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A NOTE ON LOCALLY A-PROJECTIVE GROUPS

ULRICH ALBRECHT

If A is an abelian group, then a group G is locally A-projective if every finite subset of G is contained in a direct summand P of G which is isomorphic to a direct summand of $\oplus_I A$. Under the assumption that A is a torsion-free, reduced abelian group with a semi-prime, right and left Noetherian, hereditary endomorphism ring, various results on locally A-projective groups are proved that generalize structure theorems for homogeneous, separable, torsion-free abelian groups.

1. Introduction. Since the publication of Baer's paper on torsion-free abelian groups [**6**] in 1937, many attempts have been made to give structure theorems for classes of torsion-free abelian groups reaching beyond the case of completely decomposable groups. However, even in the case of separable torsion-free abelian groups, only the homogeneous case yields some interesting results whose proofs heavily depend on the well-known structure of subgroups of the rationals $\mathbf{Q}$. Naturally, the question arises whether the results themselves depend on the consideration of subgroups of $\mathbf{Q}$ too.

A first step in answering this question was done by Arnold and Lady in 1975. In [**4**], they introduced the following generalization of the class of homogeneous, completely decomposable groups. If A is a torsion-free, reduced abelian group, then a group G is A-projective if it is isomorphic to a direct summand of $\oplus_I A$. In the case that both, A and G, are torsion-free and have finite rank, Arnold and Lady were able to show that most properties of homogeneous, completely decomposable groups still hold in the more general setting that the endomorphism ring $E(A)$ of A is right hereditary. In [**13**], Huber and Warfield showed that under these conditions on A, the ring $E(A)$ is semi-prime, right and left Noetherian, and hereditary. Using this result, the author was able to remove the finite rank condition from Arnold's and Lady's results [**1**]. These results are summarized in Lemma 3.1 of this paper.

The progress made suggests the question whether a similar generalization is possible for homogeneous, separable torsion-free groups. In [**5**], Arnold and Murley began the discussion for torsion-free abelian groups A such that $E(A)$ is a principal ideal domain, and $E(A)/I$ is torsion for all non-zero ideals I of $E(A)$. They called an abelian group G locally A-projective if every finite subset of G is contained in an A-projective direct summand of G. However, compared with [4], these conditions on A are rather restrictive even if A has finite rank. In view of the results in [1], a generalization to the case of torsion-free, reduced abelian groups A with a semi-prime, right and left Noetherian, hereditary endomorphism ring would provide a generalization of torsion-free separable groups which besides having an interest of its own as a structure theory for a rather large class of groups would also give a deeper understanding of homogeneous, separable groups.

The goal of this paper is to present such a generalization by discussing locally A-projective groups using properties of locally projective $E(A)$-modules, i.e. of $E(A)$-modules M such that every finite subset of M is contained in a projective direct summand of M. The module-theoretic results needed for this are given in §2. The key result is

PROPOSITION 2.2. *Let R be a semi-prime, right and left Noetherian, hereditary ring. An R-module M is locally projective if and only if M is isomorphic to a submodule M' of $\Pi_I R$ for some index set I such that $(\Pi_I R)/M'$ is a non-singular R-module.*

It should be remarked that Chase proved the results in §2 in the case that R is a principal ideal domain [7], but his proofs do not carry over to the general setting.

Before it is possible to use the information obtained in §2, it is necessary to consider the finite topology on the endomorphism ring of an A-projective group in order to be able to compare the results of this paper with those in [5]. In this, as in [1], it becomes apparent that the usual notion of purity does not yield the generality desired in this paper. To overcome this difficulty, the notion of almost $\{A\}_*$-purity is introduced in §4. If for a pair (A, G) of abelian groups, $S_A(G) = \Sigma\{f(A)\colon f \in \mathrm{Hom}(A, G)\}$, then a subgroup H of an abelian group G with $S_A(G) = G$ is almost $\{A\}_*$-pure in G if $S_A(H) = H$, and H is a direct summand of $H + f(A)$ for all $f \in \mathrm{Hom}(A, G)$. Special emphasis is given to the consideration of almost $\{A\}_*$-pure subgroups of locally A-projective groups. Moreover, it is outlined how almost $\{A\}_*$-purity relates to purity in this case.

Now, it is possible to formulate and prove

THEOREM 5.1. *Let A be a torsion-free, reduced abelian group with a semi-prime, right and left Noetherian, hereditary endomorphism ring. The* *following are equivalent for a torsion-free abelian group G:*

(i) *G is locally A-projective.*

(ii) *$S_A(G) = G$, and G is isomorphic to an almost $\{A\}_*$-pure subgroup of $S_A(\Pi_I A)$ for some index set I.*

The results in §4 show that in Theorem 5.1 almost $\{A\}_*$-purity can be replaced by purity if in addition A/U is torsion for all subgroups $U \cong A$ of A. This yields the exact formulation of Arnold's and Murley's result. This last condition is satisfied for all groups considered in their paper, but also for all torsion-free reduced groups of finite rank with a right hereditary endomorphism ring. In view of [**4**], a further reduction of the conditions on A seems very hard to achieve.

2. Locally projective R-modules. Many of the results of this section have been proved by Chase in [**7**] for the case that R is a principal ideal domain, but the proofs do not carry over even to Dedekind domains.

Using the notation of [**12**], the annihilator of a subset X of a left R-module M is $\operatorname{ann}_M(X) = \{r \in R: rX = 0\}$. Clearly, $\operatorname{ann}_M(X)$ is a left ideal of R. For any left R-module M, let $Z(M) = \{m \in M: \operatorname{ann}_M(m)$ is an essential left ideal of $R\}$. M is non-singular if $Z(M) = 0$. In particular, if R is a principal ideal domain, then $Z(M)$ is the torsion-submodule of M.

For the remainder of this section, R will denote a semi-prime, right and left Noetherian, hereditary ring, i.e.

(i) $I^2 \neq 0$ for all non-zero ideals I of R,

(ii) every right (left) ideal of R is finitely generated as a right (left) R-module, and

(iii) every right (left) ideal of R is projective as a right (left) R-module.

In this case, $Z(R) = 0$ by [**8**, Theorem 1.6], and $Z(M/Z(M)) = 0$ for every R-module M by [**12**, Proposition 1.23a]. Moreover, a right (left) ideal of R is essential if and only if it contains a regular element of R, i.e. it contains an element r such that $rs = 0$ or $sr = 0$ implies $s = 0$ for all $s \in R$ [**8**, Theorem 1.10 and Lemma 1.11]. Consequently, an R-module M is non-singular if and only if $rm \neq 0$ for all $0 \neq m \in M$ and all regular elements r or R.

THEOREM 2.1. *Let R be a semi-prime, right and left Noetherian, hereditary ring. If M is a finitely generated R-module, then $M = P \oplus Z(M)$ where P is a finitely generated, projective R-module.*

Proof. R has a semi-prime, Artinian right and left classical quotient ring which is also right and left maximal [**12**, