PACIFIC JOURNAL OF MATHEMATICS
Vol. 120, No. 1, 1985

A KRASNOSEL'SKII-TYPE THEOREM FOR UNIONS OF TWO STARSHAPED SETS IN THE PLANE

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Let S be a simply connected polygonal region in the plane, symmetric with respect to the x and y axes, such that each edge of S is parallel to one of these axes. Assume that for every set E consisting of 6 or fewer edges of S there exist points t_1 and t_2 collinear with the origin (and depending on E) such that every point in $\cup\{e: e \text{ in } E\}$ is visible via S from t_1 or t_2 (or both). Then S is a union of two starshaped sets. The number 6 is best possible.

Furthermore, an example reveals that there is no finite Krasnosel'skii number which characterizes arbitrary unions of two or more starshaped sets in the plane.

1. Introduction. We begin with some preliminary definitions. Let S be a set in R^d. For points x and y in S, we say x *sees* y via S (x is *visible* from y via S) if and only if the corresponding segment $[x, y]$ lies in S. Point x is *clearly visible* from y via S if and only if there is some neighborhood N of x such that y sees each point of $N \cap S$ via S. Set S is *starshaped* if and only if there is some point p in S such that p sees each point of S via S, and the set of all such points p is called the (convex) kernel of S.

A well-known theorem of Krasnosel'skii [**5**] states that if S is a nonempty compact set in R^d, then S is starshaped if and only if every $d + 1$ points of S are visible via S from a common point. Further, points of S may be replaced by boundary points of S to produce a stronger result. An interesting problem related to this concerns obtaining a Krasnosel'skii-type theorem for unions of starshaped sets in the plane R^2. This kind of problem is mentioned in [**10**, Prob. 6.6, p. 178] and in [**1**]. Moreover, using work by Lawrence, Hare, and Kenelly [**7**] concerning unions of convex sets, the following Krasnosel'skii-type results for unions of starshaped sets are obtained in [**2**]: (1) For S compact in R^2, S is a union of two starshaped sets if for every finite set F in the boundary of S there exist points s and t (depending on F) such that each point of F is clearly visible via S from at least one of s or t. If in addition set S is simply connected, then 'clearly visible' may be replaced by the weaker term 'visible'. (2) In general, for S a compact set in some linear topological space, S is a union of k starshaped sets if and only if for every finite set F in S there exist points $s_1, \ldots, s_k$ (depending on F) such that each point of F sees via S at least one of the s_i points. Unfortunately, the finiteness condition in (2) above cannot be improved, and no finite Krasnosel'skii number exists to characterize arbitrary unions of two or more compact starshaped sets, even in the plane. (See Example 4.)

Still, the problem of obtaining a finite Krasnosel'skii number for certain families of sets remains open. Since the Helly number $d + 1$ plays a fundamental role in the classical Krasnosel'skii theorem, we would expect the piercing number in **[4]** to be important in any kind of generalization. Although an example from **[5**, pp. 11–12**]** reveals that no finite n-piercing number exists for arbitrary families of compact sets when $n \geq 2$, recent work by Danzer and Grünbaum **[3]** reveals that such a piercing number does exist for families of boxes. In a similar spirit, Toussaint and El-Gindy **[9]** have shown that for polygonal regions whose edges are parallel to the coordinate axes, the classical Krasnosel'skii number 3 may be reduced to 2. Therefore, it seems reasonable to attempt to establish a generalized Krasnosel'skii theorem for such sets. This is the problem considered here.

The following terminology will be used throughout the paper: $\operatorname{Conv} S$, $\operatorname{int} S$, $\operatorname{rel\,int} S$, $\operatorname{bdry} S$, and $\ker S$ will denote the convex hull, interior, relative interior, boundary, and kernel, respectively, for set S. For distinct points x and y, $L(x, y)$ will represent the line through x and y and $\frac{1}{2}(x + y)$ will denote the midpoint of segment $[x, y]$. The reader is referred to Valentine **[10]** and to Lay **[8]** for a discussion of these concepts.

2. Preliminary results. We start with an easy lemma.

LEMMA 1. *Let S be a compact, simply connected set in R^2, with $a, b \in S$. If each boundary point of S sees via S either a or b, then each point of S sees via S either a or b.*

Proof. Let $x \in S$ to show x sees via S either a or b. If $x \in \operatorname{bdry} S$, the result is immediate, so assume that $x \in \operatorname{int} S$. Suppose $[a, x] \not\subseteq S$, and let $[a_1, a_2]$ be the component of $S \cap L(a, x)$ which contains x. Clearly $a_1 < x < a_2$, a_1 and a_2 are in $\operatorname{bdry} S$, and a sees via S neither a_1 nor a_2. Hence b sees via S both a_1 and a_2. Since S is simply connected, $\operatorname{conv}\{b, a_1, a_2\} \subseteq S$, $[b, x] \subseteq S$, and the lemma is established.

