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WHEN ALL SEMIREGULAR H-CLOSED EXTENSIONS ARE COMPACT

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It is well-known that compactifications of Tychonoff spaces are semiregular and H-closed. Katetov has determined when certain H-closed and semiregular H-closed extensions of a Hausdorff space are compact. In this paper, those Tychonoff spaces in which all semiregular, H-closed extensions are compact are characterized.

1. Introduction and preliminaries. In 1947, Katetov [$\mathbf{K}_2$] determined that the "largest" H-closed extension κX of a Hausdorff space X is compact iff X is compact. Since compact spaces are semiregular, a related problem is to determine when the semiregularization of κX (denoted $(\kappa X)_s$) is compact. This was also solved by Katetov [$\mathbf{K}_2$]. A natural extension of this problem is to determine when all of the semiregular, H-closed extensions of a space are compact.

If $\mathscr{M}(X)$ denotes the collection of all semiregular, H-closed extensions of a space X and $\mathscr{K}(X)$ denotes the collection of all compactifications of X, the problem is to determine those spaces X such that $\mathscr{M}(X) = \mathscr{K}(X)$. Since $\mathscr{M}(X) \neq \varnothing$ iff X is semiregular and $\mathscr{K}(X) \neq \varnothing$ iff X is Tychonoff, it follows that $\mathscr{M}(X) = \mathscr{K}(X) = \varnothing$ iff X is not semiregular and $\mathscr{M}(X) \neq \mathscr{K}(X)$ when X is semiregular but not Tychonoff. So, the nontrivial portion of the problem is to characterize those Tychonoff spaces X such that $\mathscr{M}(X) = \mathscr{K}(X)$. This problem is completely solved in this paper.

At first glance, the evidence points to the trivial solution that $\mathscr{M}(X) = \mathscr{K}(X)$ iff X is compact, for if D is an infinite discrete space, then $\mathscr{M}(D) \neq \mathscr{K}(D)$ (see [$\mathbf{PV}_1$]). However, additional investigation reveals that if $X = \beta\mathbf{N} \setminus \{p\}$ for some $p \in \beta\mathbf{N} \setminus \mathbf{N}$, then $\mathscr{M}(X) = \mathscr{K}(X)$.

Some preliminary definitions and concepts are needed. Throughout the paper, the word "space" will mean "Hausdorff topological space".

A space X is *H-closed* if X is closed in every space containing it as a subspace. Recall that set $A \subseteq X$ is *regular open* if $A = \operatorname{int}_X \operatorname{cl}_X A$. The *semiregularization* of a space X is the topology generated on the underlying set of X by the family of regular open subsets of S, and is denoted as X_s. A space X is *semiregular* if $X = X_s$; the space X_s is easily verified to be semiregular. Obviously, the identity function on the underlying set X, viewed as a function from space X onto the space X_s, is continuous.

A space X is *minimal Hausdorff* if there is no strictly coarser Hausdorff topology on X. A well-known result [$\mathbf{K_1}$] is that a space X is minimal Hausdorff iff X is H-closed and semiregular. If X is H-closed, then X_s is also H-closed and, hence, minimal Hausdorff [**PT**]. A space Y is an *extension* of a subspace X if $\mathrm{cl}_Y X = Y$; two extensions Y and Z of a space X are said to be *equivalent*, denoted as $Y =_X Z$, if there is a homeomorphism $h\colon Y \to Z$ such that $h(x) = x$ for each $x \in X$. Henceforth, we identify equivalent extensions of a space. Another well-known result [**S**] is that if Y is an extension of a space X, then Y_s is a semiregular extension of X_s; in particular, when X is semiregular, then Y_s is also an extension of X.

If $\mathscr{F}$ is an open filter base on a space X, the set $\bigcap\{\mathrm{cl}_X F\colon F \in \mathscr{F}\}$ is called the *adherence* of $\mathscr{F}$ in X and denoted as $\mathrm{ad}_X \mathscr{F}$. An open filter base $\mathscr{F}$ on X is *fixed* if $\mathrm{ad}_X \mathscr{F} \neq \varnothing$; otherwise it is *free*. For each space X let $X^* = X \cup \{\mathscr{U}\colon \mathscr{U}$ is a free open ultrafilter on $X\}$. The family $\{U \subseteq X\colon U$ is open in $X\} \cup \{\{\mathscr{U}\} \cup U\colon U$ is open in X, $U \in \mathscr{U}$, $\mathscr{U} \in X^* \setminus X\}$ is a base for a topology on X^*; X^* with this topology is denoted as κX. For an open set $U \subseteq X$, let $oU = U \cup \{\mathscr{U} \in X^* \setminus X\colon U \in \mathscr{U}\}$. The family $\{oU\colon U$ open in $X\}$ is a base for a topology on X^*; X^* with this topology is denoted as σX. The space $(\kappa X)_s$ is denoted by μX. We now list some results which are needed in the sequel; these results can be found in [$\mathbf{K_1}, \mathbf{K_2}, \mathbf{P}, \mathbf{PT}, \mathbf{PV_1}, \mathbf{PV_2}$].

