,k+1}$ when $i, k \notin I$.

c2. $a_{i+1,k+1} = 0$ when $i \notin I$ and $k \in I$.

c3. $a_{ik} = 0$ when $i \in I$ and $k \notin I$.

D_A is the subgroup $\{(a_{ij}) \in T | a_{ii} = a_{i+1,i+1} \text{ for } i \notin I\}$, a torus of dimension m. In order to prove the lemma, it will suffice to show that D_A is a maximal torus in $\mathrm{Cent}(D_A) \cap C_A$, where $\mathrm{Cent}(D_A)$ is the centralizer of D_A in Gl_n. We do this by showing that $\mathrm{Cent}(D_A) \cap C_A$ is upper triangular, and by noting that, since D_A already contains the diagonal piece of any upper triangular element of C_A, $B \cap C_A$ equals $D_A \cdot (U \cap C_A)$.

Since K is an infinite field,

$$\mathrm{Cent}(D_A) = \left\{ \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_m \end{pmatrix} \middle| B_i \text{ is an invertible } n_i \times n_i \text{ matrix} \right\}.$$

Take any element

$$(a_{ij}) = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_m \end{pmatrix} \text{ from } \mathrm{Cent}(D_A) \cap C_A.$$

(1) B_1 is upper triangular. In fact, for $i < i_1$ and $k = 0$, application of cl to (a_{ij}) gives the relation $0 = a_{i0} = a_{i+1,1}$; and by cl, for $i, k < i_1$, $a_{ik} = a_{i+1,k+1}$. Hence, $a_{ik} = 0$ for $0 < k < i \le i_1$.

(2) B_k is upper triangular for $1 < k \le m$. The first row (column) of B_k is the $i_k + 1$ row (column) of (a_{ij}). By c2, for $i_k < j < i_{k+1}$, $a_{j+1,i_k+1} = 0$; by repeated use of cl, $a_{j+j',i_k+j'} = 0$ for $i < j' \le i_{k+1} - j$. Hence, B_k is upper triangular.

2.2. Let N_A be the normalizer of KA in Gl_n (the stabilizer of KA under the adjoint representation). N_A operates on KA by some character χ whose kernel is C_A. Let H° denote the connected component of 1 in a group H.

LEMMA *$(N_A \cap T)^\circ$ is a maximal torus of N_A.*

Proof. Since the codimension of C_A in N_A is one at most, the codimension of D_A in a maximal torus of N_A is one at most. We show the maximality of $(N_A \cap T)^\circ$ by showing that D_A indeed has codimension one in $(N_A \cap T)^\circ$. Define $\varphi\colon G_m \to (N_A \cap T)^\circ$ by

$$\varphi(t) = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_m \end{pmatrix},$$

where B_i is the diagonal matrix

$$\begin{pmatrix} t & & & \\ & t^2 & & \\ & & \ddots & \\ & & & t^{n_i} \end{pmatrix}.$$

Since $\varphi(t)A\varphi(t)^{-1} = t^{-1}A$, $\varphi(G_m) \cap D_A$ is finite and one is the codimension of D_A in $D_A\varphi(G_m)$.

2.3. THEOREM. *Let K be algebraically closed of characteristic zero. $G_a \times_w G_m$ is universally observable for each $w > 2$.*

Proof. By the remark at the end of §2, we need only show that each embedding φ: $G_a \times_w G_m \to \mathrm{Gl}(V)$ determines a quasi-affine quotient $\mathrm{Gl}(V)/G_a \times_w G_m$. The image of $L(G_a)$ under the differential $d\varphi$ consists of nilpotent operators on V, since $\varphi(G_a)$ is unipotent. Choose a basis for V such that $d\varphi(L(G_a))$ has one of its non-zero elements A in Jordan form

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & 0 \\ & & \ddots & \\ 0 & & & A_m \end{pmatrix}.$$