We make several observations concerning Lemma 1.

First, if S is not required to be simply connected, then the result in Lemma 1 fails and in fact S is not necessarily a union of two starshaped sets. This is illustrated in [**2**, Example 1].

Next, it is interesting to notice that even when S is a polygonal region satisfying additional hypotheses, bdry$\,S$ cannot be replaced by the vertex set of S in Lemma 1. This is demonstrated in Example 1 below.

EXAMPLE 1. Let S be the polygonal region in Figure 1, with each edge of S parallel to one of the coordinate axes. Every vertex of S is clearly visible via S from either point a or point b, yet S is not a union of two starshaped sets. In fact, no point of S sees via S two members of $\{p, q, r\}$.

Finally, while the result in Lemma 1 holds for two points, it fails for three, as Example 2 illustrates.

EXAMPLE 2. Let S be the simply connected set in Figure 2. Every boundary point of S is visible via S from one of the points a, b, c. However, interior point x sees none of these points via S, and S is not a union of three starshaped sets.

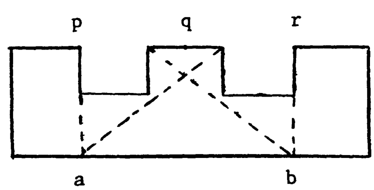

FIGURE 1

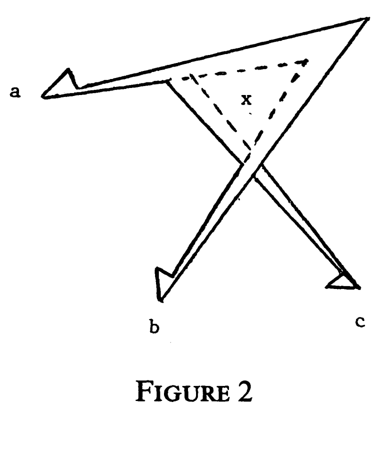

FIGURE 2

3. The main theorem. We will be concerned with the proof of the following theorem.

THEOREM 1. *Let S be a simply connected polygonal region in the plane, symmetric with respect to the x and y axes, such that each edge of S is parallel to one of these axes. Assume that for every set E consisting of 6 or fewer edges of S there exist points t_1 and t_2 collinear with the origin θ (and depending on E) such that every point in $\bigcup\{e\colon e \text{ in } E\}$ is visible via S from t_1 or t_2 (or both). Then S is a union of two starshaped sets. The number 6 is best possible.*

Proof. By Lemma 1, it suffices to show that there exist points t, t' in S such that every boundary point of S sees via S either t or t'. As in **[9]**, it is convenient to order bdry S in a clockwise direction. It is clear that this order, in turn, induces a natural order on each edge of S. Since the edges of S are parallel to the coordinate axes, we may use the terminology employed in **[9]** to classify each horizontal edge of S as a 'right' or a 'left' edge and to classify each vertical edge of S as an 'up' or a 'down' edge, according to the order it inherits from bdry S.

Observe that relative interior points of 'right' edges see via S no points above their corresponding lines. Similar statements can be made for 'left' edges and points below, 'up' edges and points to their left, 'down' edges and points to their right.

Select a 'right' edge e_R whose y coordinate is as small as possible and a 'left' edge e_L whose y coordinate is as large as possible. Let R and L denote the lines determined by e_R and e_L, respectively. Similarly, select 'up' edge e_U whose x coordinate is as large as possible and 'down' edge e_D whose x coordinate is as small as possible, and let U and D denote their associated lines. By an observation above, points of rel int e_R see via S only points on or below R. Of course, parallel statements hold for remaining edges e_L, e_U, e_D and corresponding lines. By symmetry of S, clearly $U \neq D$ and $R \neq L$.

In case line L is below line R and line D is to the right of line U, then it is easy to show that S is starshaped and ker S is exactly the rectangular region bounded by these four lines. In case line L is below line R and line D is to the left of line U, the argument is equivalent to Case 1 below. Hence throughout the remainder of the proof we will assume that line L is above line R. For convenience of notation, let M (middle) denote the closed subset of S bounded by lines L and R. Let T (top) be the closed subset of S whose points are on or above L, and let B (bottom) be the closed subset of S whose points are on or below R. For future reference, observe that each edge of S is a subset of at least one of M, T, or B.

The following notation will be helpful: Let ξ denote symmetry in the x axis and let η denote symmetry in the y axis. Let $L+$ and $L-$ represent closed halfplanes bounded by L with $\operatorname{int} L-$ containing no 'left' edge of $\operatorname{bdry} S$. Similarly, define halfplanes $U+$, $U-$ (with $\operatorname{int} U-$ containing no 'up' edge), $R+$, $R-$, $D+$, $D-$. Note that U, $U+$, $U-$ are the η-images of D, $D+$, $D-$, respectively, while R, $R+$, $R-$ are the ξ-images of L, $L+$, $L-$.