(1.1) PROPOSITION. *Let X be a space. Then*:

(a) *κX and σX are H-closed extensions of X, and the identity function from κX onto σX is continuous.*

(b) *If Y is an H-closed extension of X, then $\kappa X \geq Y$, i.e., there is a continuous function from κX onto Y which is the identity function on X.* [*It is in this sense that κX is the "largest" H-closed extension of X.*]

(c) *If X is semiregular, then μX is a minimal Hausdorff extension of X, $\mu X = (\sigma X)_s$, the identity function from σX onto μX is continuous, $\sigma X \setminus X$ is homeomorphic to $\mu X \setminus X$, the family $\{oU\colon U$ is a regular open subset of $X\}$ is a base for the topology on μX, and for an open subset $U \subseteq X$, $\mathrm{cl}_{\mu X}(oU) = (\mathrm{cl}_X U) \cup oU$.*

For a space X, the spaces κX and σX are respectively called the Katetov H-closed extension and the Fomin H-closed extension of X; if X is semiregular, μX is called the Banaschewski-Fomin-Shanin minimal Hausdorff extension of X. Let Y be an H-closed extension of a space X, and let f_Y: $\kappa X \to Y$ denote the (unique) continuous function such that $f_Y(x) = x$ for each $x \in X$ (see 1.1(b)). If X is semiregular and $Y = \mu X$, then f_Y is denoted as f_μ; since the identity function on the underlying set of κX is a continuous function from σX onto μX (see 1.1(c) above), it follows that f_μ is the identity function on X^*. For each $y \in Y \setminus X$, $f_Y^{\leftarrow}(y)$ is a subset of $\kappa X \setminus X = X^* \setminus X$ and, hence a subset of $\sigma X \setminus X$ and $\mu X \setminus X$. Let $\mathbf{P}_\mu(Y) = \{ f_Y^{\leftarrow}(y) \colon y \in Y \setminus X\}$. So $\mathbf{P}_\mu(Y)$ is a partition of $\mu X \setminus X$.

(1.2) PROPOSITION. [**P**, Th. 05; **PV**$_1$, Th. 3.1 and 3.5; **PV**$_2$, Th. 5.4]. *Let X be a semiregular space. Then*:

(a) *If Y is an H-closed extension of X, then $\mathbf{P}_\mu(Y)$ is a partition of $\mu X \setminus X$ into compact subsets.*

(b) *If $\mathbf{P}$ is a partition of $\mu X \setminus X$ into compact subsets, then there is an H-closed extension Y of X such that $\mathbf{P}_\mu(Y) = \mathbf{P}$.*

(c) *If Y and Z are H-closed extensions of X, then*

(i) $\mathbf{P}_\mu(Y) = \mathbf{P}_\mu(Y_s)$ *and*

(ii) $\mathbf{P}_\mu(Y) = \mathbf{P}_\mu(Z)$ *iff* $Y_s = Z_s$.

So, by 1.2(c), there exists a bijection between the set of minimal Hausdorff extensions of a semiregular space X and the set of partitions of $\mu X \setminus X$ into compact subsets. Let $\mathscr{M}(X)$ denote the set of all minimal Hausdorff extensions of a semiregular space X.

Let $\mathbf{P}$ be a partition of a space X into compact subsets. A set $C \subseteq X$ is $\mathbf{P}$-*saturated* if $C = \bigcup\{ B \in \mathbf{P} \colon B \subseteq C\}$. We say that $\mathbf{P}$ is *upper semicontinuous* (abbreviated as USC) if, for each open subset U of X and each $A \in \mathbf{P}$ for which $A \subseteq U$, there exists a $\mathbf{P}$-saturated open set V such that $A \subseteq U \subseteq U$. If X is a Tychonoff space, Y is a compactification of X, and g_Y: $\beta X \to Y$ is the continuous function such that $g_Y(x) = x$ for $x \in X$, then $\mathbf{P}_\beta(Y)$ is used to denote $\{ g_Y^{\leftarrow}(p) \colon p \in Y\}$. Let $\mathscr{K}(X)$ denote the set of all compactifications of X.