G_m, which operates on G_a and $L(G_a)$ by the w-power character, operates on $d\varphi(L(G_a))$ by the w-power character, that is, φ maps G_m into N_A°, where G_m operates on KA by $t \cdot A = t^w A$. Let c_g be conjugation by an element g of N_A°, mapping the torus $\varphi(G_m)$ into a subtorus of the maximal torus $(N_A \cap T)^\circ$ of N_A. With $c_g \circ \varphi$ in place of φ, $\varphi(G_m)$ is diagonal and $L(\varphi(G_a)) = d\varphi(L(G_a))$ still equals KA. Therefore, we may assume that $d\varphi(L(G_a \times_w G_m))$ is contained by the subspace $L(T) \oplus KA$ of

$$L(\mathrm{Gl}_{n_1} \times \cdots \times \mathrm{Gl}_{n_m}) = \mathrm{gl}_{n_1} \times \cdots \times \mathrm{gl}_{n_m} \qquad (\subset \mathrm{gl}_n = L(\mathrm{Gl}(V))).$$

By Theorem 8.6 of [**2**], $\varphi(G_a \times_w G_m)$ is contained by $\mathrm{Gl}_{n_1} \times \cdots \times \mathrm{Gl}_{n_m} \subset \mathrm{Gl}_n$.

Recall that $X(G)$ is the group of characters of a group G. (1) If $\mathrm{Sl}_{n_1} \times \cdots \times \mathrm{Sl}_{n_m}$ does not contain $\varphi(G_a \times_w G_m)$, then the morphism

$$G_a \times_w G_m \xrightarrow{\varphi} \mathrm{Gl}_{n_1} \times \cdots \times \mathrm{Gl}_{n_m}$$
$$\xrightarrow{\pi} \frac{\mathrm{Gl}_{n_1} \times \cdots \times \mathrm{Gl}_{n_m}}{\mathrm{Sl}_{n_1} \times \cdots \times \mathrm{Sl}_{n_m}} \cong G_m \times \cdots \times G_m$$

determines (dually) a non-trivial subgroup $(\pi \circ \varphi)^*(X(G_m \times \cdots \times G_m))$ of the infinite cyclic group $X(G_a \times_w G_m)$. Of course, the characters in this subgroup extend to characters on $\mathrm{Gl}_{n_1} \times \cdots \times \mathrm{Gl}_{n_m}$. Hence, by [**1**, Th. 9], $\mathrm{Gl}_{n_1} \times \cdots \times \mathrm{Gl}_{n_m}/\varphi(G_a \times_w G_m)$ is quasi-affine. Since $\mathrm{Gl}_{n_1} \times \cdots \times \mathrm{Gl}_{n_m}$ is linearly reductive, $\mathrm{Gl}_n/\mathrm{Gl}_{n_1} \times \cdots \times \mathrm{Gl}_{n_m}$ is affine, and so, by [**1**], $\mathrm{Gl}_n/\varphi(G_a \times_w G_m)$ is quasi-affine.

(2) We show that $\mathrm{Sl}_{n_1} \times \cdots \times \mathrm{Sl}_{n_m}$ does not contain $\varphi(G_a \times_w G_m)$, and so, (1) shows that $\mathrm{Gl}_n/\varphi(G_a \times_w G_m)$ is quasi-affine. If $\varphi\colon G_a \times_w G_m \to \mathrm{Sl}_{n_1} \times \cdots \times \mathrm{Sl}_{n_m}$ is any (supposed) injective morphism, let $\varphi_i\colon G_a \times_w G_m \to \mathrm{Sl}_{n_i}$ be the ith component of φ. We show that the kernel of φ_i contains the wth roots of unity in G_m when w is odd and the $(w/2)$th roots of unity when w is even. Hence, φ is not an embedding.

