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W^*-CATEGORIES

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A W^*-category is the categorical counterpart of a von Neumann algebra with an abstract definition equivalent to a concrete definition in terms of operators between Hilbert spaces. We develop the elementary theory of W^*-categories including modular theory and the comparison theory of objects (equivalence and quasiequivalence). We also characterize certain W^*-categories in terms of the W^*-category of projections in a von Neumann algebra, self-dual Hermitian modules for a von Neumann algebra or normal representations of a von Neumann algebra. This leads naturally to a discussion of the Morita equivalence of von Neumann algebras and of W^*-categories.

Introduction. A W^*-category is the natural generalization of a von Neumann algebra where, instead of taking the bounded linear mappings of a fixed Hilbert space as a model, we take the bounded linear mappings between a collection of Hilbert spaces. It is remarkable how easily most of the elementary results on von Neumann algebras generalize to W^*-categories. Consequently with little effort one can dispose of a relatively large body of results on W^*-categories. There are at present many interesting directions of current research where W^*-categories arise naturally: For example the representation theory of groupoids **[5]**, the harmonic analysis of the action of non-Abelian groups on von Neumann algebras **[1]**, **[9]**, **[12]**, **[21]**, **[24]** the action of group duals on von Neumann algebras **[14]**, **[22]**, and non-Abelian cohomology in an operator algebraic context **[6]**, **[23]**, **[25]**, **[28]**.

We feel that a systematic presentation of the basic theory of W^*-categories is already overdue.

Naturally the idea of using bounded linear mappings between different Hilbert spaces is such an obvious one that this paper may have many published and unpublished forerunners quite unknown to the authors. Indeed one of us (J. E. R.) has been toying with the idea of writing such a paper for many years but initially felt that the time was not yet ripe for such a development. In any case the roots of this development go right back to the beginnings of the theory of operator algebras and perhaps the basic example of mappings between different Hilbert spaces are the intertwining operators of representation theory. The set of such intertwining operators forms a W^*-category and has been studied from this point of view by Rieffel [20]. Many of his results are in fact of a general nature.

Another source of examples and hence of motivation is comparison theory. To describe the comparison of projections in a von Neumann algebra M, one takes the set of projections of M as the objects of a W^*-category $\mathscr{P}(M)$ and, for each pair of projections $e, f \in M$, the set of $t \in M$ such that $t = te = ft$ as the set of arrows from e to f in $\mathscr{P}(M)$. The comparison theory of weights of Connes and Takesaki [6] follows the same pattern and the W^*-category of weights on M, $\mathscr{W}(M)$ is discussed in §8.

The natural starting point for a systematic discussion of W^*-categories is the elementary theory of C^*-categories. Without prejudice as to the eventual importance of C^*-categories in their own right we confine ourselves to a bare minimum of results. With the aid of a G. N. S.-construction we show that every C^*-category has a concrete realization and that a suitably defined category of functors between two C^*-categories is again a C^*-category.

Specializing to the case of W^*-categories, we first prove the analogue of Sakai's result for von Neumann algebras namely that if W^*-categories are defined as C^*-categories admitting a predual, this predual is unique and each W^*-category has a concrete realization as a weakly closed $*$-subcategory of a suitable category of Hilbert spaces.

At this stage, it is already clear that there will be little difficulty in generalizaing results from the theory of von Neumann algebras. Modular theory, treated in §3, even benefits from the added generality as the Connes 2×2-matrix argument is typical of W^*-category techniques. Readers quite unfamiliar with category theory may find the notions of commutant and centre, given in §4, strange at first sight but should feel more at home with the comparison theory of §6 and become reconciled to the definition of centre at the latest in the treatment of quasiequivalence in §7.

The smooth way in which the basic theory develops indicates that a W^*-category is never very far from being a von Neumann algebra. In fact it can always be thought of as pieces of some large von Neumann algebra. We prove two results characterizing certain W^*-categories up to equivalence. A W^*-category with a maximal object and sufficient subobjects is equivalent to some $\mathscr{P}(M)$ (see Prop. 6. 4). A σ-finite W^*-category with sufficient subobjects and countable direct sums is equivalent to the full subcategory of some $\mathscr{P}(M)$ whose objects are the σ-finite projections of M (Thm. 7.14).

A rather different approach to classifying W^*-categories makes contact with the work of Rieffel **[20]** and emerges from studying the quasiequivalence of objects in a W^*-category. Two objects in a W^*-category are said to be quasiequivalent if they have the same central supports. A generator in the sense of category theory coincides with an object of central support 1. We then prove that any W^*-category with a generator, direct sums and sufficient subobjects is equivalent to a W^*-category of Hermitian self-dual modules (Prop. 7.6). Knowing a generator of a W^*-category $\mathfrak{A}$ essentially determines the $*$-functors from $\mathfrak{A}$ (see Corollary 7.7 and Theorem 7.13). In particular it determines the W^*-category up to Morita equivalence (Corollary 7.8). Two W^*-categories are Morita equivalent if and only if they have faithful representations whose commutants are isomorphic as von Neumann algebras (Prop. 7.9). In fact any W^*-category can be regarded as a full subcategory of the category of representations of the commutant of some faithful representation.

The aim of this paper is to provide a basic stock of results on W^*-categories so that they can, in future, be used freely to simplify arguments and clarify concepts. We have deliberately omitted certain topics such as the standard representation of a W^*-category or the type of a W^*-category as being superfluous to this aim. We have also limited ourselves to a single example, the W^*-category of weights, in our attempt to illustrate the virtues of a systematic use of W^*-categories.

In preparing this paper for publication we have made minor changes in the 1978 preprint version and, in particular, have added a number of more recent references. Our opinion that the really interesting directions for research into W^*-categories involve additional structure can meanwhile be underlined by certain concrete achievements. First, Woronowicz **[29]** has been able to characterize the W^*-categories of representations of a C^*-algebra in a fixed Hilbert space using a topology on the set of objects. These topological W^*-categories determine the C^*-algebras up to isomorphism and thus provide a duality theory for C^*-algebras. Secondly, a tensor product structure brings us to the monoidal W^*-categories introduced in **[22]**. The motivating examples here are the monoidal W^*-category Rep G of continuous unitary representations of a locally compact group G and the monoidal W^*-category whose objects are the endomorphisms of a von Neumann algebra. An interesting first step towards understanding monoidal W^*-categories is to strengthen the classical Tannaka-Krein duality theory by characterizing Rep G for G compact as an abstract monoidal W^*-category as it was done recently by Doplicher and

Roberts. One is also led to consider W^*-categories carrying an action of a monoidal W^*-category. For example, the harmonic analysis of the action of a non-Abelian group on a von Neumann algebra leads to W^*-categories carrying an action of Rep G. This structure can be used to study the Connes invariant Γ [**10**].

1. C^*-categories. This section treats the elementary properties of C^*-categories in so far as they are relevant to the study of W^*-categories.

Let $\mathfrak{A}$ be a category whose objects are denoted by $A, B, \ldots$.

Let (A, B) denote the set of arrows (morphisms) from A to B:

1.1. DEFINITION. $\mathfrak{A}$ is called a complex $*$-category if:

A1 Each (A, B) is a complex vector space and the composition of arrows is bilinear.

A2 There is an involutive antilinear contravariant endofunctor $*$ of $\mathfrak{A}$ which preserves objects. The image of x under $*$ will be denoted by x^*. It follows that each (A, A) is a $*$-algebra with identity.

A3 For each $x \in (A, B)$, x^*x is a positive element of the $*$-algebra (A, A), i.e. $x^*x = y^*y$ for some $y \in (A, A)$. Furthermore $x^*x = 0$ implies $x = 0$.

It follows that the mapping $(x, y) \to x^*y$ from $(A, B) \times (A, B) \to (A, A)$ is a (A, A)-valued inner product on the right (A, A)-module (A, B) where (A, A) acts on (A, B) by composition of arrows ([**15**], [**20**]).

A $*$-category $\mathfrak{A}$ is called a normed $*$-category if:

A4 Each (A, B) is a normed space and $\|xy\| \leq \|x\| \, \|y\|$,

A normed $*$-category $\mathfrak{A}$ is called a Banach $*$-category[1] if:

A5 Each (A, B) is a Banach space.

A Banach $*$-category $\mathfrak{A}$ is called a C^*-category if:

A6 For each arrow x of $\mathfrak{A}$, $\|x\|^2 = \|x^*x\|$.

It follows that each (A, A) is a C^*-algebra with identity. A6 shows that the norm on a C^*-category is uniquely determined by the norms on the C^*-algebra (A, A). In fact we can say more: let $\mathfrak{A}$ be a $*$-category where each (A, A) is a C^*-algebra, then $\mathfrak{A}$ can be made into a normed $*$-category satisfying A6 (but not A5 in general) in a unique way by setting $\|x\| = \|x^*x\|^{1/2}$. To see that this does define a norm we may, for example, regard (A, B) as right (A, A)-pre-Hilbert module and argue as in [**15**, Proposition 2.3]. Thus we only need to prove that $\|xy\| \leq \|x\| \, \|y\|$

[1] An example of a Banach $*$-category is the category $L^1(G)$ associated with a groupoid G defined by Connes in [**5**, IV]

because $\|x^*\| = \|x\|$ is then a direct consequence of $\|x\|^2 = \|x^*x\|$. If $y \in (A, B)$, $b \to y^*by$ is a positive linear map of (B, B) into (A, A). As a map of C^*-algebras, this is bounded with norm $\|y^*\|$. If $x \in (B, C)$ and $y \in (A, B)$ we deduce

$$\|xy\|^2 = \|y^*x^*xy\| \leq \|y^*y\| \, \|x^*x\| = \|y\|^2\|x\|^2.$$

Of course any C^*-algebra with identity can be considered as a C^*-category with a single object.

1.2. EXAMPLE. Let $\mathfrak{H}$ denote the category with Hilbert spaces[2] as objects and all bounded linear mappings as arrows. Then $\mathfrak{H}$, with the usual definitions of $*$ and $\| \; \|$ is a C^*-category.

1.3. EXAMPLE. Let A be a C^*-algebra and Rep(A) the category whose objects are the non-degenerate representations of A on Hilbert spaces and whose arrows are the intertwining operators between these representations. Then Rep(A) is a C^*-category.

1.4. EXAMPLE. Let A be a C^*-algebra. A right Hermitian A-module X is a Banach right A-module with an A-valued inner product $\langle \, , \rangle$, conjugate linear in the first variable and linear in the second, such that, for all $x, y \in X$ and $a \in A$,

(1) $\langle x, x \rangle \geq 0$

(2) $\langle x, y \rangle^* = \langle y, x \rangle$

(3) $\langle x, ya \rangle = \langle x, y \rangle a$

(4) $\|x\|^2 = \|\langle x, x \rangle\|$.