(1.3) PROPOSITION. *Let X be a Tychonoff space. Then*:

(a) [**N**, Prop. 1] *If $Y \in \mathscr{K}(X)$, then $\mathbf{P}_\beta(Y)$ is an USC partition of βX.*

(b) [**N**, Prop. 1] *If $\mathbf{P}$ is an USC partition of βX and $\{\{x\} \colon x \in X\} \subseteq \mathbf{P}$, then for some $Y \in \mathscr{K}(X)$, $\mathbf{P} = \mathbf{P}_\beta(Y)$.*

(c) $\mu X \geq \beta X$.

Proof. Part (c) follows from 1.1(b), the fact that $(\kappa X)_s = \mu X$, and the following fact (see [**K**$_1$]): if Z is a space and f: $Z \to R$ is a continuous function into a regular space R, then f: $Z_s \to R$ is also continuous. □

(1.4) PROPOSITION. *Let X be a Tychonoff space for which $\mu X =_X \beta X$. Then $\mathscr{K}(X) = \mathscr{M}(X)$ iff, for each partition $\mathbf{P}$ of $\mu X \setminus X$ into compact subsets, the partition $\hat{\mathbf{P}} = \mathbf{P} \cup \{\{x\}: x \in X\}$ is an USC partition of βX.*

Proof. Suppose $\mathscr{K}(X) = \mathscr{M}(X)$ and $\mathbf{P}$ is a partition of $\mu X \setminus X$ into compact subsets. Then $\mathbf{P} = \mathbf{P}_\mu(Y)$ for some $Y \in \mathscr{M}(X)$ by 1.2 (b,c). But $Y \in \mathscr{K}(X)$ by hypothesis. Since $g_Y \circ f_\mu(x) = x$ for each $x \in X$, it follows that $g_Y \circ f_\mu = f_Y$. Hence $\hat{\mathbf{P}} = \mathbf{P}_\beta(X)$ and $\hat{\mathbf{P}}$ is an USC partition of βX. Conversely, to prove that $\mathscr{M}(X) = \mathscr{K}(X)$, first note that $\mathscr{K}(X) \subseteq \mathscr{M}(X)$ as every compactification of X is minimal Hausdorff. Now, suppose $Y \in \mathscr{M}(X)$. Then by hypothesis, $\widehat{\mathbf{P}_\mu(Y)}$ is an USC partition of βX. So, there is some $Z \in \mathscr{K}(X)$ such that $\mathbf{P}_\beta(Z) = \widehat{\mathbf{P}_\mu(Y)}$. In particular, $\mathbf{P}_\mu(Z) = \mathbf{P}_\mu(Y)$ so $Z_s =_X Y_s$ by 1.2(c), which implies that $Z = Y$. □

A point x in a space X is called *extremally disconnected* in X if for each pair of disjoint open sets U, V of X, $x \notin \mathrm{cl}_X U \cap \mathrm{cl}_X V$. A subset $A \subseteq X$ is said to be *regularly nowhere dense* in X if there are disjoint open sets U and V in X such that $A \subseteq \mathrm{cl}_X U \cap \mathrm{cl}_X V$.

(1.5) Let X be a Tychonoff space. The following are equivalent:

(a) $\mu X = \beta X$,

(b) every closed, regularly nowhere dense subset of X is compact, and

(c) every point of $\beta X \setminus X$ is extremally disconnected in βX.