Let

$$\varphi_i(t) = \begin{pmatrix} t^{m_1} & & 0 \\ & t^{m_2} & \\ 0 & & t^{m_{n_i}} \end{pmatrix} \quad \text{for } t \in G_m.$$

Since $\varphi_i(G_m)$ operates on $(d\varphi_i)(L(G_a)) = KA_i$ by $\varphi_i(t)A_i\varphi_i(t)^{-1} = t^w A_i$, we have $t^{m_j - m_{j+1}} = t^w$ for $1 \leq j < n_i$, or in terms of exponents $m_j = m_1 - jw$ for $1 < j \leq n_i$. Since $\varphi_i(G_a \times_w G_m)$ consists of special linear transformations,

$$0 = \sum_{j=1}^{n_i} m_i = \sum_{j=0}^{n_i - 1} (m_1 - jw) = n_i m_1 - \frac{(n_i)(n_i - 1)}{2} w,$$

and so $2m_1 = (n_i - 1)w$.

If w is odd, w divides m_i, for each i, and $\mathrm{Ker}\,\varphi_i$ contains the wth roots of unity. If w is even, $w/2$ divides m_i for each i, and $\mathrm{Ker}\,\varphi_i$ contains the $(w/2)$th roots of unity.

2.4. COROLLARY. *Universal observability is not a property of isogeny classes.*

Proof. For any $w \neq 0$, there is the isogeny $G_a \times_w G_m \to G_a \times_1 G_m$ mapping (a, t) to (a, t^w). For $w > 2$, $G_a \times_w G_m$ is universally observable, but $G_a \times_1 G_m$ is not, since it may be embedded in PGl_2 as a Borel subgroup, in which case the quotient is projective.

3. We situate $G_a \times_w G_m$ in GL_2 as the kernel of a character on a Borel subgroup, and then show that there are *not* universally observable subgroups of Gl_n $(n > 2)$ whose situation is analogous to that of $G_a \times_w G_m$ in Gl_2.

Let B be the upper triangular subgroup of GL_2, and let w be a positive integer. Let χ_w be the character on B defined by

$$\chi_w \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = t_1^{w-1} t_2.$$

The kernel of χ_w coincides with the image of the embedding $G_a \times_w G_m \to \mathrm{Gl}_2$ taking (a, t) to $\left(\begin{smallmatrix} t & a \\ 0 & t^{1-w} \end{smallmatrix}\right)$. In the second section we showed that Ker χ_w $(w > 2)$ is a universally observable group. There are two particular facts to mention about this situation. (1) Ker χ_w does not contain the center of Gl_2; and (2) $G_a \times_w G_m$ is isogenous to a Borel subgroup of PGl_2. In contrast, for χ a character on the upper triangular subgroup of Gl_n $(n > 2)$, for Ker χ to be universally observable, it must, among other things, contain the center of Gl_n (Th. 3.3); consequently, it is not isogeneous to a Borel subgroup of $\mathrm{PG1}_n$. (For $n = 2$, a prescription for the universal observability of Ker χ which contains the requirement that Ker χ should contain the center of Gl_2 would refer only to the direct product $G_a \times G_m$.) Furthermore, we will see in the proof of the Theorem in §4 that, for those subgroups Ker χ of Gl_n $(n > 2)$ which are universally observable, the characters of Ker χ on the one-dimensional modules in C (Ker χ) form a group (cf. §1).

3.1. Let K be an algebraically closed field of *any* characteristic. Fix a character λ on B, the upper triangular subgroup of Gl_n $(n > 2)$, and let i: Ker $\lambda \to \mathrm{GL}_n$ be the inclusion map. We will test Ker λ for observability against those embeddings of Ker λ in Gl_n obtained from skewing i by a character χ on B.

Let λ_i be the ith fundamental weight on B,

$$\lambda_i \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} = t_1 \cdots t_i.$$

For any character $\chi = \Sigma_{i=1}^n a_i \lambda_i$, let a_χ stand for the "signature" $\Sigma_{i=1}^n i a_i$ of χ. For each character χ: $B \to \mathrm{Gl}(K\chi)$, form the representation $P_\chi = i \otimes \chi$: $B \to \mathrm{Gl}(K^n \otimes K\chi) = \mathrm{Gl}(K^n)$.