Such modules have been treated by Paschke [15], Rieffel [20] and Kasparov [11].

We consider these modules as the objects of a category h^* mod A. An arrow T from X to Y in h^* mod A is a linear map from X to Y such that

(5) $$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| < +\infty, \qquad x \in X.$$

(6) $$T(x \cdot a) = (Tx) \cdot a \qquad x \in X \quad \text{and} \quad a \in A.$$

(7) there is a map $T^*\colon Y \to X$ such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \qquad x \in X \quad \text{and} \quad y \in Y.$$

[2] Strictly speaking we mean here all Hilbert spaces in some universe as we prefer this way of avoiding Russell's paradox. The universe will not be specified even when more than one universe is needed as the gain in precision would be outweighted by the notational complexity. The reader unfamiliar with these concepts might consult [2] [13].

$h^*\operatorname{mod} A$ is a C^*-category. A module X in $h^*\operatorname{mod} A$ is said to be self-dual if every bounded A-module homomorphism from X to A has an adjoint.

Here we consider A itself as an object of $h^*\operatorname{mod} A$ under right multiplication and with the inner product defined by $\langle a, b\rangle = a^*b$. It then follows that every bounded A-module homomorphism from X to an Hermitian A-module has an adjoint [**15**; Prop. 3.4]. The full subcategory of self-dual modules is again a C^*-category, denoted $H\operatorname{mod}(A)$.

Corresponding to the concept of a morphism of $*$-algebras, we have

1.5. Definition. Let $\mathfrak{A}$, $\mathfrak{B}$ be $*$-categories, a $*$-functor $F\colon \mathfrak{A} \to \mathfrak{B}$ is a linear functor such that $F(a^*) = F(a)^*$, $a \in \mathfrak{A}$.

$*$-functors of C^*-categories like morphisms of C^*-algebras are norm decreasing $\|F(a)\| \le \|a\|$, $a \in \mathfrak{A}$.

1.6. Definition. Let $\mathfrak{A}$ be a $*$-category and $\mathfrak{J} \subset \mathfrak{A}$. Let $(A, B)_{\mathfrak{J}} = (A, B) \cap \mathfrak{J}$. Then $\mathfrak{J}$ is called a left ideal if $(A, B)_{\mathfrak{J}}$ is a linear subspace of (A, B) and $a \in (A, B)_{\mathfrak{J}}$, $b \in (B, C)$ imply $ba \in (A, C)_{\mathfrak{J}}$. A right ideal is defined similarly. $\mathfrak{J}$ is a two-sided ideal if it is both a left and right ideal.

$\mathfrak{J} \subset \mathfrak{A}$ determines an equivalence relation on the arrows of $\mathfrak{A}$: $x \sim y$, if $x - y \in \mathfrak{J}$. If $\mathfrak{J} = \mathfrak{J}^*$ is an ideal of $\mathfrak{A}$, the set of equivalence classes $\mathfrak{A}/\mathfrak{J}$ can be made into a $*$-category in a unique way by requiring the canonical map $x \to \hat{x}$ of $\mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ to be a $*$-functor. $\mathfrak{A}/\mathfrak{J}$ has the same objects as $\mathfrak{A}$. If $\mathfrak{A}$ is a normed $*$-category, $\mathfrak{A}/\mathfrak{J}$ can be made into a normed $*$-category, by defining

$$\|\hat{x}\| = \inf_{y \in \hat{x}} \|y\|$$

Arguing as for C^*-algebras, see e.g. [**7**: 1.8.2], one can show

1.7. Proposition. *Let $\mathfrak{A}$ be a C^*-category and $\mathfrak{J}$ a closed, two-sided ideal of $\mathfrak{A}$. Then $\mathfrak{J}^* = \mathfrak{J}$ and $\mathfrak{A}/\mathfrak{J}$ is a C^*-category.*

We next introduce representations of $*$-categories and establish an analogue of the usual Gelfand-Naimark-Segal construction.

1.8. Definition. A representation of a $*$-category $\mathfrak{A}$ is a $*$-functor F: $\mathfrak{A} \to \mathfrak{H}$. If A is an object of $\mathfrak{A}$, we say that $\xi \in F(A)$ is cyclic for F if $\{F(a)\xi\colon a \in (A, B)\}$ is dense in $F(B)$ for each object B of $\mathfrak{A}$.

1.9. Proposition. *Let $\mathfrak{A}$ be a C^*-category, A an object of $\mathfrak{A}$ and ϕ a positive linear form on the C^*-algebra (A, A). There is a representation F_ϕ of*

$\mathfrak{A}$ *with cyclic vector* $\xi_\phi \in F_\phi(A)$ *such that*

$$\phi(a) = \langle \xi_\phi, F_\phi(a)\xi_\phi \rangle, \qquad a \in (A, A).$$

If F is another representation of $\mathfrak{A}$ with cyclic vector $\xi \in F(A)$ such that $\phi(a) = \langle \xi, F(a)\xi \rangle$, $a \in (A, A)$, there is a unique natural (unitary) equivalence u: $F_\phi \to F$ with $u_A \xi_\phi = \xi$.

Proof. We define a semi-definite scalar product on (A, B) by

$$\langle a, b \rangle = \phi(a^*b).$$

Let $F_\phi(B)$ be the associated Hilbert space and $a \to \hat{a}$ the canonical map of (A, B) into $F_\phi(B)$. We now have:

$$\|\widehat{ba}\|^2 = \phi(a^*b^*ba) \leq \phi(a^*a)\|b^*b\|$$

$$= \|\hat{a}\|^2\|b^*b\|, \qquad a \in (A, B), b \in (B, C).$$

Thus $\hat{a} \to \widehat{ba}$ extends to a unique bounded linear operator $F_\phi(b)$: $F_\phi(B) \to F_\phi(C)$. It is easy to see that F_ϕ is a representation of $\mathfrak{A}$ with cyclic vector $\xi_\phi = \hat{1}_A$ and $\phi(a) = \langle \xi_\phi, F_\phi(a)\xi_\phi \rangle$. To complete the proof note that $\langle \xi, F(a)\xi \rangle = \langle \xi_\phi, F_\phi(a)\xi_\phi \rangle$ implies

$$\langle F_\phi(a)\xi, F_\phi(b)\xi \rangle = \langle F(a)\xi, F(b)\xi \rangle, \qquad a, b \in (A, B)$$

Thus there is a unitary u_B: $F_\phi(B) \to F(B)$.
with $u_B F_\phi(a)\xi_\phi = F(a)\xi$, $a \in (A, B)$.

u: $F_\phi \to F$ is the desired natural equivalence of functors.

If $\mathfrak{A}$ and $\mathfrak{B}$ are categories, the functor category $[\mathfrak{A}, \mathfrak{B}]$ has as objects the functors from $\mathfrak{A}$ to $\mathfrak{B}$ and as arrows the natural transformations between them. For C^*-categories we modify this procedure:

1.10. Definition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-categories and F and G $*$-functors from $\mathfrak{A}$ to $\mathfrak{B}$. Let t: $F \to G$ be a natural transformation. We set $\|t\| = \sup_A \|t_A\|$ where A runs over the objects of $\mathfrak{A}$ and say that t is bounded if $\|t\| < \infty$.

Note that t^*, defined by $(t^*)_A = (t_A)^*$, is a natural transformation t^*: $G \to F$ and is bounded wherever t is bounded. Furthermore, one can easily prove

1.11. Proposition. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* C^**-categories. The category* $(\mathfrak{A}, \mathfrak{B})$ *whose objects are the* $*$*-functors from* $\mathfrak{A}$ *to* $\mathfrak{B}$ *and whose arrows are the bounded natural transformations between them is a* C^**-category.*

1.12. EXAMPLE. If $\mathfrak{A}$ is a C^*-category, so is $(\mathfrak{A}, \mathfrak{H})$ its category of representations. As a special case we get the C^*-category Rep(A) of Example 1.3.

1.13. REMARK. Let A be a C^*-algebra and H: Rep$(A) \to \mathfrak{H}$ the evaluation $*$-functor, i.e. $H(\pi)$ is the Hilbert space of the representation π of A and if $t \in (\pi, \pi')$, $H(t)$: $H(\pi) \to H(\pi')$ is the concrete intertwining operator. Then (H, H) the set of bounded natural transformation from H to H is isomorphic to the universal enveloping algebra of A, see [**20**; Cor. 2.7].

1.14. PROPOSITION. *Every C^*-category $\mathfrak{A}$ may be realized as a concrete C^*-category, i.e. there is a faithful embedding functor F: $\mathfrak{A} \to \mathfrak{H}$.*

Proof. Let $F = \bigoplus_\phi F_\phi$ where ϕ runs over all positive linear functionals of all objects of $\mathfrak{A}$. F is a faithful functor since $F(a) = 0$ for $a \in (A, B)$ implies that $\phi(a^*a) = 0$ for all positive linear functionals on (A, A) and hence that $a = 0$.

The following simple construction will prove to be useful in the sequel. Let $A_1, A_2, \ldots, A_n$ be objects of a C^*-category $\mathfrak{A}$. Let $M(A_1, A_2, \ldots, A_n)$ denote the $*$-algebra of matrices $x = (x_{ij})$ with $x_{ij} \in (A_j, A_i)$, $i, j = 1, 2, \ldots, n$, where the algebraic operations are defined from those of $\mathfrak{A}$ by the usual rules of matrix algebra. If F is a faithful representation of $\mathfrak{A}$, $M(A_1, A_2, \ldots, A_n)$ may be regarded in the obvious way as an operator algebra on $\bigoplus_{i=1}^n F(A_i)$. If $x = y \otimes e_{ij}$ with $y \in (A_j, A_i)$, the operator norm of x coincides with $\|y\|$. Thus the operator norm on $M(A_1, A_2, \ldots, A_n)$ induces the product topology on $\times_{i,j}(A_j, A_i)$ and makes $M(A_1, A_2, \ldots, A_n)$ into a C^*-algebra.

If I is a finite set indexing objects of $\mathfrak{A}$ and $J \subset I$ then we have a morphism $f = f(J \subset I)$ of the corresponding matrix C^*-algebras defined by

$$f(x)_{ij} = x_{ij} \quad \text{if } i, j \in J$$
$$f(x)_{ij} = 0 \quad \text{otherwise.}$$

Thus as $A_1, A_2, \ldots, A_n$ vary we obtain an inductive system of matrix C^*-algebras $M(A_1, A_2, \ldots, A_n)$.

2. W^*-categories. In this section we define W^*-categories and derive a few elementary results about them.

2.1. DEFINITION. A W^*-category $\mathfrak{A}$ is a C^*-category where each Banach space (A, B) is the dual of a Banach space $(A, B)_*$. We say that $(A, B)_*$ is a predual for (A, B).