Proof. The proof of the equivalence of (a) and (b) is in [$\mathbf{K}_2$]. To show (a) implies (c), it suffices to show for disjoint open sets U and V of βX that $\mathrm{cl}_{\beta X} U \cap \mathrm{cl}_{\beta X} V \subseteq X$. Note that $\mathrm{cl}_{\beta X} U = \mathrm{cl}_{\mu X} U = \mathrm{cl}_{\mu X}(U \cap X) = \mathrm{cl}_X(U \cap X) \cup o(U \cap X)$; the first equality is by (a) and the last equality is by 1.1(c). Since $o(U \cap X) \cap o(V \cap X) = o(U \cap V \cap X) = \varnothing$, it follows that $\mathrm{cl}_{\beta X} U \cap \mathrm{cl}_{\beta X} V = (\mathrm{cl}_X(U \cap X) \cap \mathrm{cl}_X(V \cap X)) \cup (o(U \cap X) \cap o(V \cap X)) \subseteq X$. Conversely, to show that (c) implies (b), suppose U and V are disjoint open subsets of X. Let $R = \beta X \setminus \mathrm{cl}_{\beta X}(X \setminus U)$ and $T = \beta X \setminus \mathrm{cl}_{\beta X}(X \setminus V)$. Note that $R \cap X = U$, $T \cap X = V$, and $R \cap T \cap X \subseteq U \cap V = \varnothing$; as X is dense in βX this implies that $R \cap T = \varnothing$. By (c), $\mathrm{cl}_{\beta X} R \cap \mathrm{cl}_{\beta X} T \subseteq X$. Since $\mathrm{cl}_X U \cap \mathrm{cl}_X V \subseteq \mathrm{cl}_{\beta X} R \cap \mathrm{cl}_{\beta X} T$, it follows that $\mathrm{cl}_X U \cap \mathrm{cl}_X V$ is compact. This completes the proof of (b). □

Let X be a Tychonoff space. A point $p \in \beta X \setminus X$ is called a *remote point of* βX if for each closed, nowhere dense subset $A \subseteq X$, $p \notin \mathrm{cl}_{\beta X} A$.

(1.6) [**vD**] Let X be a Tychonoff space. Then:

(a) If X is second countable, non-pseudocompact and has no isolated points, then βX has remote points.

(b) If p is a remote point of βX, then p is an extremally disconnected point of βX.

2. Main result. We can now prove the main result of this paper.

(2.1) THEOREM. *Let X be a Tychonoff space. Then $\mathcal{M}(X) = \mathcal{K}(X)$ iff the following are true*:

(a) *every closed, regularly nowhere dense subset of X is compact*,

(b) *$\beta X \setminus X$ is discrete, and*

(c) *if $\beta X \setminus X$ is infinite, then $\mathrm{cl}_{\beta X}(\beta X \setminus X)$ is the one-point compactification of $\beta X \setminus X$.*

Proof. Suppose $\mathcal{M}(X) = \mathcal{K}(X)$. Since $\mu X \in \mathcal{M}(X)$, then by 1.3(c), $\mu X = \beta X$. By 1.5, (a) is true. If $\beta X \setminus X$ is finite, then both (b) and (c) are satisfied. So, suppose $\beta X \setminus X$ is infinite. Then $\beta X \setminus X$ has at least one accumulation point in βX. Assume, by way of contradiction, that p and q are distinct accumulation points of $\beta X \setminus X$ in βX. Let U_p and U_q be open neighborhoods of p and q, respectively, such that $\mathrm{cl}_{\beta X} U_p \cap \mathrm{cl}_{\beta X} U_q = \varnothing$. There is an infinite set $A = \{x_n \colon n \in \mathbf{N}\} \subseteq U_p \setminus X$ and an infinite set $B = \{y_n \colon n \in \mathbf{N}\} \subseteq U_q \setminus X$. Let $f \colon \beta\mathbf{N} \to \mathrm{cl}_{\beta X} A$ and $g \colon \beta\mathbf{N} \to \mathrm{cl}_{\beta X} B$ be continuous functions such that $f(n) = x_n$ and $g(n) = y_n$ for $n \in \mathbf{N}$. Let $\alpha \in \beta\mathbf{N} \setminus \mathbf{N}$. So, $f(\alpha)$ and $g(\alpha)$ are distinct accumulation points of A and B, respectively. Choose $k \in \mathbf{N}$ so that $f(\alpha) \neq x_n$ and $g(\alpha) \neq y_n$ if $n > k$. Consider the partition

$$\begin{aligned} \mathbf{P} = {} & \{\{x_n, y_n\} \colon n \in \mathbf{N} \setminus \{1, 2, \ldots, k\}\} \cup \{\{x_i\} \colon 1 \le i \le k\} \\ & \cup \{\{y_i\} \colon 1 \le i \le k\} \cup \{\{y\} \colon y \in \beta X \setminus (X \cup A \cup B)\} \end{aligned}$$