The proof will be accomplished by considering two cases, determined by the positions of U and D.

Case 1. For the moment, assume that line D lies to the right of line U.

First we make some observations.

(1) In Case 1, S is convex in horizontal direction. That is, for $x, y \in S$ and $[x, y]$ horizontal, it follows that $[x, y] \subseteq S$. This holds because for each horizontal line A, each component of $A \cap S$ must have its left endpoint on some 'up' edge (in $U+$) and its right endpoint on some 'down' edge (in $D+$). But $D+$ and $U+$ are disjoint, so $A \cap S$ has at most one component. Analogous reasoning shows that $S \cap D- \cap U-$ is convex in vertical direction and contains the rectangular region $D- \cap U- \cap L+ \cap R+$.

(2) Each region bounded by a simple closed curve in S is contained in S (since S is simply connected). In particular, if x in S sees via S points a and b of edge e, then x sees via S the interval $[a, b]$. Thus (3) below holds.

(3) For $x \in S$ and e any edge of $\operatorname{bdry} S$, the subset of e seen by x is a compact interval (possibly empty).

Now let lines L and R meet the y axis at points p and q, respectively. We will show that for e any edge in $\operatorname{bdry} S$, each point of e sees (always via S) either p or q.

(4) The set $\operatorname{bdry} S \cap \operatorname{int} L-$ consists of 'up' and 'right' edges in $U+$, one 'right' edge crossing $U- \cap D-$, and 'right' and 'down' edges in $D+$. By (1) and (2), it follows that each point of $S \cap L-$ sees p. By ξ-symmetry, each point of $S \cap R-$ sees q.

(5) For e in the remaining part $S \cap L+ \cap R+$ (and not on L or R), select t_1, t_2 collinear with θ to satisfy the hypothesis of the theorem for the edge set

$$\{e, \xi(e), \eta(e), \xi\eta(e), e_R, e_L\}.$$

By definition of e_L and e_R, we may assume $t_1 \in T = S \cap L-$ and $t_2 \in B = S \cap R-$.

(6) If e is a 'left' edge, t_2 cannot see the relative interior of e and of $\eta(e)$, so t_1 sees the whole of e and of $\eta(e)$. By η-symmetry of S, $\eta(t_1)$ also sees the whole of e and of $\eta(e)$. By (1) and (2), $\operatorname{conv}\{t_1, \eta(t_1), e, \eta(e)\}$ is in S. But this set contains p, so p sees each point of e. Similarly, if e is a 'right' edge, q sees each point of e.

(7) If $e = [a, b]$ is vertical and preceded and followed by 'left' ('right') edges, then by (6), point p (point q, respectively) sees a and b, so it sees the whole of e by (3).

(8) If e is an 'up' ('down') edge preceded by a 'right' ('left') edge and followed by a 'left' ('right') edge, then p sees the upper endpoint of e and q sees the lower endpoint of e by (6). Hence $\operatorname{conv}\{e, p, q\}$ has its boundary in S and lies in S by (2). This means that each point of e sees p as well as q.

(9) For the remaining cases we may assume by η-symmetry of S that e is a 'down' edge in $D+ \cap L+ \cap R+$ preceded by a 'right' edge and followed by a 'left' edge. Let a denote the upper endpoint of e and b the lower endpoint of e. Then p sees b and q sees a by (6). To complete the proof, by (3) it suffices to show that some point x_0 of e sees p as well as q.

Now each point of e and of $\xi\eta(e)$ sees t_1 or t_2, and by (3) there are some $x_1 \in e$ and some $w_1 \in \xi\eta(e)$ which see t_1 as well as t_2. So $Q \equiv \operatorname{conv}\{x_1, t_1, w_1, t_2\}$ has its boundary in S and is contained in S by (2). In particular, if we let $\{p_1\} \equiv L \cap [t_1, t_2]$ and $\{q_1\} \equiv R \cap [t_1, t_2] = \xi\eta(p_1)$, then $\operatorname{conv}\{x_1, p_1, q_1\} \subseteq Q \subseteq S$ and x_1 sees p_1 as well as q_1. We assert that some $x_2 \in e$ sees $\{p_2\} \equiv \eta(p_1)$ as well as $q_2 \equiv \eta(q_1)$. In fact by (5) and (3) some $\eta(x_2) \in \eta(e)$ and some $w_2 \in \xi(e) = \xi\eta\eta(e)$ see t_1 and t_2, so by