It follows from a result of Sakai [**27**; 1.13] that each (A, A) is a von Neumann algebra and that $(A, A)_*$ is unique as a subspace of the dual of (A, A). As we shall see Sakai's result extends to W^*-categories.

2.2. EXAMPLE. The category $\mathfrak{H}$ of Hilbert spaces is a W^*-category because, if H_1, H_2 are objects of $\mathfrak{H}$, the Banach space of bounded linear forms ϕ_t on (H_1, H_2) defined by $\phi_t(a) = \mathrm{Tr}(ta)$ for each $t \in (H_2, H_1)$ with $\mathrm{Tr}(|t|) < +\infty$ is a predual for (H_1, H_2).

2.3. EXAMPLE. Let $\mathfrak{A}$ be a $*$-subcategory of $\mathfrak{H}$ such that each $(H_1, H_2)_{\mathfrak{A}}$ is closed in $(H_1, H_2)_{\mathfrak{H}}$ in the weak operator topology. It is easy to verify that $(H_1, H_2)_{\mathfrak{A}}$ is closed in $(H_1, H_2)_{\mathfrak{H}}$ for the weak $*$-topology on (H_1, H_2) viewed as the dual of the Banach space of trace-class operators from H_2 to H_1. The quotient of this predual of $(H_1, H_2)_{\mathfrak{H}}$ by the polar of $(H_1, H_2)_{\mathfrak{A}}$ is a predual for $(H_1, H_2)_{\mathfrak{A}}$. Thus $\mathfrak{A}$ is a W^*-subcategory of $\mathfrak{H}$, i.e. a concrete W^*-category.

2.4. EXAMPLE. Let A be a C^*-algebra then $\mathrm{Rep}(A)$ is a W^*-category as it satisfies the assumptions of Example 2.3.

2.5. EXAMPLE. Let $\mathfrak{A}$ be a C^*-category. If $t\colon F \to G$ is an arrow of $(\mathfrak{A}, \mathfrak{H})$ and A an object of $\mathfrak{A}$, then $t \to t_A$ is a $*$-functor from $(\mathfrak{A}, \mathfrak{H})$ to $\mathfrak{H}$. Setting $\Phi(t) = \oplus_A t_A$, we get a faithful $*$-functor Φ. Clearly $\Phi(F, G)$ is a weakly closed subspace of $(\Phi(F), \Phi(G)) = (\oplus_A F(A), \oplus_A G(A))$ thus $(\mathfrak{A}, \mathfrak{H})$ is a W^*-category.

The key lemma enabling us to extend results on von Neumann algebras to results on W^*-categories uses the 2×2-matrix algebras $M(A_1, A_2)$ defined at the end of §1.

2.6. LEMMA. *A C^*-category $\mathfrak{A}$ is a W^*-category if and only if $M(A_1, A_2)$ is a von Neumann algebra for each pair of objects A_1, A_2 of $\mathfrak{A}$.*

Proof. The C^*-algebra $M(A_1, A_2)$ may be identified with the Banach space direct sum $\oplus_{i,j=1}^2 (A_j, A_i)$ up to an equivalence of norms. Hence $M(A_1, A_2)$ has a predual if and only if (A_j, A_i) has a predual for $i, j = 1, 2$.

2.7. COROLLARY (*Polar decomposition of arrows.*) *Let $\mathfrak{A}$ be a W^*-category, every $a \in (A, B)$ can be expressed uniquely in the form $a = u|a|$, where u is a partial isometry of (A, B) with $u^*u = s(|a|)$, the support of $|a|$, and $uu^* = s(|a^*|)$.*

Proof. (A, B) is a direct summand of $M(A, B)$ and it suffices to use the existence and uniqueness of the polar decomposition of a as an element of $M(A, B)$ (see e.g. [**27**; Thm. 1.12.1])

2.8. COROLLARY. (*Polar decomposition of linear functionals.*) *If* $\mathfrak{A}$ *is a* W^**-category, every element* $f \in (A, B)_*$ *may be expressed uniquely in the form*

$$f(a) = \phi(ua), \qquad a \in (A, B).$$

where $\phi \in (A, A)_*$ *is positive with* $\|\phi\| = \|f\|$ *and* $u \in (B, A)$ *is a partial isometry with final projection* $s(\phi)$, *the support of* ϕ.

Proof. Comparing f with the projection mapping $M(A, B) \to (A, B)$, we get a canonical extension of f to an element $\tilde{f}$ of the predual of $M(A, B)$. It suffices to compute the polar decomposition of $\tilde{f}$, $\tilde{f}(x) = \tilde{\phi}(u, x)$, $x \in M(A, B)$ and verify that $\tilde{\phi}$ is the canonical extension of a $\phi \in (A, A)_*$ and $u \in (B, A) \subset M(A, B)$ (see [**8**; Ch. 1, §4 Thm. 4], [**27**; Thm. 1.14.4]).

Combining Corollary 2.8 and Sakai's result for von Neumann algebras, we see that $(A, B)_*$ is unique as a subspace of $(A, B)^*$.

Let $\mathfrak{A}$ be a W^*-category, then just as for a von Neumann algebra, there are various topologies we may put on the (A, B). Thus the σ-topology is the weak topology of the dual system $((A, B), (A, B)_*)$ and the s-topology is generated by the seminorms $a \to \phi(a^*a)^{1/2}$ where ϕ runs through the positive elements of $(A, A)_*$. These topologies coincide with those induced on (A, B) by the corresponding topologies on $M(A, B)$.

Thus we have

2.9. COROLLARY. *The set of* σ*-continuous and* s*-continuous linear functionals on* (A, B) *coincide.*

The same reasoning applies to the Mackey topology and the s^*-topology (see [**27**; §1.3]) We may also consider the strong and weak operator topologies on the (A, B) induced by a representation F. These, too, coincide with those induced by the corresponding topologies on $M(A, B)$ provided $M(A, B)$ is represented in the obvious way on $F(A) \oplus F(B)$. In particular, we deduce

2.10. COROLLARY. *The Kaplansky Density Theorem holds for* W^**-categories.*

2.11. DEFINITION. Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a $*$-functor between W^*-categories, we say that F is *normal* if it induces a normal morphism of (A, A) into $(F(A), F(A))$ for each object A of $\mathfrak{A}$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are W^*-categories we will use the notation $(\mathfrak{A}, \mathfrak{B})$ to denote the category whose objects are the normal $*$-functors. This will not cause any confusion in what follows.

A consequence of Corollary 2.9 and the corresponding result for von Neumann algebras is the following [**20**; Prop. 4.7]

2.12. PROPOSITION. *Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a $*$-functor between W^*-categories then the following properties are equivalent*

(a) *F is a normal $*$-functor*

(b) *F is σ-continuous, i.e. the induced mapping of (A, A') into $(F(A), F(A'))$ is σ-continuous for each pair A, A' of objects of $\mathfrak{A}$.*

(c) *F is s-continuous.*

2.13. PROPOSITION. *Every W^*-category $\mathfrak{A}$ may be realized as a concrete W^*-category, i.e. there is a faithful normal embedding $*$-functor $F\colon \mathfrak{A} \to \mathfrak{H}$.*

Proof. If ϕ in Proposition 1.8 is in $(A, A)_*$, F_ϕ is a normal $*$-functor. Hence we may argue as in Proposition 1.13 using only normal positive linear functionals.

In the light of the proposition, we could define W^*-categories to be weakly closed $*$-subcategories of $\mathfrak{H}$. We sketch at this point an alternative characterization of W^*-categories. In a C^*-category $\mathfrak{A}$, each (A, B) is a right Hermitian (A, A)-module (see Example 1.4). In a W^*-category each (A, B) is even a self-dual module, in fact:

2.14. PROPOSITION. *Let A, B, C be objects of a W^*-category $\mathfrak{A}$ and consider (A, B) and (A, C) as right Hermitian (A, A)-modules. Then if T: $(A, B) \to (A, C)$ is a bounded (A, A)-module homomorphism, there is an $x \in (B, C)$ such that*

$$T(b) = xb, \qquad b \in (A, B).$$

This is proved as Lemma 2.1 of [**26**] drawing on the ideas of [**20**; Thm. 6.4]. We now have

2.15. PROPOSITION. *A C^*-category $\mathfrak{A}$ is a W^*-category if and only if each (A, A) is a von Neumann algebra and and (A, B) is self-dual as an (A, A)-module.*

Proof. The necessity is proved in Proposition 2.14; the sufficiency follows from [**15**; Prop. 3.8].

We have seen in this section how the matrix algebras $M(A, B)$ associated with a W^*-category $\mathfrak{A}$ provide an easy way of generalizing from von Neumann algebras to W^*-categories. For some purposes it is useful to replace $\mathfrak{A}$ by a single von Neumann algebra $M(\mathfrak{A})$. $M(\mathfrak{A})$ is defined as the W^*-inductive limit of the von Neumann algebras $M(A_1, A_2, \ldots, A_n)$ as $\{A_1, A_2, \ldots, A_n\}$ runs over the family of finite subsets of objects of $\mathfrak{A}$ ordered under inclusion. An arrow of $\mathfrak{A}$ can be regarded as an element of $M(\mathfrak{A})$, these elements generate $M(\mathfrak{A})$ as a von Neumann algebra. If e_A denotes the identity on A considered as an element of $M(\mathfrak{A})$ then $\{e_A\}$ is a partition of the identity in $M(\mathfrak{A})$. The W^*-category $\mathfrak{A}$ may be recovered from $M(\mathfrak{A})$ together with $\{e_A\}$ by defining

$$(A, B) = \{x \in M(\mathfrak{A}): x = xe_A = e_B x\}$$

and using the algebraic operations on $M(\mathfrak{A})$ to define the corresponding operations on $\mathfrak{A}$. If F is any normal representation of $\mathfrak{A}$, there is an associated normal representation $\tilde{F}$ of $M(\mathfrak{A})$ on $\oplus_A F(A)$. If F is faithful so is $\tilde{F}$.

As we shall see, many of the properties of $\mathfrak{A}$ are conveniently expressed in terms of $M(\mathfrak{A})$ so it is perhaps wise to point out two of the principal defects of trying to replace $\mathfrak{A}$ by $M(\mathfrak{A})$. $M(\mathfrak{A})$ is unsuited to discussing $*$-functors and is often too large: each (A, A) may be σ-finite without $M(\mathfrak{A})$ being σ-finite.

The relation between $M(\mathfrak{A})$ and $\mathfrak{A}$ will become clearer in §7.

3. Modular theory. A positive element of a W^*-category $\mathfrak{A}$ must be understood as a positive element of (A, A) for some object A of $\mathfrak{A}$. Thus if $\mathfrak{A}^+$ denotes the set of positive elements of $\mathfrak{A}$, $x \in \mathfrak{A}^+$ implies $\lambda x \in \mathfrak{A}^+$ for $\lambda \geq 0$ and $x, y \in \mathfrak{A}^+$ imply $x + y \in \mathfrak{A}^+$ whenever the sum is defined. Once this definition of $\mathfrak{A}^+$ has been accepted, the notion of weight and of modular automorphism proceed smoothly along the lines established for von Neumann algebras.