of compact subsets of $\beta X \setminus X = \mu X \setminus X$. By 1.4, $\hat{\mathbf{P}} = \mathbf{P} \cup \{\{x\} \colon x \in X\}$ is an USC partition of βX. Let $T = \beta X \setminus \mathrm{cl}_{\beta X} U_q$. Evidently $f(\alpha) \in T$, so there is a $\hat{\mathbf{P}}$-saturated open set $V \subseteq \beta X$ such that $f(\alpha) \in V \subseteq T$. By the continuity of f there is an infinite set $C \in \alpha$ such that $f[C] \subseteq V$. So, there is some $m \in C$ such that $m > k$. Hence, $\{x_m, y_m\} \subseteq V$ as V is $\hat{\mathbf{P}}$-saturated. This is impossible as $y_m \in B \subseteq \mathrm{cl}_{\beta X} U_q$ and $V \cap \mathrm{cl}_{\beta X} U_q = \varnothing$. This completes the proof that $\beta X \setminus X$ has precisely one accumulation point in βX. Thus, $\mathrm{cl}_{\beta X}(\beta X \setminus X) = (\beta X \setminus X) \cup \{p\}$ where p is the accumulation point of $\beta X \setminus X$. Also, this shows that $\mathrm{cl}_{\beta X}(\beta X \setminus X)$ is a one-point compactification of the discrete space $\beta X \setminus (X \cup \{p\})$. By showing that $p \notin \beta X \setminus X$, we will have shown that (b) and (c) are satisfied. Assume, by way of contradiction, that $p \in \beta X \setminus X$. Let $\{x_n \colon n \in \mathbf{N}\}$ be a faithfully indexed infinite subset of $\beta X \setminus (X \cup \{p\})$. Since $\{x_n \colon n \in \mathbf{N}\}$ is discrete and βX

is regular, it is straightforward to obtain a family $\{U_n: n \in \mathbf{N}\}$ of pairwise disjoint open sets of βX such that $x_n \in U_n$. Let $U_e = \bigcup\{U_n: n \text{ even}\}$ and $U_0 = \bigcup\{U_n: n \text{ odd}\}$. Then $U_e \cap U_0 = \varnothing$. But $p \in \mathrm{cl}_{\beta X} U_e \cap \mathrm{cl}_{\beta X} U_0$ so p is not an extremally disconnected point of βX, which contradicts 1.5. So we have that $p \notin \beta X \setminus X$ and (b) and (c) are satisfied.

Conversely, suppose (a), (b), and (c) are satisfied. By 1.5, $\beta X = \mu X$. Let $\mathbf{P}$ be a partition of compact subsets of $\mu X \setminus X$. By 1.4, it suffices to show that $\hat{\mathbf{P}} = \mathbf{P} \cup \{\{x\}: x \in X\}$ is an USC partition of βX. First note that if $A \in \mathbf{P}$, then A is a finite set as A is a compact subset of the discrete space $\mu X \setminus X = \beta X \setminus X$. If $\beta X \setminus X$ is a finite set, then any partition of $\beta X \setminus X$, in particular $\mathbf{P}$, is an USC partition of $\beta X \setminus X$; if X is a locally compact space and $\mathbf{P}$ is an USC partition of $\beta X \setminus X$, it easily follows that $\hat{\mathbf{P}}$ is an USC partition of βX. So, suppose $\beta X \setminus X$ is infinite. Then $\mathrm{cl}_{\beta X}(\beta X \setminus X) = (\beta X \setminus X) \cup \{p\}$. To show $\hat{P}$ is an USC partition of βX, let U be an open subset of βX. There are three cases.

Case 1. $A \subseteq U$ where $A \in \mathbf{P}$. Since $\beta X \setminus X$ is discrete, there is an open set U_A in βX such that $U_A \cap (\beta X \setminus X) = A$. Now, $A \subseteq U_A \cap U \subseteq U$ and $U_A \cap U$ is $\hat{\mathbf{P}}$-saturated.

Case 2. $p \in U$. Since $(\beta X \setminus X) \setminus U$ is finite, there exist $n \in \mathbf{N}$ and sets $A_1, \ldots, A_n \in \mathbf{P}$ such that $(\beta X \setminus X) \setminus U \subseteq A_1 \cup \cdots \cup A_n$. Now, $p \in U \setminus (A_1 \cup \cdots \cup A_n) \subseteq U$ and evidently $U \setminus (A_1 \cup \cdots \cup A_n)$ is $\hat{\mathbf{P}}$-saturated.