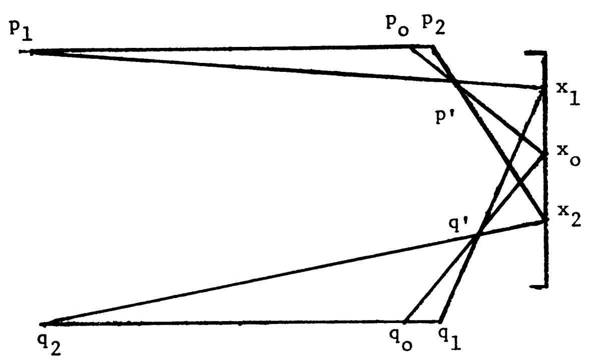

FIGURE 3

If $x_0 = \lambda x_1 + (1-\lambda)x_2$ *and* $\lambda \in [0,1]$, *then* $\lambda p_1 + (1-\lambda)p_2 \in [p_0, p_1]$ *and* $\lambda q_1 + (1-\lambda)q_2 \in [q_0, q_2]$.

the previous reasoning conv$\{\eta(x_2), p_1, q_1\}$ is in S. It follows by η-symmetry of S that conv$\{x_2, p_2, q_2\} \subseteq S$. In particular, $x_2 \in e$ sees p_2 and q_2.

Note that $p = \frac{1}{2}(p_1 + p_2)$ and $q = \frac{1}{2}(q_1 + q_2)$. Now there are two subcases to consider:

In case $[x_1, p_1]$, $[x_2, p_2]$ intersect and $[x_1, q_1]$, $[x_2, q_2]$ also intersect, then we have the situation in Figure 3 (up to permutation of indices 1 and 2). There points p', p_i, x_i are collinear and q', q_i, x_i are collinear, $i = 0, 1, 2$. The assertion of the caption of Figure 3 is verified by elementary geometry and shows for $\lambda = \frac{1}{2}$ that $p = \frac{1}{2}(p_1 + p_2) \in [p_0, p_1]$ and $q = \frac{1}{2}(q_1 + q_2) \in [q_0, q_2]$. By convexity in horizontal direction, we conclude that conv$\{x_0, p_0, p, q, q_0\} \subseteq S$ and in particular that $x_0 \in e$ sees p and q.

In case $[x_1, p_1]$, $[x_2, p_2]$ do not intersect, neither do $[x_1, q_1]$, $[x_2, q_2]$. Then conv$\{x_1, p_1, p_2, x_2\}$ and conv$\{x_1, q_1, q_2, x_2\}$ have their boundaries in S. In particular, letting $x_0 \equiv \frac{1}{2}(x_1 + x_2) \in e$, $[x_0, p] \cup [x_0, q] \subseteq S$. Again $x_0 \in e$ sees p as well as q, the desired result.

We conclude that every point of edge e sees via S either p or q, and hence every boundary point of S sees via S either p or q. Therefore, by Lemma 1, every point of S sees via S either p or q, and S is indeed a union of two starshaped sets. The proof for Case 1 is established.

Case 2. Assume that line D lies to the left of line U.

The following observations will be helpful.

(1) If V is a vertical line meeting S, each component of $V \cap S$ has its top endpoint on a 'right' edge (in R_+) and its bottom endpoint on a 'left' edge (in L_+). Thus $L_- \cap S$ is convex in vertical direction and contains its orthogonal projection onto L. Similarly $U_- \cap S$ is convex in horizontal direction and contains its orthogonal projection onto U. ξ- and η-symmetry give corresponding statements on $R_- \cap S$ and $D_- \cap S$.

For convenience of notation, let A denote the closed rectangular region determined by lines L, R, D, U. Let $A_1, A_1', A_2, A_2', A_3, A_3', A_4, A_4'$ denote the eight closed unbounded regions determined by L, R, D, U labeled in a clockwise direction about A, with A_i meeting A only at the common vertex a_i, $1 \leq i \leq 4$, and with $A_1 = L_- \cap U_-$. (See Figure 4.) Then we observe:

(2) A is contained in S. In fact, choose points v_1 and v_2 which satisfy the hypothesis of Theorem 1 for the collection of edges $\{e_L, \eta(e_L), e_R, \eta(e_R), e_U, \xi(e_U)\}$. From preliminary comments, for $x \in$ (rel int e_L) $\cup$ (rel int $\eta(e_L)$), $y \in$ (rel int e_R) $\cup$ (rel int $\eta(e_R)$), and $z \in$ (rel int e_U) $\cup$ (rel int $\xi(e_U)$), x sees via S at most points of L_-, y sees at most points of R_-,

z sees at most points of U_-. Hence for an appropriate labeling, we have $v_1 \in L_-$ and each point of $e_L \cup \eta(e_L)$ sees v_1, $v_2 \in R_-$ and each point of $e_R \cup \eta(e_R)$ sees v_2. Furthermore, by the collinearity of v_1, θ, v_2, one and only one of the points v_1, v_2 sees $e_U \cup \xi(e_U)$. If this point is v_1, then $v_1 \in A_1 = L_- \cap U_-$ and v_1 sees e_L, $\eta(e_L)$, e_U, $\xi(e_U)$ via S. By observation (1) it follows that S contains the subset of bdry$A$ lying in the first quadrant. By the symmetry of S, bdry$A \subseteq S$, and since S is simply connected, $A \subseteq S$. Similarly, we can conclude $A \subseteq S$ if v_2 sees $e_U \cup \xi(e_U)$.

From observations (1) and (2) we get:

(3) If V is a vertical line in $D_+ \cap U_+$, then $V \cap S$ is a segment. Hence $D_+ \cap U_+ \cap S$ is convex in vertical direction. Similarly, $L_+ \cap R_+ \cap S$ is convex in horizontal direction.

For future reference, observe that since $A \subseteq S$, no relative interior point of any edge e_L, e_R, e_U, e_D can meet A. Therefore, by the symmetry of S, we may assume that these edges are labeled so that $e_L \subseteq A_1$, $e_U \subseteq A_2$, $e_R \subseteq A_3$, and $e_D \subseteq A_4$. Thus $\eta(e_L) \subseteq A_4$, $\xi(e_U) \subseteq A_1$, $\eta(e_R) \subseteq A_2$, and $\xi(e_D) \subseteq A_3$.

To complete the proof in Case 2, we will show that each boundary point d of S sees via S two consecutive points from $(a_1, a_2, a_3, a_4, a_1)$, the vertex set of A. Let e be an edge of S containing point d. Clearly e lies in one of the closed halfplanes determined by each line L, R, D, and U. Since $A \subseteq S$,either $e \subseteq A_i$ or $e \subseteq A_i'$ for some i, $1 \leq i \leq 4$. First consider the situation in which $e \subseteq A_i$ for some i, and for convenience of notation, assume $i = 1$. Choose points t_1 and t_2 satisfying the hypothesis of Theorem 1 for $\{e, \eta(e), e_L, e_U, e_R, e_D\}$.

Using earlier comments, it is clear that for an appropriate labeling of t_1 and t_2, one of the following must occur: Either $t_1 \in A_4$ and $t_2 \in A_2$ or $t_1 \in A_1$ and $t_2 \in A_3$. In the first case, using observation (1), each point of e which sees t_1 must see a_4 and a_1 as well. Each point of e which sees t_2 must see a_1 and a_2. Thus point d sees two consecutive vertices of A. In the second case, each point of $\eta(e)$ which sees t_1 must see a_4 and a_1, while each point of $\eta(e)$ which sees t_2 must see a_4 and a_3. By the η-symmetry of S, this forces each point of e to see either a_1 and a_4 or a_1 and a_2, again the desired result.

Now consider the situation in which $e \in A_i'$ for some i, and by symmetry of our assumptions we may assume $i = 1$. We select points t_1 and t_2 satisfying our hypothesis for the collection of edges $\{e, \xi(e), e_L, e_R, e_U, e_D\}$. As before, for an appropriate labeling, one of the following must occur: Either $t_1 \in A_4$ and $t_2 \in A_2$ or $t_1 \in A_1$ and $t_2 \in A_3$.

There are three cases to investigate, depending on the classification of edge e.

Case A. Here we examine the case in which e is a horizontal edge in A_1'. Assume e is a 'left' edge. Then $\xi(e)$ is a 'right' edge. If $t_1 \in A_4$ and $t_2 \in A_2$, then $\operatorname{conv}\{t_2 \cup \xi(e)\} \subseteq S$. By symmetry, $\operatorname{conv}\{\xi(t_2) \cap e\} \subseteq S$, and in particular e sees some point of $L \cap A_1$. The same holds if $t_1 \in A_1$ and $t_2 \in A_3$. But since $L_+ \cap R_+ \cap S$ is convex in horizontal direction and η-symmetric, it follows that e sees both a_4 and a_1 via S, the desired result. Symmetrically, if e is a 'right' edge, e sees both a_2 and a_3 via A.

Case B. Now we assume that e is a 'down' edge in A_1' and either preceded by a 'left' edge or followed by a 'right' edge (or both). If e is preceded by a 'left' edge f, then by Case A each point of f and in particular the top point of e sees via S both a_4 and a_1. From observation (3), it follows that every point of e sees both a_4 and a_1 via S. Similarly, if e is followed by a 'right' edge, the bottom point of e sees both a_2 and a_3. Hence every point of e sees both a_2 and a_3 via S.