LEMMA. *P_χ is an embedding if and only if $a_\chi = 0$.*

Proof.

$$P_\chi(b) \text{ equals } b \cdot \begin{pmatrix} \chi(b) & & 0 \\ & \ddots & \\ 0 & & \chi(b) \end{pmatrix}, \quad \text{for } b \in B.$$

Evidently, P_χ is an embedding exactly when P_χ: $B \to B$ is an automorphism. Since P_χ stabilizes T and operates identically on B_u, P_χ is an automorphism of B exactly when its restriction to T is an automorphism of T. P_χ^*: $X(T) \to X(T)$, the endomorphism of the character group dual to P_χ: $T \to T$, has the formula $P_\chi^*(\beta) = \beta + a_\beta \chi$. When $a_\chi = 0$, one may check that the inverse to P_χ^* has the formula $(P_\chi^*)^{-1}(\beta) = \beta - a_\beta \chi$. When $a_\chi \neq 0$, one may check that χ is *not* an element in the image of P_χ^*.

3.2. LEMMA. *Let $\beta \in X(T)$. $P_\chi(\mathrm{Ker}(\beta \circ P_\chi)) = \mathrm{Ker}\,\beta$, when $a_\chi \neq -1$.*

Proof. We may regard this statement as equally about characters on B, or characters on T, since P_χ acts identically on B_u and since characters vanish on B_u. One may check that if φ is a surjective endomorphism of T, then $\varphi(\mathrm{Ker}(\beta \circ \varphi)) = \mathrm{Ker}\,\beta$. So it will suffice to show that P_χ: $T \to T$ is surjective. The kernel of P_χ lies in the center of Gl_n, and P_χ at a central element

$$\begin{pmatrix} S & & & 0 \\ & S & & \\ & & \ddots & \\ 0 & & & S \end{pmatrix}$$

has the value

$$\begin{pmatrix} S^{1+a_\chi} & & 0 \\ & \ddots & \\ 0 & & S^{1+a_\chi} \end{pmatrix}.$$

Hence, $\mathrm{Ker}\,P_\chi$ is finite when $1 + a_\chi \neq 0$, and so, P_χ is surjective.

COROLLARY. *Let $\lambda \in X(T)$. If $a_\chi = 0$, then $P_\chi(\mathrm{Ker}\,\lambda) = \mathrm{Ker}((P_\chi^*)^{-1}(\lambda))$.*

Proof. The corollary follows from the lemma with $\beta = (P_\chi^*)^{-1}(\lambda)$, using the definition of P_χ^*: $P_\chi^*(\beta) = \beta \circ P_\chi$.

3.3. Let K be algebraically closed. Let λ be a character on the upper triangular subgroup B of Gl_n $(n \geq 3)$.

THEOREM. *Ker λ is universally observable if and only if the signature a_λ of λ is zero, and $\pm\lambda|_{T \cap \mathrm{Sl}_n}$ lies in the dominant chamber.*

REMARK. $a_\lambda = 0$ if and only if $\mathrm{Ker}\,\lambda$ contains the center of GL_n.

Proof. Suppose that $a_\lambda \neq 0$. Let b be an integer and let χ_b be the character $-2b\lambda_1 + b\lambda_2$. Since $a_{\chi_b} = 0$, P_{χ_b} is a embedding, and, by the corollary in §3.2, $P_{\chi_b}(\operatorname{Ker}\lambda) = Ker((P^*_{\chi_b})^{-1}(\lambda)) = \operatorname{Ker}(\lambda - a_\lambda\chi_b)$. Since a_λ is non-zero, the coefficients of λ_1 and λ_2 in $\lambda - a_\lambda\chi_b$ have opposite signs for large b, and so, for large b, $\pm(\lambda - a_\lambda\chi_b)|_{T\cap \mathrm{Sl}_n}$ lies not in the dominant chamber. Hence, by [4, Theorem 2.3],

$$Sl_n/\operatorname{Ker}\big((\lambda - a_\lambda\chi_b)|_{B\cap \mathrm{Sl}_n}\big)$$

is not quasi-affine. Neither is $\mathrm{Gl}_n/\operatorname{Ker}(\lambda - a_\lambda\chi_b)$ quasi-affine. If it were,