Case 3. $x \in U$ where $x \in \beta X \setminus \mathrm{cl}_{\beta X}(\beta X \setminus X)$. Then $x \in U \setminus \mathrm{cl}_{\beta X}(\beta X \setminus X) \subseteq U$ and $U \setminus \mathrm{cl}_{\beta X}(\beta X \setminus X)$ is $\hat{\mathbf{P}}$-saturated. □

For each cardinal $\lambda > 0$, we now give an example of a noncompact, Tychonoff space X such that $\mathscr{M}(X) = \mathscr{K}(X)$ and $|\beta X \setminus X| = \lambda$.

(2.2) EXAMPLE. Let $p \in \beta\mathbf{N} \setminus \mathbf{N}$ and $X = \beta\mathbf{N} \setminus \{p\}$. Then $\kappa X \setminus X$ is a singleton and $(\kappa X)_s = \mu X = \beta\mathbf{N}$. So, if Y is the topological sum of n copies of X, where $n \in \mathbf{N}$, then Y is an example of a space with the properties that $\mathscr{M}(Y) = \mathscr{K}(Y)$ and $|\beta Y \setminus Y| = n$.

(2.3) EXAMPLES. Let λ be an infinite cardinal. Let D be a discrete space of cardinality λ, and let $\mathscr{L}$ be a partition of D into countable infinite subsets such that $|\mathscr{L}| = \lambda$. For each $d \in D$, let I_d be a copy of the unit interval $[0,1]$. Let Y denote the topological sum of the I_d's—i.e., $Y = \oplus\{I_d: d \in D\}$. For each $L \in \mathscr{L}$, let $Y_L = \oplus\{I_d: d \in L\}$, and put $X = Y \cup \{\infty\}$. A subset U of X is defined to be open if (1) $U \cap Y$ is open in Y, and (2) if $\infty \in U$, then there is a finite subset $\mathscr{F}$ of $\mathscr{L}$ such that $X \setminus \bigcup\{Y_L: L \in \mathscr{F}\} \subseteq U$. Clearly this defines a Tychonoff topology on X. Here are some results that will be useful in obtaining the desired example.

(a) If $L \in \mathscr{L}$, then Y_L is clopen in X; in particular, $\mathrm{cl}_{\beta X} Y_L = \beta Y_L$.

(b) $\{\mathrm{cl}_{\beta X} Y_L: L \in \mathscr{L}\}$ is a family of pairwise disjoint clopen subsets of βX.

(c) $\beta X = \{\infty\} \cup [\bigcup\{\mathrm{cl}_{\beta X} Y_L: L \in \mathscr{L}\}]$.

(d) A point $p \in \beta X$ is a remote point of βX iff for some $L \in \mathscr{L}$, p is a remote point of βY_L.

Proof. The proofs of (a) and (b) are straightforward. To prove (c), let $p \in \beta X \setminus X$. There is an open set U in βX such that $\infty \in U$ and $p \notin \mathrm{cl}_{\beta X} U$. There is a finite set $\mathscr{F} \subseteq \mathscr{L}$ such that $X \setminus \bigcup\{Y_L: L \in \mathscr{F}\} \subseteq U$. Since $\beta X = [\bigcup\{\mathrm{cl}_{\beta X} Y_L: L \in \mathscr{F}\}] \cup \mathrm{cl}_{\beta X}(X \setminus \bigcup\{Y_L: L \in \mathscr{F}\})$, then $p \in \mathrm{cl}_{\beta X} Y_L$ for some $L \in \mathscr{F}$. The remainder of the proof of (c) is easy. To prove (d), let p be a remote point of βX. By (c), $p \in \mathrm{cl}_{\beta X} Y_L$ for some $L \in \mathscr{L}$. If A is a closed, nowhere dense subset of Y_L, then A is a closed, nowhere dense subset of X. So, $p \notin \mathrm{cl}_{\beta X} A$ which implies that $p \notin \mathrm{cl}_{\beta Y_L} A$ as $\beta Y_L = \mathrm{cl}_{\beta X} Y_L$ by (a). Hence, p is a remote point of βY_L. Conversely, suppose p is a remote point of βY_L ($= \mathrm{cl}_{\beta X} Y_L$) for some $L \in \mathscr{L}$, and let A be a closed, nowhere dense subset of X. Then $B = A \cap Y_L$ is a closed, nowhere dense subset of Y_L. Since $\mathrm{cl}_{\beta X} Y_L$ is a neighborhood of p in βX, then $p \notin \mathrm{cl}_{\beta Y_L} B$ iff $p \notin \mathrm{cl}_{\beta X} B$ iff $p \notin \mathrm{cl}_{\beta X} A$. So, p is a remote point of βX.