Case C. It remains to consider the case in which e is a 'down' edge in A_1' which is preceded by a 'right' edge and followed by a 'left' edge. For convenience of notation, let $e = [a, b]$ where a is above b. Because e_L is the first edge in A_1', $a \notin L$. Similarly, $b \notin D$. Recall that every point of the 'down' edges e and $\xi(e)$ (which may coincide) sees via S either t_1 or t_2. By the ξ-symmetry of S, every point of e and $\xi(e)$ sees via S either $\xi(t_1) \equiv s_1$ or $\xi(t_2) \equiv s_2$. There exist points x_1 and x_2 on e such that x_1 sees via S both t_1 and t_2 while x_2 sees via S both s_1 and s_2. Without loss of generality, we may assume that $t_1 \in A_4$ and $t_2 \in A_2$. Hence $s_1 \in A_3$ and $s_2 \in A_1$. Observe that none of the segments $[a, t_1]$, $[a, s_2]$, $[b, s_1]$, $[b, t_2]$ lie in S. (See Figure 4.) Both a and x_1 see t_2 via S, so $\operatorname{conv}\{a, x_1, t_2\} \subseteq S$. In particular, every point of $[a, x_1]$ sees some point of $R \cap A_2$, and by observations (2) and (3), it follows that every point of $[a, x_1]$ sees both a_2 and a_3 via S. Similarly, since both b and x_2 see s_2 via S, we conclude that every point of $[x_2, b]$ sees both a_4 and a_1. If x_2 is above x_1 or if $x_1 = x_2$, the argument is finished. If x_1 is above x_2, it remains to show that every point of $[x_1, x_2]$ sees two consecutive vertices of A, and it is helpful to consider two complementary subcases.

Subcase C1. Here we assume that $[t_1, t_2]$ meets the segment $[a_4, a_1]$. Because of $\theta \in [t_1, t_2]$ and ξ-symmetry, $[t_1, t_2]$ as well as $[s_1, s_2]$ meet both

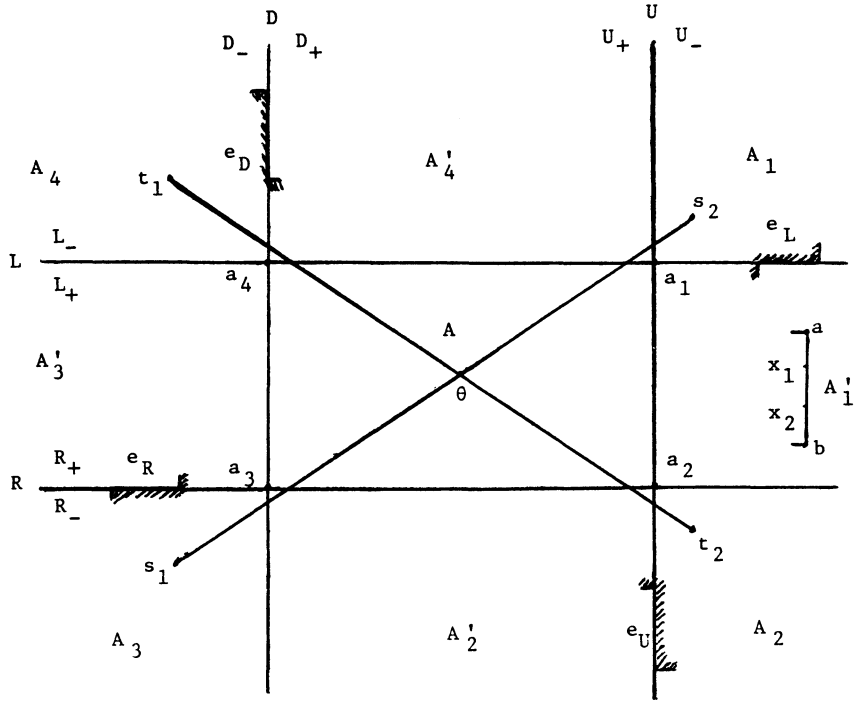

FIGURE 4

segments $[a_4, a_1]$ and $[a_2, a_3]$. Then $[x_1, t_1]$ meets D in a point $y_1 \in A_4 \cap D$ and $[x_2, s_1]$ meets D in a point $y_2 \in A_3 \cap D$. By construction, $[x_1, t_1] \subseteq S$ and $[x_2, s_1] \subseteq S$, so using observation (3), we conclude that $[x_1, y_1] \cup [y_1, y_2] \cup [y_2, x_2] \cup [x_2, x_1] \subseteq S$. But S is simply connected, so $\operatorname{conv}\{x_1, y_1, y_2, x_2\} \subseteq S$. Since a_3 and a_4 are on $[y_1, y_2]$, it follows that every point of $[x_1, x_2]$ sees via S both a_3 and a_4.