3.1. DEFINITION. A weight ϕ on $\mathfrak{A}$ is a mapping of $\mathfrak{A}^+$ into $[0, \infty]$ satisfying

$$\phi(x + y) = \phi(x) + \phi(y), \quad x, y \in \mathfrak{A}^+$$
$$\phi(\lambda x) = \lambda\phi(x), \quad x \in \mathfrak{A}^+, \lambda \geq 0$$

with the usual convention that $0.\infty = 0$.

Thus a weight on $\mathfrak{A}$ is just a field $A \to \phi_A$ where ϕ_A is a weight on (A, A) for each object A of $\mathfrak{A}$.

$$\mathfrak{M}_\phi^+ = \{x \in \mathfrak{A}^+ : \phi(x) < \infty\}$$

is an order ideal of $\mathfrak{A}^+$ and

$$\mathfrak{N}_\phi = \{x \in \mathfrak{A} : \phi(x^*x) < +\infty\}$$

is a left ideal of $\mathfrak{M}$ since the inequalities of [**17**; Lemma 1.1] are valid here. If $\mathfrak{M}_\phi$ denotes the linear span of elements of the form b^*a with $b, a \in \mathfrak{M}_\phi$ then $\mathfrak{M}_\phi^+ = \mathfrak{M}_\phi \cap \mathfrak{A}_+$ but $\mathfrak{M}_\phi$ is not, in general, the linear span of $\mathfrak{M}_\phi^+$.

3.2. LEMMA. *$\mathfrak{M}_\phi \cap (A, B)$ is the linear span of elements of the form b^*a with $a \in \mathfrak{M}_\phi \cap (A, A)$ and $b \in \mathfrak{M}_\phi \cap (B, A)$.*

Proof. Let $x \in \mathfrak{M}_\phi \cap (A, C)$ and $y \in m\mathrm{M}_\phi \cap (B, C)$. If $x = u|x|$ is the polar decomposition of x, then $ux = |x|$ and since $\mathfrak{M}_\phi$ is a left ideal, $|x| \in \mathfrak{M}_\phi \cap (A, A)$. Now $y^*x = y^*u|x| = (u^*y)^*|x|$ and $u^*y \in \mathfrak{M}_\phi \cap (B, A)$. This implies the lemma.

A weight ϕ on $\mathfrak{A}$ is called *faithful* if $\phi(x) = 0$ implies $x = 0$, *semifinite* if $\mathfrak{H}_\phi$ is σ-dense in $\mathfrak{A}$ and *normal* if there are weights ω_α with $\omega_{\alpha,A} \in (A, A)_*$ for each A and $\phi = \sup_\alpha \omega_\alpha \phi$ is faithful, semifinite or normal if and only if each ϕ_A is faithful, semifinite or normal.

Modular theory for W^*-categories can be deduced from modular theory for von Neumann algebras by passing from a weight ϕ on a W^*-category $\mathfrak{A}$ to weights $\phi_{a_1 A_2 \cdots A_n}$ on the associated inductive system of matrix von Neumann algebras $M(A_1, A_2, \ldots, A_n)$ where $A_1, A_2, \ldots, A_n$ are (not necessarily distinct) objects of $\mathfrak{A}$. This is of course a natural extension of the matrix techniques of Connes [**4**; 1.2.1]. We define

$$\phi_{a_1 A_2 \cdots A_n}(x) = \sum_{i=1}^{n} \phi_{A_i}(x_{ii}), \qquad x \in M(A_1, A_2, \ldots, A_n)^+.$$

The weights $\phi_{a_1 A_2 \cdots A_n}$ are compatible with the inductive system and if ϕ is a faithful, normal semifinite weight, so is each $\phi_{a_1 A_2 \cdots A_n}$.

3.3. LEMMA. *$x \in \mathfrak{N}_{\phi_{A_1 A_2 \cdots A_n}}$ or $\mathfrak{M}_{\phi_{A_1 A_2 \cdots A_n}}$ if and only if each matrix entry $x_{ij} \in \mathfrak{N}_\phi$ or $\mathfrak{M}_\phi$ respectively.*

Proof. The first result is an immediate consequence of

$$\phi_{a_1 A_2 \cdots A_n}(x^*x) = \sum_{i,j=1}^{n} \phi_{A_i}(x_{ji}^* x_{ji}).$$

It follows that $x \in \mathfrak{M}\phi_{a_1A_2\cdots A_n}$ implies that each matrix entry $x_{ij} \in \mathfrak{M}_\phi$. To prove the converse, it suffices to consider the case that only one matrix entry is non-zero. This is easily dealt with using Lemma 3.2.

Now any W^*-category $\mathfrak{A}$ has a faithful, normal semifinite weight ϕ and we shall associate an action σ^ϕ of $\mathbf{R}$ on $\mathfrak{A}$ satisfying the KMS condition. An action σ of $\mathbf{R}$ on $\mathfrak{A}$ assigns to $t \in \mathbf{R}$ an invertible $*$-functor σ_t of $\mathfrak{A}$ into $\mathfrak{A}$ such that

$$\sigma_t\sigma_t = \sigma_{t+t'}, \qquad t, t' \in \mathbf{R}.$$

We deal here with actions leaving the objects of $\mathfrak{A}$ invariant and such that $t \to \sigma_t(x)$ is σ-continuous for each arrow x of $\mathfrak{A}$. A weight ϕ on $\mathfrak{A}$ satisfies the KMS condition with respect to such an action σ of $\mathbf{R}$ if, given any $a, b \in (A, B) \cap \mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*$, there is a bounded continuous function F defined on $0 \leq \operatorname{Im}(z) \leq 1$ and holomorphic in $0 < \operatorname{Im}(z) < 1$ with

$$F(t) = \phi\big(\sigma_t(a)b^*\big), \quad F(t+i) = \phi\big(b^*\sigma_t(a)\big), \qquad t \in \mathbf{R}.$$

We now have

3.4. Proposition. *A faithful, normal semifinite weight ϕ on a W^*-category satisfies the KMS condition with respect to a unique action σ^ϕ of $\mathbf{R}$ on $\mathfrak{A}$.*

Proof. We use the analogous result for von Neumann algebras [**3**; Props. 4.4 and 4.8] and consider the modular actions $\sigma^{\phi_{A_1\cdots A_n}}$ on $M(A_1,\ldots,A_n)$. The strategy of the proof should be clear if one realizes that σ and $\sigma^{\phi_{A_1\cdots A_n}}$ are related by

$$(*) \qquad \sigma^{\phi_{A_1\cdots A_n}}(x)_{ij} = \sigma_t^\phi(x_{ij}), \qquad x \in M(A_1,\ldots,A_n).$$

In the proof, $(*)$ will be used to define σ^ϕ so what must be established is that the left-hand side of $(*)$ depends only on x_{ij}, i.e. neither on the other entries of x nor on the choice of A_k for $k \neq i, j$. Now an explicit computation using Lemma 3.2 shows that $1_{A_k} \otimes e_{kk}$ is in the centralizer of $\phi_{A_1\cdots A_n}$, $k = 1,2,\ldots,n$. Hence by [**18**; Thm. 3.6], these elements are invariant under the modular action. This implies first that the left-hand side of $(*)$ is independent of the other entries of x and secondly that $M(A_i, A_j)$ considered in the obvious way as a submatrix algebra of $M(A_1,\ldots,A_n)$ is globally invariant under the modular action. Since, furthermore, $\phi_{A_iA_j}$ is the restriction of $\phi_{A_1\cdots A_n}$ to $M(A_i, A_j)$, $\sigma^{\phi_{A_iA_j}}$ agrees with $\sigma^{\phi_{A_1\cdots A_n}}$ on $M(A_i, A_j)$. Thus σ_t^ϕ is well defined by $(*)$ and now that this has been established, the properties of σ^ϕ follow from the corresponding properties of $\sigma^{\phi_{A_1 \cdots A_n}}$. The uniqueness of σ^ϕ follows similarly with the aid of Lemma 3.3.

Just as in the case of a von Neumann algebra, the centralizer of ϕ coincides with the fixed points of σ^ϕ.

3.5. PROPOSITION. *Let ϕ be a faithful, normal semifinite weight on a W^*-category $\mathfrak{A}$ then the following two conditions on an arrow x of $\mathfrak{A}$ are equivalent.*

(a) $\sigma_t^\phi(x) = x,\ t \in \mathbf{R}$

(b) $x\mathfrak{M}_\phi \subset \mathfrak{M}_\phi$, $\mathfrak{M}_\phi x \subset \mathfrak{M}_\phi$ *and* $\phi(xy) = \phi(yx)$
for all $y \in \mathfrak{M}_\phi$ *such that* xy *and* yx *are defined and belong to* $\mathfrak{A}^+$.

Proof. Simple computations using $(*)$, Lemma 3.3 and the definition of the weights $\phi_{A_1 \cdots A_n}$ allow one to pass to the associated matrix algebras $M(A_1, A_2, \ldots, A_n)$ and apply [**18**; Thm. 3.6].

As we shall see when we discuss the W^*-category of weights on a von Neumann algebra, the existence and basic properties of Radon-Nikodým derivatives can be easily deduced from Proposition 3.4. Here we deduce it instead from the corresponding results for von Neumann algebras [**4**].

3.6. PROPOSITION. *If ϕ and ψ are two faithful, normal semifinite weights on a W^*-category $\mathfrak{A}$, there is a natural unitary transformation $(D\psi : D\phi)_t$ from σ_t^ϕ to σ_t^ψ defined by*

$$(D\psi : D\phi)_{t,A} = (D\psi_A : D\phi_A)_t.$$

Consequently, $(D\psi : D\phi)_{t,A}$ satisfies the 1-cocycle identity

$$(D\psi : D\phi)_{t_1+t_2,A} = (D\psi : D\phi)_{t_1,A}\, \sigma_{t_1}^\phi\big((D\psi : D\phi)_{t_2,A}\big).$$

Proof. Let A_1, A_2 be objects of $\mathfrak{A}$, then as we have seen in the proof of Proposition 3.4, $1_{A_1} \otimes e_{11}$ is invariant under $\sigma_t^{\phi_{A_1 A_2}}$ and $\sigma_t^{\psi_{A_1 A_2}}$. Thus by [**4**; Thm. 1.2.1] $(D\psi_{A_1 A_2} : D\phi_{A_1 A_2})_t$ commutes with $1_{A_1} \otimes e_{11}$ so that $(D\psi_{A_1 A_2} : D\phi_{A_1 A_2})_t$ must be the diagonal matrix

$$\sum_{i=1,2} \big(D\psi_{A_i} : D\phi_{A_i}\big)_t \otimes e_{ii}.$$

Applying $(*)$ and [**4**; Thm. 1.2.1] we deduce that

$$(D\psi : D\phi)_{t,A_2}\sigma_t^\phi(a) = \sigma_t^\psi(a)(D\psi : D\phi)_{t,A_1}, \qquad a \in (A_1, A_2),$$

so that $(D\psi : D\phi)_t$ is a natural unitary transformation from σ_t^ϕ to σ_t^ψ as required.