By 1.6, βY_L has a remote point, say p_L, for each $L \in \mathscr{L}$. Let $Z = \beta X \setminus \{p_L: L \in \mathscr{L}\}$. Since $X \subseteq Z \subseteq \beta X$, then $\beta Z = \beta X$ and $|\beta Z \setminus Z| = \lambda$. By (b), $\beta Z \setminus Z$ is a discrete subset of βZ. By 1.6, each point of $\beta Z \setminus Z$ is extremally disconnected in βZ; hence, by 1.5, every closed, regularly nowhere dense subset of Z is compact. Clearly, $\{\mathrm{cl}_{\beta X}(X \setminus \bigcup\{Y_L: L \in \mathscr{F}\}): \mathscr{F}$ is a finite subset of $\mathscr{L}\}$ is a clopen neighborhood base of ∞ in $\beta X = \beta Z$. But, for each finite subset $\mathscr{F}$ of $\mathscr{L}$,

$$\mathrm{cl}_{\beta X}(X \setminus \bigcup\{Y_L: L \in \mathscr{F}\}) \supseteq \{p_L: L \in \mathscr{L} \setminus \mathscr{F}\};$$

this shows that $(\beta Z \setminus Z) \cup \{\infty\} = \mathrm{cl}_{\beta Z}(\beta Z \setminus Z)$ is the one-point compactification of $\beta Z \setminus Z$.

So, Z is a Tychonoff space satisfying (a), (b) and (c) of 2.1; hence, $\mathscr{M}(Z) = \mathscr{K}(Z)$ and $|\beta Z \setminus Z| = \lambda$. □

Let $\mathbf{Q}$ denote the space of rational numbers. Another example of a Tychonoff space X with the properties that $\mathscr{M}(X) = \mathscr{K}(X)$ and $|\beta X \setminus X| = \aleph_0$ can be obtained by letting $X = \beta\mathbf{Q} \setminus \{d_n: n \in \mathbf{N}\}$ where $\{d_n: n \in \mathbf{N}\}$ is a sequence of remote points of $\beta\mathbf{Q}$ converging to some point of

$\mathbf{Q}$. That there is a sequence of remote points of $\beta\mathbf{Q}$ that converge to a point of $\mathbf{Q}$ follows from the result in **[vD]** that the set of remote points of $\beta\mathbf{Q}$ is dense in $\beta\mathbf{Q} \setminus \mathbf{Q}$ and, thus in $\beta\mathbf{Q}$.

We are indebted to J. Vermeer for this different example. Let λ be an infinite cardinal, $Y = \oplus\{\mathbf{N}_\alpha\colon \alpha < \lambda\}$ where $\mathbf{N}_\alpha$ is a copy of $\mathbf{N}$, and $X = Y \cup \{\infty\}$. A subset $U \subseteq X$ is defined to be open if $U \cap Y$ is open in Y and if $\infty \in U$, there is a finite subset $F \subset \lambda$ such that $\mathbf{N}_\alpha \subseteq U$ for $\alpha \in \lambda \setminus F$. Let $p_\alpha \in \beta\mathbf{N}_\alpha \setminus \mathbf{N}_\alpha$ and $Z = \beta X \setminus \{p_\alpha\colon \alpha < \lambda\}$. Using the above technique, it follows that $\mathscr{M}(Z) = \mathscr{K}(Z)$ and $|\beta Z \setminus Z| = \lambda$. Another interesting example pointed out by J. Roitman is to let $\mathscr{R}$ be a maximal almost disjoint family of infinite subsets of $\mathbf{N}$ and $X = \mathbf{N} \cup \{\infty\}$ where $U \subseteq X$ is defined to be open if $\infty \in U$ implies there is a finite subset $\mathscr{F} \subseteq \mathscr{R}$ such that $R \subseteq U$ for $R \in \mathscr{R} \setminus \mathscr{F}$. For each $R \in \mathscr{R}$, let $p_R \in \mathrm{cl}_{\beta X} R \setminus R$ $(= \beta R \setminus R)$, and $Z = \beta X \setminus \{p_R\colon R \in \mathscr{R}\}$. Then $\mathscr{M}(Z) = \mathscr{K}(Z)$, $|\beta Z \setminus Z| = |\mathscr{R}|$, and $Z \setminus \{\infty\}$ is not the topological sum of $\{\beta R \setminus \{p_R\}\colon R \in \mathscr{R}\}$.