$$\mathrm{Gl}_n/\operatorname{Ker}\big((\lambda - a_\lambda\chi_b)|_{B\cap \mathrm{Sl}_n}\big)$$

would be quasi-affine, since $\operatorname{Ker}((\lambda - a_\lambda\chi_b)|_{B\cap \mathrm{Sl}_n})$ is normal in $\operatorname{Ker}(\lambda - a_\lambda\chi_b)$. Then $\mathrm{SL}_n/(\operatorname{Ker}((\lambda - a_\lambda\chi_b)|_{B\cap \mathrm{Sl}_n}))$ would be quasi-affine. Therefore, $\operatorname{Ker}\lambda$ is not universally observable when $a_\lambda \neq 0$.

Suppose that $\pm\lambda|_{T\cap \mathrm{Sl}_n}$ is not in the dominant chamber. Arguing as above, $\mathrm{GL}_n/\operatorname{Ker}\lambda$ is already non-quasi-affine.

Conversely, suppose that $\pm\lambda|_{T\cap Sl_n}$ lies in the dominant chamber and that $a_\lambda = 0$. Since $a_\lambda = 0$, the kernel of λ contains the center of Gl_n, evidently, and so, $\operatorname{Ker}\lambda = B_u \cdot \operatorname{Ker}\lambda|_{T\cap \mathrm{Sl}_n} \cdot \operatorname{Center}(\mathrm{Gl}_n)$. In the next section, we show that the characters of $\operatorname{Ker}\lambda$ which appear in $C(\operatorname{Ker}\lambda)$ form a subgroup. Hence, by Proposition 1.4, $\operatorname{Ker}\lambda$ is universally observable. (More accurately, $(\operatorname{Ker}\lambda)^\circ$ is universally observable. By [1], it follows that $\operatorname{Ker}\lambda$ is universally observable).

4. Let P be the parabolic subgroup of Sl_n which is associated to a character χ on $T \cap \mathrm{Sl}_n$ (T = diagonal subgroup of Gl_n). P is generated by $T \cap \mathrm{Sl}_n$ and by the one-parameter unipotent subgroups of Sl_n corresponding to the roots α such that the coroot of α has non-negative value at χ. Let P_u be the unipotent radical of P and B the upper triangular subgroup of Sl_n.

THEOREM. *The subgroup* $P_u \cdot \operatorname{Ker}\chi \cdot \operatorname{Center}(\mathrm{Gl}_n)$ *of* Gl_n *is universally observable.*

COROLLARY. *If* $\pm\chi$ *lies in the dominant chamber, then* $B_u \cdot \operatorname{Ker}\chi \cdot \operatorname{Center}(\mathrm{Gl}_n)$ *is universally observable.*

The corollary follows from the Theorem since B is the parabolic subgroup associated to characters in the dominant chamber.

Proof of the Theorem. Suppose that H is a group such that the monoid S of characters of H on $C(H)$ is a group. Let G be a connected group of the form $H \cdot C$ where H is a subgroup of G, and C is a central subgroup, and where the multiplication map $H \times C \to G$ is separable. From the last condition, which is equivalent to requiring that the differential of the multiplication map at the neutral element be surjective, it follows that the associated map $\mathrm{hy}(H) \otimes \mathrm{hy}(C) \to \mathrm{hy}(G)$ is surjective (for instance, by [3, §2, proposition]).

A character χ on G appears in the operation of G on the one-dimensional modules in $C(G)$ exactly when $\chi|_H$ appears in the operation of H on $C(G)$, since C operates trivially on $\mathrm{hy}(G)$. Furthermore, the operations of H on $C(H)$ and on $C(G)$ give the same monoid of characters on H. In fact, any H-submodule of $\mathrm{hy}(G)$ is a quotient of a direct sum of H-submodules of $\mathrm{hy}(H)$, since $\mathrm{hy}(H) \otimes \mathrm{hy}(C) \to \mathrm{hy}(G)$ is surjective, and since $C(H)$ is closed under quotients by Lemma 1.2.