Finally, as a trivial consequence of [**4**; Thm. 1.2.4] we have

3.7. PROPOSITION. *Let ϕ be a faithful, normal semifinite weight on a W^*-category $\mathfrak{A}$ and $u_{t,A}$ a unitary of (A, A) for each $t \in \mathbf{R}$ and object A of $\mathfrak{A}$ with*

$$u_{t_1+t_2,A} = u_{t_1,A}\sigma_{t_1}^{\phi}(u_{t_2,A}), \qquad t_1, t_2 \in \mathbf{R}.$$

Then there is a unique faithful, normal semifinite weight ψ on $\mathfrak{A}$ with $(D\psi : D\phi)_{t,A} = u_{t,A}$.

4. Commutant, bicommutant and centre. A representation F of a C^*-category $\mathfrak{A}$ is an object in the W^*-category $(\mathfrak{A}, \mathfrak{H})$. This leads to

4.1. DEFINITION. The commutant F' of a representation F of a C^*-category $\mathfrak{A}$ is the von Neumann algebra (F, F). The bicommutant F'' of F is the W^*-category with the same objects as $\mathfrak{A}$ but where

$$(A, B)_{F''} = \{a: F(A) \to F(B): t_B a = a t_A, t \in F'\}.$$

This definition is consistent with the usual terminology if $\mathfrak{A}$ has a single object, i.e. if $\mathfrak{A}$ is a C^*-algebra, but it does not do full justice to the spatial aspects of the notion of commutant. The elements of (F, F) are bounded fields of linear operators $A \to t_A$: $F(A) \to F(A)$ over the set of objects of $\mathfrak{A}$.

The appropriate setting for bringing out such aspects and stating a version of the double commutant theorem is to consider a field F of Hilbert spaces over a set S, i.e. $F(A)$ is a Hilbert space for each $A \in S$. There are two algebraic systems associated with F; the first is the von Neumann algebra $\mathfrak{F}(F)$ of bounded fields of linear operators $A \to t_A$: $F(A) \to F(A)$ with $\|t\| = \sup_{A \in S} \|t_A\| < +\infty$ and the usual algebraic operations.

The second is the W^*-category $\mathfrak{H}(F)$ with S as the set of objects, where (A, B) is the set of all bounded linear mappings from $F(A)$ to $F(B)$ with the usual algebraic operations.

We now define the commutant so as to be a Galois connexion between the subsets of $\mathfrak{H}(F)$ and $\mathfrak{F}(F)$. If $X \subset \mathfrak{H}(F)$, we define

$$X' = \{t \in \mathfrak{F}(F): x t_A = t_B x, x \in X \cap (A, B), A, B \in S\}.$$

Conversely, if $Y \subset \mathfrak{F}(F)$ we define

$$Y' = \bigcup_{A, B \in S} \{x \in (A, B): x t_A = t_B x, t \in Y\}.$$

If $X = X^*$, X' is a von Neumann subalgebra of $\mathfrak{F}(F)$ and if $Y = Y^*$, Y' is a W^*-subcategory of $\mathfrak{H}(F)$.

We now give a version of the von Neumann double commutant theorem.

4.2. THEOREM. *If $\mathfrak{A}$ is a $*$-subcategory of $\mathfrak{H}(F)$ with the same objects as $\mathfrak{H}(F)$, then $\mathfrak{A}''$ is the W^*-subcategory generated by $\mathfrak{A}$. If M is a $*$-subalgebra containing the identity of $\mathfrak{F}(F)$ then M'' is the von Neumann subalgebra of fields generated by M.*

Proof. Consider the Hilbert space $F(S) = \oplus_{A \in S} F(A)$. Elements of $\mathfrak{F}(F)$ and arrows of $\mathfrak{H}(F)$ can also be regarded in the obvious way as bounded linear operators on $F(S)$. Let e_A denote the projection on $F(S)$ corresponding to 1_A. Let $a \in (A, B)_{\mathfrak{A}''}$ then since $\mathfrak{A}''$ is contained in the double commutant of $\mathfrak{A}$ regarded as a set of operators on $F(S)$, there is a net x_α in the $*$-algebra generated by $\mathfrak{A}$ which converges to a in, say, the weak operator topology. Then $e_B x_\alpha e_A$ is a net in $\mathfrak{A}$ converging weakly to a. Thus $\mathfrak{A}''$ is the W^*-subcategory generated by $\mathfrak{A}$. Now the commutant of M looked at as a von Neumann algebra on $F(S)$ is just the von Neumann algebra generated by M'. Hence the bicommutant of M as a von Neumann algebra on $F(S)$ is just M''. Hence M'' is the von Neumann subalgebra of $\mathfrak{F}(F)$ generated by M, by the usual double commutant theorem.

4.3. REMARK. Given a normal representation F of a W^*-category $\mathfrak{A}$, we associated at the end of §2 a normal representation $\tilde{F}$ of $M(\mathfrak{A})$ on the Hilbert space $\oplus_A F(A)$. If F' is regarded as above as a von Neumann algebra on $\oplus_A F(A)$, then F' is just the commutant of $\tilde{F}$.

4.4. DEFINITION. Let $\mathfrak{A}$ be a C^*-category, then $Z(\mathfrak{A})$, the centre of $\mathfrak{A}$, is the set of bounded natural transformations from $1_{\mathfrak{A}}$ to $1_{\mathfrak{A}}$, where $1_{\mathfrak{A}}$ is the identity functor of $\mathfrak{A}$.

Now by Proposition 1.10, $1_{\mathfrak{A}}$ is an object in the C^*-category $(\mathfrak{A}, \mathfrak{A})$, thus $Z(\mathfrak{A})$ is the C^*-algebra $(1_{\mathfrak{A}}, 1_{\mathfrak{A}})$ in this C^*-category. It is clearly an Abelian C^*-algebra. If $\mathfrak{A}$ is a W^*-category, then $(\mathfrak{A}, \mathfrak{A})$ is a W^*-category and $Z(\mathfrak{A})$ is an Abelian von Neumann algebra.

4.5. PROPOSITION. *Let $\mathfrak{A}$ be a W^*-category and $M(\mathfrak{A})$ the associated von Neumann algebra. Then $Z(\mathfrak{A})$ is isomorphic to the centre of $M(\mathfrak{A})$.*

Proof. If $c \in Z(\mathfrak{A})$, then by definition

$$tc_A = c_B t, \quad t \in (A, B), \qquad \sup_A \|c_A\| < \infty.$$

Thus recalling that the arrows of $\mathfrak{A}$ may be regarded as a generating set of $M(\mathfrak{A})$, we see that $c \to \oplus_A c_A$ is an isomorphism of $Z(\mathfrak{A})$ onto the centre of $M(\mathfrak{A})$.

5. Support and central support. In this section we define the support and central support of an arrow in a W^*-category $\mathfrak{A}$ and prove a few elementary results concerning them.

5.1. DEFINITION. If $a \in (A, B)$ then $s(a)$, the support of a, is the smallest projection e in (A, A) with $ae = a$, and $c(a)$, the central support of a is the smallest projection e in $Z(\mathfrak{A})$ with

$$a = ae_A = e_B a.$$

Just as for von Neumann algebras, we have the following elementary properties, $ab = 0$ if and only if $s(a)s(b^*) = 0$, $s(a) = s(a^*a)$, $c(a) = c(a^*)$, $c(a) = c(s(a))$. Indeed, if $a, s(a)$ and $c(a)$ are considered as elements of $M(\mathfrak{A})$, $s(a)$ and $c(a)$ are the support and central support of a in the von Neumann algebra $M(\mathfrak{A})$ respectively. Since $M(A, B)$ is a reduced algebra of $M(\mathfrak{A})$, the support of a in $M(A, B)$ is still $s(a)$ and its central support is the diagonal matrix with entries $c(a)_A$ and $c(a)_B$.

The following result enables one to compute central supports in terms of supports.

5.2. PROPOSITION. *If* $a \in (A, A')$

$$c(a)_B = \sup_{t \in (B, A)} s(at).$$

Proof. If $t \in (B, A)$ then at $c(a)_B = ac(a)_A t = at$ so $c(a)_B \geq s(at)$. Now let $z_B = \sup_{t \in (B, A)} s(at)$, then since $az_A = a$ and $c(a)_B \geq z_B$ we need only show that z is in $Z(\mathfrak{A})$. Given $b \in (C, B)$, we have $at(bz_C - b) = 0$ for each $t \in (B, A)$. Thus $z_B(bz_C - b) = 0$ and replacing b by b^* and taking adjoints, we deduce that $z_B b = bz_C$ so that $z \in Z(\mathfrak{A})$ as required.

5.3. COROLLARY. *For any projection f of $Z(\mathfrak{A})$ we have*

$$c(f_A) = fc(A)$$

where we have written $c(A)$ in place of $c(1_A)$.

Proof. If $t \in (B, A)$ we have

$$s(f_A t) = s(t f_B) = s(t) f_B$$
$$c(f_A)_B = \sup_{t \in (B,A)} s(f_A t) = \sup_{t \in (B,A)} s(t) f_B = c(A)_B f_B$$

as required.

As a first application of these ideas, we characterize the W^*-categories which are factors. An object A of $\mathfrak{A}$ is called a zero object if $(A, A) = 0$ or equivalently if $c(A) = 0$. $\mathfrak{A}$ is called connected if one cannot partition the objects of $\mathfrak{A}$ into two sets S_1 and S_2 such that $A_1 \in S_1$ and $A_2 \in S_2$ implies $(A_1, A_2) = 0$ and neither set consists entirely of zero objects.

5.4. PROPOSITION. *A W^*-category $\mathfrak{A}$ is a factor if and only if it is connected and each (A, A) is a factor.*

Proof. Consider any partition of the objects of $\mathfrak{A}$ into two sets S_1 and S_2 such that $A_1 \in S_1$ and $A_2 \in S_2$ implies $(A_1, A_2) = 0$, then defining $z_A = 1_A$ if $A \in s_1$ and $z_A = 0$ if $A \in S_2$ will define a non-trivial projection in the centre of $\mathfrak{A}$ unless either S_1 or S_2 consists entirely of zero objects. If e is a projection in the centre of (A, A), then by Proposition 5.2, $c(e)_A = e$. Hence if $\mathfrak{A}$ is a factor, $\mathfrak{A}$ must be connected and each (A, A) must be a factor. Conversely if e is a projection in $Z(\mathfrak{A})$ and each (A, A) is a factor, let S_1 be the set of objects A of $\mathfrak{A}$ with $e_A = 1_A$ and S_2 the set of objects A with $e_A = 0$. If $\mathfrak{A}$ is connected, either S_1 consists of zero objects giving $e = 0$ or S_2 consists of zero objects givin $e = 1$.