2.4. Remark. Property 2.1(a) is an internal property of a Tychonoff space X and 2.1(c) translates into this internal property: either there exists $n \in \omega$ such that given any collection of $n + 1$ pairwise disjoint zero-sets of X, at least one is compact, or else X is locally compact at all but one point. To obtain an internal condition on X that is equivalent to 2.1(b) is more involved, and it seems difficult to formulate a simple condition that does not involve mention of z-filters. However, it is possible to formulate an involved internal condition as follows. [The reader is referred to **[GJ]** or **[W]** for relevant background information about Stone-Čech compactifications.]

(2.5) Proposition. *Let λ be an infinite cardinal and let X be a Tychonoff space. The following are equivalent*:

(1) *$\beta X \setminus X$ is a discrete space of cardinality λ and*

(2) *there are families $\{Z_i\colon i < \lambda\}$ and $\{H_i\colon i < \lambda\}$ of zero-sets of X with the following properties*:

(a) *for each $i < \lambda$, Z_i is not compact, but if A and B are disjoint zero-sets of X contained in Z_i, then at least one of A or B is compact,*

(b) *for each $i < \lambda$, $Z_i \cap H_i = \varnothing$ and if $S \in Z(X)$ and $S \cap (Z_i \cup H_i) = \varnothing$, then S is compact,*

(c) *if $i < j < \lambda$, then $Z_i \cap Z_j$ is compact, and*

(d) *if $\mathscr{F}$ is a family of noncompact zero-sets of X and if $F \cap G$ is compact whenever F and G are distinct members of $\mathscr{F}$, then $|\mathscr{F}| \le \lambda$.*

Sketch of proof. To show (1) implies (2), let $\beta X \setminus X = \{d_i \colon i < \lambda\}$. For each $i < \lambda$, find $S_i \in Z(\beta X)$ and $T_i \in Z(\beta X)$ such that $d_i \in \operatorname{int}_{\beta X} S_i$, $(\beta X \setminus X) \setminus \{d_i\} \subseteq \operatorname{int}_{\beta X} T_i$, and $S_i \cap T_i = \varnothing$. Let $Z_i = S_i \cap X$ and $H_i = T_i \cap X$. Evidently, $\mathrm{cl}_{\beta X} Z_i \setminus Z_i = \{d_i\}$ and (a) follows from this. As $\beta X \setminus X \subseteq \operatorname{int}_{\beta X} S_i \cup \operatorname{int}_{\beta X} T_i$, (b) follows readily, and (c) follows from (a) and the fact that $d_i \neq d_j$ if $i \neq j$. If $F, G \in Z(X)$, $p_F \in \mathrm{cl}_{\beta X} F \setminus X$, $p_G \in \mathrm{cl}_{\beta X} G \setminus X$, and $F \cap G$ is compact, then $p_F \neq p_G$; hence, (d) follows from the fact that $|\beta X \setminus X| \leq \lambda$. Conversely, to show (2) implies (1), let $\{Z_i \colon i < \lambda\}$ and $\{H_i \colon i < \lambda\}$ be families of zero-sets of X satisfying (a)–(d). It follows from 2(a) that $|\mathrm{cl}_{\beta X} Z_i \setminus X| = 1$ for $i < \lambda$. Let $\{d_i\} = \mathrm{cl}_{\beta X} Z_i \setminus X$. By 2(b) $\{d_i\} = (\beta X \setminus X) \setminus \mathrm{cl}_{\beta X} H_i$, which shows that $\beta X \setminus X$ is discrete. If $i \neq j$, then $d_i \neq d_j$ by (c), and so $|\beta X \setminus X| \geq \lambda$. It follows in a similar way from (d) (and the fact that $\beta X \setminus X$ is discrete) that $|\beta X \setminus X| \leq \lambda$. □

A space X is *Urysohn* if each pair of points are contained in disjoint closed neighborhoods.

(2.6) THEOREM. *Let X be a space. Then $\mathscr{M}(X_s) = \mathscr{K}(X_s)$ iff every H-closed extension of X is Urysohn.*

Proof. The proof follows from these two facts: (i) a space Y is compact iff X is H-closed, semiregular, and Urysohn and (ii) a space Y is Urysohn iff Y_s is Urysohn. The first fact is from [**K**$_1$] and the second fact is straightforward to prove. □

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