Taking $H = P_u \cdot \mathrm{Ker}\,\chi$ and $C = \mathrm{Center}(\mathrm{Gl}_n)$ (which satisfies the hypothesis of separability since $\mathrm{Sl}_n \cdot C$ does), we will show that $P_u \cdot \mathrm{Ker}\,\chi \cdot \mathrm{Center}(\mathrm{Gl}_n)$ is universally observable by showing that the monoid of characters of $P_u \cdot \mathrm{Ker}\,\chi$ on $C(P_u \mathrm{Ker}\,\chi)$ is a group, and hence, so is the monoid of characters of $P_u \cdot \mathrm{Ker}\,\chi \cdot \mathrm{Center}(\mathrm{Gl}_n)$ on

$$C(P_u \cdot \mathrm{Ker}\,\chi \cdot \mathrm{Center}(\mathrm{Gl}_n))$$

by the previous paragraph.

Let W be the Weyl group of Sl_n relative to $T \cap \mathrm{Sl}_n$, an w an element which conjugates B into P. If we replace χ by its transform under w^{-1}, χ will be a dominant weight relative to B. χ has the form $\Sigma_{i\in J} n_i\lambda_i$, $n_i > 0$, for some subset J of $\{1,\ldots,n\}$. P_u is generated by one parameter subgroups G_α, the coroot of α being positive at χ, and $L(P_u)$ is spanned by $\{E_{jk} | j \le i < k \text{ for some } i \in J\}$, where E_{jk} is the matrix whose single non-zero entry is a 1 in the jkth position.

Let A be the group of characters generated by

$$\left\{(\alpha_j + \cdots + \alpha_{k-1})|_{P_u \mathrm{Ker}\,\chi} | j \le i < k \text{ for some } i \in J\right\}.$$

The character monoid M for $P_u \mathrm{Ker}_\chi$ relative to $C(P_u \mathrm{Ker}_\chi)$ is a submonoid of A, since $\alpha_j + \cdots + \alpha_{k-1}$ is the weight of $T \cap \mathrm{Sl}_n$ operating on $KE_{jk} \subset L(P_u \mathrm{Ker}\,\chi)$. We will know M for a group once we show that M contains a subgroup of finite index in A.

Fix i in J, and take the highest non-zero exterior power of the $P_u \mathrm{Ker}\,\chi$-module spanned by the E_{jk}, $j \le i < k$. Since $\alpha_j + \cdots + \alpha_{k-1}$ is the weight of E_{jk}, a_l is a term in the weight of E_{jk} exactly when $j \le l < k$. Hence, the $T \cap \mathrm{Sl}_n$-weight of the exterior power is $(n-i)\Sigma_{j=1}^{i} j\alpha_j + i\Sigma_{j=i+1}^{n-1}(n-j)\alpha_j$, which equals $n\lambda_i$.

Since χ is zero on $P_u \mathrm{Ker}\,\chi$, $-nn_i\lambda_i$ equals $\Sigma_{j\ne i} nn_j\lambda_j$, which is an element of M. That is, $-n_i(\Sigma_{j\le i<k} \mathrm{wt}(E_{jk}))$ is an element of M, for any i

in J. Evidently, $\mathrm{wt}(E_{j'k'}), j' \leq i < k'$, lies in M. Hence

$$-n_i\big(\mathrm{wt}(E_{jk})\big) = -n_i\Bigg(\sum_{j' \leq i < k'} \mathrm{wt}(E_{j'k'})\Bigg) + n_i\Bigg(\sum_{\substack{j' \leq i < k' \\ (j', k') \neq (j, k)}} \mathrm{wt}(E_{j'k'})\Bigg)$$

lies in M. Therefore, M contains the subgroup (of finite index) of A generated by

$$\left\{\Big(\prod_{i \in J} n_i\Big)(\alpha_j + \cdots + \alpha_{k-1})|_{P_u \operatorname{Ker} \chi} \big| j \leq i < k \text{ for some } i \in J\right\}.$$

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