We close this section by noting how supports and central supports transform under a normal $*$-functor.

5.5. PROPOSITION. *Let F be a normal $*$-functor from $\mathfrak{A}$ to $\mathfrak{B}$ then*

$$F(s(a)) = s(F(a))$$
$$F(c(a)_B) \leq c(F(a))_{F(B)}$$

and the inequality can be replaced by an equality if F is full.

Proof. To show that F preserves supports, note that $s(a) = s(a^*a)$ so that the result follows from the corresponding result for von Neumann algebras. The inequality for central supports now follows from Proposition 5.2 since F is normal and $F(A, B) \subset (F(A), F(B))$. When $F(A, B) = (F(A), F(B))$ for all objects A and B of $\mathfrak{A}$, F is full by definition and we have equality.

This has the following two simple consequences.

5.6. COROLLARY. *Let F be a normal ∗-functor from* $\mathfrak{A}$ *to* $\mathfrak{B}$ *then if a and b are arrows of* $\mathfrak{A}$ *with* $c(b) \leq c(a)$, $c(F(b)) \leq c(F(a))$.

Proof. Let $b \in (B, B')$. Then $c(b) \leq c(a)$ is by definition equivalent to $bc(a)_B = b$ and hence to $c(a)_B \geq s(b)$. Now by Proposition 5.5, we have

$$c(F(a))_{F(B)} \geq F(c(a)_B) \geq F(s(b)) = s(F(b)),$$

so

$$c(F(b)) \leq c(F(a)).$$

5.7. PROPOSITION. *Let* $F\colon \mathfrak{A} \to \mathfrak{B}$ *be a normal ∗-functor and let* e_A *denote the support of the morphism* $(A, A) \to F(A, A)$ *then* $A \to e_A$ *is a projection in the centre of* $\mathfrak{A}$ *called the support of F.*

Proof. Let $a \in (A, A')$ and suppose $F(a) = 0$ then $F(s(a)) = 0$ so $s(a)e_A = 0$ hence $ae_A = 0$. We deduce that for any $a \in (A, A')$, $ae_A = e_{A'}ae_A$ which, replacing a by a^* and taking adjoints, implies that $A \to e_A$ is a projection in the centre of $\mathfrak{A}$.

6. Comparison theory. Up till now we have largely been concerned with deriving the analogues of the basic results of von Neumann algebra theory. Here, in treating comparison theory, we deal with techniques which ave already played an important role in the theory of von Neumann algebras but which properly belong to the theory of W^*-categories.

6.1. DEFINITION. Let A and B be objects in a W^*-category, we way that A and B are equivalent, $A \sim B$ if (A, B) contains a unitary operator and that A is a subobject of B written $A \leq B$ or $B \geq A$ if (A, B) contains an isometry.

The equivalence of objects in a W^*-category corresponds to the usual notion of isomorphism of objects in a category since the polar part of an invertible arrow is unitary. In category theory, subobjects are defined in terms of monic arrows, i.e. monomorphisms, and the notion here is different. However, $m \in (A, B)$ is monic if and only if $s(m) = 1_A$, hence if and only if its polar part is an isometry. $A \leq B$ and $B \leq A$ imply $A \sim B$. $A \leq B$ if and only if the corresponding projections, e_A and e_B, satisfy $e_A \prec e_B$ in $M(\mathfrak{A})$.

If A is a C^*-category, then π and π' satisfy $\pi \leq \pi'$ as objects of the W^*-category Rep(A) if and only if π is unitary equivalent to a subrepresentation of π'. The other important motivating example is

6.2. EXAMPLE. Let M be a von Neumann algebra and let $\mathscr{P}(M)$ denote the W^*-category whose objects are the projections of M and where an arrow from e to e' in $\mathscr{P}(M)$ is a triple $(e'|t|e)$ where $t \in M$ and $e't = t = te$. The algebraic structure and norm are defined in the obvious manner, e.g.

$$(e''|t'|e')(e'|t|e) = (e''|t't|e), \quad (e'|t|e)^* = (e|t^*|e')$$

and $\|(e'|t|e)\| = \|t\|$ and we shall write t in place of $(e'|t|e)$ where this causes no confusion. The comparison of objects in $\mathscr{P}(M)$ is not just the usual Murray and von Neumann comparison theory of projections in M.

The above construction works equally well for an arbitrary W^*-category $\mathfrak{A}$ and the comparison of objects of $\mathscr{P}(\mathfrak{A})$, the W^*-category of projections in $\mathfrak{A}$, is the comparison theory of projections in $\mathfrak{A}$. By Proposition 5.4, $\mathscr{P}(\mathfrak{A})$ is a factor if and only if $\mathfrak{A}$ is a factor.

6.3. DEFINITION. Two W^*-categories $\mathfrak{A}$ and $\mathfrak{B}$ are said to be equivalent if there is a $*$-functor F: $\mathfrak{A} \to \mathfrak{B}$ which is an equivalence of categories.

A $*$-functor F: $\mathfrak{A} \to \mathfrak{B}$ is an equivalence if F is full and faithful and if each object B of $\mathfrak{B}$ is isomorphic (hence equivalent) to an object of the form $F(A)$. In particular, an equivalence of W^*-categories is automatically normal since F induces a $*$-isomorphism of (A, A) and $(F(A), F(A))$ (cf. [**20**; Prop. 7.3]). There are other ways of describing the equivalence of categories (see e.g. [**13**; §IV. 4 Thm. 1]). In particular, two W^*-categories $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent if and only if there are normal $*$-functors F: $\mathfrak{A} \to \mathfrak{B}$ and G: $\mathscr{B} \to \mathfrak{A}$ such that $GF \sim 1_{\mathfrak{A}}$ in $(\mathfrak{A}, \mathfrak{A})$ and $FG \sim 1_{\mathfrak{B}}$ in $(\mathfrak{B}, \mathfrak{B})$. This can be easily proved directly or it can be derived from the catgorical result by noting that a natural isomorphism can be turned into a natural unitary equivalence by taking polar decompositions object by object.

The Morita equivalence of C^*-algebras and von Neumann algebras studied by Rieffel is the equivalence of the corresponding W^*-categories of representations [**20**; Defn. 7.4]. We shall adopt the same terminology for W^*-categories.

We take a first step towards classifying W^*-categories up to equivalence by identifying those W^*-categories $\mathfrak{A}$ which are equivalent to $\mathscr{P}(M)$ for some von Neumann algebra M. We begin by isolating the relevant properties of $\mathscr{P}(M)$. An object A of $\mathfrak{A}$ will be said to be *maximal* if every object of $\mathfrak{A}$ is a subobject of A. The identity of M is a maximal object of $\mathscr{P}(M)$. $\mathfrak{A}$ will be said to have *sufficient subobjects* if given any projection e of $\mathfrak{A}$, there is an isometry w of $\mathfrak{A}$ with $ww^* = e$. $\mathfrak{A}$ has sufficient subobjects if and only if it is equivalent to $\mathscr{P}(\mathfrak{A})$. The category $\mathfrak{H}$ of Hilbert spaces has sufficient subobjects and if $\mathfrak{A}$ has sufficient subobjects then so has $(\mathfrak{B}, \mathfrak{A})$ for any $*$-category $\mathfrak{B}$. In particular Rep(A) has sufficient subobjects for any C^*-algebra A.

6.4. PROPOSITION. *A W^*-category $\mathfrak{A}$ with maximal object A and sufficient subobjects is equivalent to $\mathscr{P}(M)$ with $M = (A, A)$.*

Proof. Given any object B of $\mathfrak{A}$ there is an isometry w_B in (B, A). We may define a functor F: $\mathfrak{A} \to \mathscr{P}(M)$ by setting $F(B) = w_B w_B^*$ and $F(b) = w_C b w_B^*$ for $b \in (B, C)$. F is a full and faithful $*$-functor. Since $\mathfrak{A}$ has sufficient subobjects, any projection of M is equivalent to some $F(B)$. Hence F is an equivalence of W^*-categories.

With the example of representations in mind, we discuss irreducibles, infinite multiplicity and direct sums in the context of W^*-categories. An object A of a W^*-category $\mathfrak{A}$ is *irreducible* if $(A, A) \simeq \mathbf{C}$. If A and B are irreducibles then either A and B are equivalent or $(A, B) = 0$. A factor contains at most one equivalence class of irreducibles. Irreducibles are obviously "atoms" for the relation $\leq$ and if $\mathfrak{A}$ has sufficient subobjects any atom is irreducible.

We say that A is a direct sum of A_i, $i \in I$, and write $A = \oplus_{i \in I} A_i$ if there are isometries $w_i \in (A_i, A)$ with $\sum_{i \in I} w_i w_i^* = 1_A$. A is determined up to equivalence by the A_i. If $A_i \simeq B$ for each $i \in I$ we write $A = |I| B$ where $|I|$ denotes the cardinality of I. We say that $\mathfrak{A}$ has finite or countable direct sums if given objects A_i, $i \in I$, with I finite or countable respectively, there exists an object $A = \oplus_{i \in I} A_i$ of $\mathfrak{A}$. $\mathfrak{H}$ has countable direct sums and $\mathscr{P}(M)$ has finite or countable direct sums if and only if M is properly infinite. If $\mathfrak{A}$ has finite or countable direct sums, so does $(\mathfrak{B}, \mathfrak{A})$ for any $*$-category $\mathfrak{B}$. Just as we were able to pass from $\mathfrak{A}$ to $\mathscr{P}(\mathfrak{A})$ if we wanted to have sufficient subobjects, so we can enlarge $\mathfrak{A}$ so that it has finite or countable direct sums but we refrain from giving an explicit construction. Note that

$$\text{if } A = \bigoplus_{i \in I} A_i, \quad \text{then } c(A) = \sup_i c(A_i).$$

6.5. PROPOSITION. *The following conditions on an object A of a W^*-category are equivalent*:

(a) (A, A) *is properly infinite*.

(b) $A \sim A \oplus A$.

(c) $A = \infty A$.

(d) $A = \infty B$, *for some object* B *of* $\mathfrak{A}$.

We omit the simple proof of this result which serves to characterize the objects of infinite multiplicity in $\mathfrak{A}$.

7. Quasiequivalence. In this section we deal with the quasiequivalence of objects in a W^*-category. This is of course a weaker notion than equivalence useful whenever it suffices to resolve problems up to questions of multiplicity.