Subcase C2. Here we assume that $[t_1, t_2]$ meets the segment $[a_3, a_4]$ in its relative interior. (See Figure 5.) Then $[\theta, t_1] \cap L \equiv p_1 \in A_4 \cap L$, $[\theta, s_1] \cap R \equiv q_2 \in A_3 \cap R$, $[\theta, s_2] \cap L \equiv p_2 \in A_1 \cap L$, and $[\theta, t_2] \cap R \equiv q_1 \in A_2 \cap R$. By construction, $[x_2, s_2]$ and $[x_1, t_2]$ lie in S. Thus by observations (2) and (3), $[\theta, p_2] \cup [\theta, q_1] \subseteq S$. By η-symmetry, $[\theta, p_1] \cup [\theta, q_2] \subseteq S$ as well. Since x_1 sees both t_1 and t_2 via s, it follows from observation (3) that x_1 sees p_1 and q_1 via S and also sees a_2 and a_3 via S. Similarly, we conclude that x_2 sees p_2 and q_2 via S and also sees a_1 and a_4 via S. So we have again the situation of Figure 3. Determine $\lambda \in [0, 1]$ such that $a_1 = \lambda p_1 + (1 - \lambda)p_2$. Then $a_3 = \lambda q_1 + (1 - \lambda)q_2$. The conclusion drawn from Figure 3 in Case 1, Step 9, yields here that the point $x_0 \equiv \lambda x_1 + (1 - \lambda)x_2$ sees both a_1 and a_3. Likewise, determine $\lambda' \in [0, 1]$

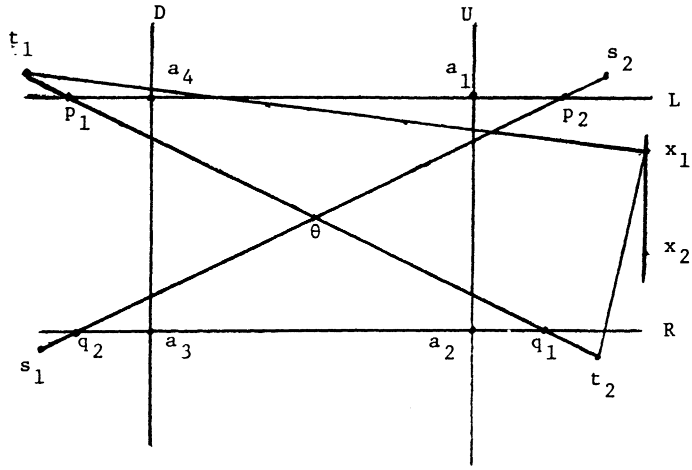

FIGURE 5

such that $a_4 = \lambda' p_1 + (1 - \lambda') p_2$. Then $a_2 = \lambda' q_1 + (1 - \lambda') q_2$, and as before the point $x_0' \equiv \lambda' x_1 + (1 - \lambda') x_2$ sees both a_2 and a_4. We have found that both x_1 and x_0 see a_3, and both x_0 and x_2 see a_1. Hence every point of $[x_1, x_0]$ sees a_3 and every point of $[x_0, x_2]$ sees a_1. Both x_1 and x_0' see a_2 and both x_0' and x_2 see a_4, so every point of $[x_1, x_0']$ sees a_2 and every point of $[x_0', x_2]$ sees a_4. In conclusion, every point of $[x_1, x_2]$ sees via S either a_2 or a_4, and every point of $[x_1, x_2]$ sees via S either a_1 or a_3. We have the required result in Subcase C2.

We have proved that each boundary point of S sees via S two consecutive vertices from $(a_1, a_2, a_3, a_4, a_1)$. Therefore, each boundary point of S sees via S either a_1 or a_3. By Lemma 1, each point of S sees via S either a_1 or a_3, and S is a union of two starshaped sets in Case 2. This finishes the proof of Theorem 1.

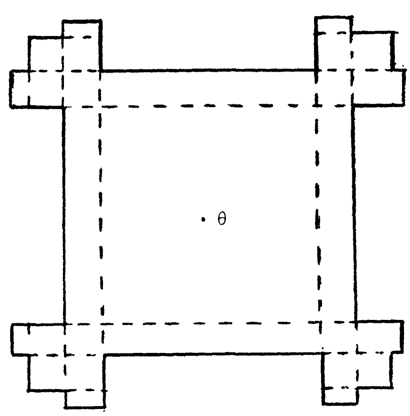

FIGURE 6

To see that the number 6 in Theorem 1 is best possible, consider the following example.

EXAMPLE 3. Let S be the simply connected set in Figure 6. Set S is symmetric with respect to the x and y axes, and for every set E consisting of 5 or fewer edges of S, there correspond points t_1 and t_2 collinear with θ such that every point in $\bigcup\{e\colon e \text{ in } E\}$ is visible via S from t_1 or t_2. (Of course, t_1 and t_2 are not necessarily symmetric with respect to the origin.) However, S is not a union of two starshaped sets.