7.1. DEFINITION. Let A and B be objects in a W^*-category $\mathfrak{A}$. We say that A is quasicontained in B if $c(A) \leq c(B)$ and that A and B are quasiequivalent if $c(A) = c(B)$. If $c(A)c(B) = 0$, then A and B are said to be disjoint.

There are competing definitions of quasicontainment which, however, reduce to the above definition if $\mathfrak{A}$ has sufficient subobjects.

7.2. PROPOSITION. *If* $\mathfrak{A}$ *is a* W^**-category with sufficient subobjects then the following conditions on objects* A *and* B *of* $\mathfrak{A}$ *are equivalent.*

(a) $c(A) \leq c(B)$.

(b) *Given* $0 \neq A' \leq A$, *there exists* $0 \neq A'' \leq A'$ *with* $A'' \leq B$.

(c) $A = \oplus_{i \in I} B_i$ *with* $B_i \leq B$.

We omit the simple proof which in any case is comparable with similar proofs in the theory of representations (see e.g. [7; §5.1]).

We now consider objects A of $\mathfrak{A}$ which are quasimaximal, i.e. such that $c(A) \geq c(B)$ for all objects B of $\mathfrak{A}$. This is clearly equivalent to demanding that $c(A) = 1$ but there are many other ways of characterizing such objects.

7.3. PROPOSITION. *Let* $\mathfrak{A}$ *be a* W^**-category then the following conditions on an object* A *are equivalent*

(a) $c(A) = 1$

(b) A *is a generator, i.e. for all* $b \in (B, C)$ *with* $b \neq 0$ *there exists* $a \in (A, B)$ *with* $ba \neq 0$. (*Equivalently,* A *is a cogenerator since* $\mathfrak{A}$ *is self-dual.*)

(c) *Given any object* B *of* $\mathfrak{A}$ *there are partial isometries* $w_i^B \in (B, A)$ *with* $\sum_i w_i^{B*} w_i^B = 1_B$.

(d) *The map* $z \to z_A$ *is an isomorphism of* $Z(\mathfrak{A})$ *with the centre of* (A, A).

If $\mathfrak{A}$ has sufficient subobjects then these conditions are also equivalent to
(e) $(A, B) \neq 0$ *for each non-zero object* B *of* $\mathfrak{A}$.
(f) *Every object* B *is a direct sum of subobjects of* $\mathfrak{A}$.

Proof. The equivalence of (a) and (b) is clear from Proposition 5.2 (b) implies (c) is a standard argument using Zorn's Lemma and the existence of polar decompositions and the converse is trivial. (d) implies (a) trivially since $c(A)_A = 1_A$. The normal morphism $z \to z_A$ is always surjective since, for any projection e in the centre of (A, A), $c(e)_A = e$. If $c(A) = 1$, it is injective by Corollary 5.3. The equivalence of (e) and (b) and of (f) and (a) is clear when $\mathfrak{A}$ has sufficient subobjects.

If $\mathfrak{A}$ does not have a generator we can easily adjoin one without enlarging its centre by adding on $M(\mathfrak{A})$ as a single object. The easiest way to do this formally is to replace $\mathfrak{A}$ by the full subcategory of $\mathscr{P}(M(\mathfrak{A}))$ whose objects are either the identity of $M(\mathfrak{A})$ or the projection e_A associated with some object A of $\mathfrak{A}$.

7.4. Corollary. *Let* A *be a generator of a* W^**-category* $\mathfrak{A}$ *and* F *a normal* $*$*-functor from* $\mathfrak{A}$ *to a* W^**-category* $\mathfrak{B}$, *then* (F, F) *and* $F(A, A)'$ *are isomorphic as von Neumann algebras.*

Proof. The evaluation map $\eta \to \eta_A$ from (F, F) to $F(A, A)'$ is certainly a continuous linear mapping. Now for each object B of $\mathfrak{A}$ consider the partial isometries $w_i^B \in (B, A)$ given by Proposition 7.3c then, if $\eta_A = 0$ we get

$$\eta_B = \eta_B \sum_i F(w_i^B)^* F(w_i^B) = \sum_i F(w_i^B)^* \eta_A F(w_i^B) = 0$$

so that the evaluation map is injective; but it is also surjective, since, for each $\xi \in F(A, A)'$, there is an element $\eta \in (F, F)$ given by

$$\eta_B = \sum_i F(w_i^B)^* \xi F(w_i^B)$$

and $\eta_A = \xi$.

7.5. Proposition. *Let* $\mathfrak{A}$ *be a* W^**-category, then* $Z(\mathfrak{A})$ *is isomorphic to* $Z(\mathfrak{A}, \mathfrak{H})$ *and an object of* $(\mathfrak{A}, \mathfrak{H})$ *is a generator if and only if it is faithful.*

Proof. Although it is not difficult to give a direct proof, we deduce this result from [**20**; Props. 1.3 and 2.1] by noting that $(\mathfrak{A}, \mathfrak{H})$ is equivalent to the category of representation of $M(\mathfrak{A})$ and that an object F of

$(\mathfrak{A}, \mathfrak{H})$ is faithful if and only if $\tilde{F}$, the associated representation of $M(\mathfrak{A})$ is faithful and that $\mathfrak{A}$ and $M(\mathfrak{A})$ have isomorphic centres (Prop. 4.5).

We are now interested in characterizing the W^*-categories which are equivalent to $H\,\mathrm{mod}\, M$ for some von Neumann algebra M (Example 1.4). Clearly $H\,\mathrm{mod}\, M$ has sufficient subobjects and direct sums and Proposition 7.3e shows that M, considered as self-dual module over M is a generator of $H\,\mathrm{mod}\, M$. Thus, by Proposition 7.3e any self-dual Hermitian M-module is equivalent to a module of the form $e(H \otimes M)$, where H is an Hilbert space and e is a projection of $\mathfrak{B}(H) \otimes M$.

7.6. PROPOSITION. *A W^*-category $\mathfrak{A}$ with a generator A having direct sums and sufficient subobjects is equivalent to $H\,\mathrm{mod}\, M$ with $M = (A, A)$.*[3]

Proof. Let B and C be objects of $\mathfrak{A}$ and let $F_A(B)$ denote (A, B) considered as a right Hermitian (A, A)-module. By Proposition 2.14 it is a self-dual module. Now, for each $t \in (B, C)$ let $F_A(t)$: $F_A(B) \to F_A(C)$ be defined by $F_A(t)r = tr$, $r \in (A, B)$. Clearly, F_A is a $*$-functor; it is full by Proposition 2.14 again and normal by Proposition 2.12. Proposition 7.3b shows that F_A is faithful. Finally, since $F_A(A) = (A, A)$ is a generator of $H\,\mathrm{mod}(A, A)$, Proposition 7.3f together with the fact that $\mathfrak{A}$ has sufficient subobjects show that any object of $H\,\mathrm{mod}(A, A)$ is unitarily equivalent to an object in the image of F_A, so that F_A is an equivalence of W^*-categories.

7.7. COROLLARY. *Let $\mathfrak{A}$ be a W^*-category, then $(\mathfrak{A}, \mathfrak{H})$ is equivalent to $H\,\mathrm{mod}(F, F)$, for each faithful object of $(\mathfrak{A}, \mathfrak{H})$.*

Proof. Combine Proposition 7.5 with Proposition 7.6.

7.8. COROLLARY. *If A is a generator of a W^*-category $\mathfrak{A}$, then $\mathfrak{A}$ is Morita equivalent to any full subcategory containing A as an object. The von Neumann algebras (A, A) and $M(\mathfrak{A})$ are Morita equivalent.*

Proof. If $\mathfrak{B}$ is any full subcategory of $\mathfrak{A}$ containing A as an object $(\mathfrak{B}, \mathfrak{H})$ is equivalent to $H\,\mathrm{mod}\, F(A, A)'$ by Corollaries 7.4 and 7.7. We

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[3] At this point recourse to a universe is needed for a precise formulation: the hom-sets (B, A) need to be in the universe used to define $H\,\mathrm{mod}\, M$ and $\mathfrak{A}$ must have direct sums indexed by sets from that universe. In Corollary 7.7, the set of arrows of $\mathfrak{A}$ need to be in the universe in question.

have already noted that $(\mathfrak{A}, \mathfrak{H})$ is equivalent to the category of representations of $M(\mathfrak{A})$ in the course of proving Proposition 7.5.

We can also deduce the characterization of Morita equivalence suggested by Connes (see [**20**; pg. 92]) and that this is even true in the context of W^*-categories.

7.9. PROPOSITION. *Two W^*-categories $\mathfrak{A}$ and $\mathfrak{B}$ are Morita equivalent if and only if $\mathfrak{A}$ and $\mathfrak{B}$ have faithful representations whose commutants are isomorphic as von Neumann algebras.*

Proof. An equivalence of $(\mathfrak{A}, \mathfrak{H})$ and $(\mathfrak{B}, \mathfrak{H})$ must map a generator F into a generator G and induce an isomorphism of the von Neumann algebras (F, F) and (G, G). The converse is a consequence of Corollary 7.7.

7.10. REMARK. If A and B are quasiequivalent objects in a W^*-category then (A, A) and (B, B) are Morita equivalent. In fact, considering the full subcategory with objects A and B we get a W^*-category where A and B are generators and (A, B) is then a self-dual (A, A)-(B, B)-equivalence bimodule in the terminology of [**20**].

7.11. COROLLARY. *Let π be a faithful normal representation of a von Neumann algebra M, then the W^*-category of the normal representations of M is equivalent to $H \bmod \pi(M)'$.*

7.12. REMARK. Corollary 7.11 and Proposition 7.6 also characterize the W^*-category of normal representations of a von Neumann algebra since a W^*-category with a generator A, having direct sums and sufficient subobjects is equivalent to $(M, \mathfrak{H})$ for $M = (A, A)'$.

The following result can be viewed as a generalization of Corollary 7.8 and can be proved using Proposition 7.3c.

7.13. THEOREM. *Let A be a generator of $\mathfrak{A}$ then the restriction $*$-functor $(\mathfrak{A}, \mathfrak{K}) \to (A, \mathfrak{K})$ is full and faithful for any W^*-category $\mathfrak{K}$. Here we have also denoted by A the full subcategory of $\mathfrak{A}$ with A as a single object. If $\mathfrak{K}$ has sufficient subobjects and direct sums of sufficiently high cardinality, this functor is an equivalence.*

We now demonstrate a further property of generators.

7.14. PROPOSITION. *Let F: $\mathfrak{A} \to \mathfrak{B}$ be a normal $*$-functor and A a generator of $\mathfrak{A}$. Then F is faithful or full if and only if the induced mapping $(A, A) \to (F(A), F(A))$ is injective or surjective respectively.*

Proof. Suppose $(A, A) \to (F(A), F(A))$ is injective, then by Propositions 5.7 and 7.3d F has support 1 and is thus faithful. The converse is trivial.

Suppose $(A, A) \to (F(A), F(A))$ is surjective and $b \in (F(B), F(C))$, then using Proposition 7.3c

$$b = \sum_{i,j} F(w_j^C)^* F(w_j^C) b F(w_i^B)^* F(w_i^B)$$

but $F(w_j^C) b F(w_i^B)^* \in (F(A), F(A)) = F(A, A)$ so $b \in F(B, C)$ and F is full. The converse is again trivial.

The last part of this section is devoted to σ-finite W^*-categories $\mathfrak{A}$, i.e. (A, A) is σ-finite for each object A of $\mathfrak{A}$. In this situation there is a technique for passing from quasiequivalence to equivalence which has been employed several times in [**6**].

7.15. Proposition. *If A is an object of infinite multiplicity in a σ-finite W^*-category $\mathfrak{A}$, then $c(B) \leq c(A)$ implies $B \leq A$.*

Proof. $c(B) \leq c(A)$ implies that 1_A considered as a projection in the σ-finite von Neumann algebra $M(A, B)$ is properly infinite with central support 1. Thus 1_A is equivalent to 1 in $M(A, B)$ (see for example the proof of [**27**; Prop. 2.2.14.]). Regarding 1_B as a projection in $M(A, B)$, we conclude that $A \leq B$.

7.16. Corollary. *Any generator with infinite multiplicity in a σ-finite W^*-category is maximal. Any σ-finite W^*-category with countable direct sums and a generator has a maximal element.*

Let M be a properly infinite von Neumann algebra, then as we have already pointed out $\mathscr{P}(M)$ has sufficient subobjects and countable direct sums. Hence the full subcategory $\mathscr{P}_\sigma(M)$ whose objects are the σ-finite projections in M is a σ-finite W^*-category with sufficient subobjects and countable direct sums.

7.17. Theorem. *Let $\mathfrak{A}$ be a σ-finite W^*-category with sufficient subobjects and countable direct sums. Then every σ-finite projection of $Z(\mathfrak{A})$ is of the form $c(A)$ for some object A of $\mathfrak{A}$. Furthermore $\mathfrak{A}$ is equivalent to $\mathscr{P}_\sigma(M(\mathfrak{A}))$.*

Proof. The identity of $Z(\mathfrak{A})$ is the supremum of $c(A)$ as A varies over the objects of $\mathfrak{A}$. Thus if f is a σ-finite projection of $Z(\mathfrak{A})$, there are objects $B_1, B_2, \ldots$ of $\mathfrak{A}$ with $f \leq \sup_j c(B_i)$. Hence setting $B = \oplus_i B_i$, $f \leq c(B)$. Since $\mathfrak{A}$ has sufficient subobjects there is an isometry $w \in (A, B)$ with $ww^* = f_B$. Then $c(A) \leq c(B)$ and $c(A)_B = f_B$ so by Corollary 5.3, $c(A) = f$. Since $\mathfrak{A}$ is σ-finite, the full and faithful $*$-functor taking $a \in (A, B)$ onto $(e_B|a|e_A)$ takes values in $\mathscr{P}_\sigma(M(\mathfrak{A}))$. Let e be a σ-finite projection of $M(\mathfrak{A})$; its central support $c(e)$ is a σ-finite projection of $ZM(\mathfrak{A})$. Hence by the first part of the theorem there is an object A which we may suppose to be of infinite multiplicity with $c(e_A) = c(e)$. Thus by Proposition 7.15, e is a subobject of e_A in $\mathscr{P}_\sigma(M(\mathfrak{A}))$. Since $\mathfrak{A}$ has sufficient subobjects e is equivalent to an object in the image of $\mathfrak{A} \to \mathscr{P}_\sigma(M(\mathfrak{A}))$ so that this functor is an equivalence.

8. The W^*-category of weights on a von Neumann algebra. Let M be a von Neumann algebra; we define a W^*-category $\mathscr{V}(M)$ whose objects are the normal semifinite weights on M. If ϕ and ϕ' are two such weights with supports $s(\phi)$ and $s(\phi')$ respectively, then an arrow from ϕ to ϕ' in $\mathscr{V}(M)$ is a triple $(\phi'|x|\phi)$ where $x \in M$ and $x = xs(\phi) = s(\phi')x$.

The remaining structure is defined as for $\mathscr{P}(M)$ (Example 6.2) so that $(\phi'|x|\phi) \to (s(\phi')|x|s(\phi))$ becomes an equivalence from $\mathscr{V}(M)$ to $\mathscr{P}(M)$. $\mathscr{V}(M)$ differs from $\mathscr{P}(M)$ in that it carries a canonical faithful normal semifinite weight Φ: $\Phi_\phi = \phi \upharpoonright M_{s(\phi)}$. We can construct the corresponding modular action σ^Φ as in Proposition 3.4. On the algebra $(\phi, \phi)_{\mathscr{V}(M)} \simeq M_{s(\phi)}$, σ^Φ of course coincides with σ^ϕ. If ϕ and ϕ' are faithful weights on M, then

$$\sigma_t^\Phi(\phi'|1|\phi) = \left(\phi'\middle|(D\phi' : D\phi)_t\middle|\phi\right)$$

and the basic properties of Radon-Nikodým derivatives, such as the chain rule, can be easily deduced from this formula.

The W^*-category $\mathscr{W}(M)$ of weights on M is defined to be the fixed points of $\mathscr{V}(M)$ under σ^Φ and is the object of study in this section. The canonical weight Φ induces a canonical faithful normal semifinite trace on $\mathscr{W}(M)$.

We now make use of Proposition 3.5 to give an alternative description of the isometries of $\mathscr{W}(M)$.

8.1. PROPOSITION. *$(\phi'|u|\phi)$ is an isometry of $\mathscr{W}(M)$ if and only if*

$$u^*u = s(\phi), \qquad uu^* \in M_{\phi'}$$

and

$$\phi(x) = \phi'(uxu^*), \qquad x \in M_+.$$

Proof. If $(\phi'|u|\phi)$ is an isometry of $\mathscr{W}(M)$, $u^*u = s(\phi)$, $uu^* \in M_{\phi'}$ and applying Proposition 3.5, $(\phi|x|\phi) \in \mathfrak{M}_\Phi$ is equivalent to

$$(\phi'|u|\phi)(\phi|x|\phi)(\phi|u^*|\phi') \in \mathfrak{M}_\Phi \quad \text{and} \quad \phi'(uxu^*) = \phi(x).$$

Conversely, if $u^*u = s(\phi)$ and $uu^* \in M_{\phi'}$ then $(\phi'|u|\phi)$ is an isometry of $\mathscr{P}(M)$. Furthermore, $\mathfrak{N}_\Phi(\phi'|u|\phi) \subset \mathfrak{N}_\Phi$ since $uu^* \in M_{\phi'}$ and $(\phi|u^*|\phi')\mathfrak{N}_\Phi \subset \mathfrak{N}_\Phi$ trivially.

The conditions in Proposition 3.5b can now be verified by a simple calculation so that $(\phi'|u|\phi)$ is invariant under σ^Φ completing the proof.

This result tells us that $\phi \prec \phi'$ in the sense of Connes and Takesaki [**6**] if and only if $\phi \leq \phi'$ as objects of $\mathscr{W}(M)$.

8.2. PROPOSITION. *$\mathscr{W}(M)$ has sufficient subobjects.*

Proof. Let $(\phi|e|\phi)$ be a projection in $\mathscr{W}(M)$ then $e \in M_\phi$ and if ϕ_e is the weight on M with support e defined by $\phi_e(x) = \phi(exe)$, we may apply Proposition 8.1 to deduce that $(\phi|e|\phi_e)$ is an isometry in $\mathscr{W}(M)$ with final projection $(\phi|e|\phi)$.

As far as direct sums go we have

8.3. PROPOSITION. *$\phi = \oplus_i \phi_i$ in $\mathscr{W}(M)$ if and only if $\phi = \sum_i \psi_i$ where ψ_i are weights with pairwise orthogonal supports and $\psi_i \simeq \phi_i^i$. $\mathscr{W}(M)$ has countable direct sums if and only if M is properly infinite.*

Proof. Let $(\phi|w_i|\phi_i)$ be isometries of $\mathscr{W}(M)$ realizing $\phi = \oplus_i \phi_i$ and set $\psi_i = \phi_{e_i}$ where $e_i = w_i w_i{}^*$. The ψ_i have pairwise orthogonal supports and $(\psi_i|w_i|\phi_i)$ are unitaries of $\mathscr{W}(M)$. Now $\sum_i \psi_i(x) = \sum_i \phi(w_i w_i^* x w_i w_i^*) = \phi(x)$ since $w_i w_i^* \in M_\phi$. The converse is equally simple.

If M is properly infinite and ϕ_i are objects of $\mathscr{W}(M)$, $i = 1, 2, \ldots$, then let $(e|w_i|s(\phi_i))$ be isometries of $\mathscr{P}(M)$ realizing $e = \oplus_i s(\phi_i)$. Setting $\phi = \sum_i \phi_{e_i}$, where $e_i = w_i w_i^*$, then $\phi_{e_i} \simeq \phi_i$ so $\phi = \oplus_i \phi_i$. The converse is trivial.

If we now apply the theory of W^*-categories developed here we obtain the following result designed to resemble [**6**; Thm. 1.11].

8.4. THEOREM. *Let M be a σ-finite properly infinite von Neumann algebra. Every σ-finite projection of $Z\mathscr{W}(M)$ is of the form $c(\phi)$ for some weight ϕ.*

(i) *Writing $\check{\phi} = \infty\phi$,*

$$c(\check{\phi}) = c(\phi)$$

$$c(\phi_1) \leq c(\phi_2) \quad \textit{if and only if } \check{\phi}_1 \leq \check{\phi}_2$$

(ii) *The map* $e \to c(\phi_e)$ *from central projections of* M_ϕ *extends uniquely to an isomorphism of* $Z(M_\phi)$ *and* $Z(\mathscr{W}(M))_{c(\phi)}$ *and*

$$c(c(\psi)_\phi) = c(\phi)c(\psi)$$

(iii) *For any sequence* $\{\phi_n\}$ *of normal semifinite weights with pairwise orthogonal supports,* $c(\sum_{n=1}^{\infty}\phi_n) = \sup_n c(\phi_n)$.

No formal proof is needed here since the main points are contained in Corollary 5.3, Proposition 7.15 and Theorem 7.17. This result reveals our debt to the techniques developed in [6]. We hope it also demonstrates that W^*-categories provide the right framework for the comparison theory of weights and that there is no more natural representation of the pair $\{p_M, \mathscr{P}_M\}$ of [6], unique up to isomorphism than as $\{c, Z\mathscr{W}(M)\}$.

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