We conclude the paper with an example adapted from [**5**, pp. 11–12] which reveals that no finite Krasnosel'skii number exists to characterize arbitrary unions of two or more compact starshaped sets, even in the plane. Hence the Krasnosel'skii-type result (2) in our introduction ([**2**, Theorem 2]) is best possible. Moreover, some restrictions like those in Theorem 1 must be imposed on our sets to obtain better results.

EXAMPLE 4. Let n be fixed, $n \geq 2$. For $1 \leq i \leq 2n$, let C_i, C_i' be arcs on the unit circle defined in polar coordinates as follows:

$$C_i = \left\{(1, \theta)\colon (i - n + 1)\frac{\pi}{2n} \leq \theta \leq (i + n - 1)\frac{\pi}{2n}\right\},$$
$$C_i' = \left\{(1, \theta)\colon (i + n + 1)\frac{\pi}{2n} \leq \theta \leq (i + 3n - 1)\frac{\pi}{2n}\right\}.$$

Let a_i and b_i denote the endpoints of C_i, with a_i' and b_i' the endpoints of C_i'. Finally, let $S_n = \bigcup\{\operatorname{conv} C_i \cup \operatorname{conv} C_i'\colon 1 \leq i \leq 2n\}$. (Figure 7 illustrates set S_n when $n = 2$.)

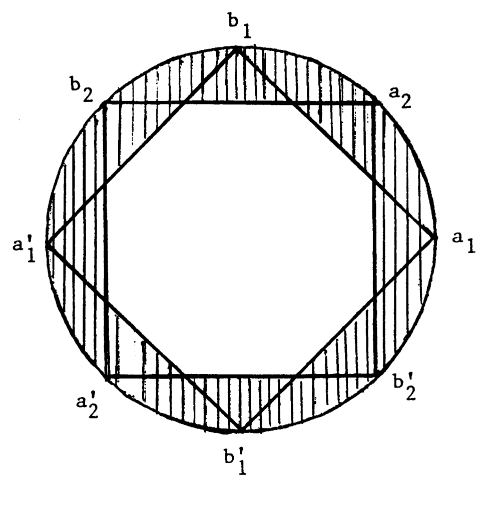

FIGURE 7

Using an argument from [**5**, p. 11], every $2n - 1$ of the pairs of arcs C_i, C_i' have a common pair of antipodal points. Hence for every $(2n - 1)$-member subset T of S_n, there correspond two points s, t (depending on T) such that each point of T is visible via S from s or t. However, there is no line meeting all $4n$ arcs, and it is not hard to show that there are no two points s', t' satisfying the condition above for the set of midpoints

$$T' = \left\{\tfrac{1}{2}(a_i + b_i), \tfrac{1}{2}(a_i' + b_i'): 1 \leq i \leq 2n\right\}.$$

Since n is arbitrary, it is clear that no finite Krasnosel'skii number exists for the general case. Thus the result in [**2**, Theorem 2] is best possible.

The author would like to thank the referee for providing the improved and shortened version of the proof of Theorem 1, Case 1, and for carefully reworking the text and figures of Theorem 1, Case 2 so that an assertion in Subcase C2 could be justified efficiently.

References

[1] Marilyn Breen, *An example concerning unions of two starshaped sets in the plane*, Israel J. Math., **17** (1974), 347–349.

[2] ______, *Clear visibility and unions of two starshaped sets in the plane*, Pacific J. Math., (to appear).

[3] Ludwig Danzer and Branko Grünbaum, *Intersection properties of boxes in R^d*, Combinatorica, **2** (3) (1982), 237–246.

[4] Ludwig Danzer, Branko Grünbaum and Victor Klee, *Helly's theorem and its relatives*, Proc. Symposia in Pure Math., Vol. VII (Convexity) (1963), 101–180.

[5] Hugo Hadwiger, Hans Debrunner, and Victor Klee, *Combinatorial Geometry in the Plane*, Holt, Rinehart and Winston, New York, 1964.

[6] M. A. Krasnosel'skii, *Sur un critère pour qu'un domain soit étoilé*, Math. Sb., **19** (61) (1946), 309–310.

[7] J. F. Lawrence, W. R. Hare, Jr., and John W. Kenelly, *Finite unions of convex sets*, Proc. Amer. Math. Soc., **34** (1972), 225–228.

[8] Steven R. Lay, *Convex Sets and Their Applications*, John Wiley, New York, 1982.

[9] Godfried T. Toussaint and Hossan El-Gindy, *Traditional galleries are star-shaped if every two paintings are visible from some common point*, Amer. Math. Monthly, (to appear).

[10] F. A. Valentine, *Convex Sets*, McGraw Hill, New York, 1964.

Received February 21, 1984.

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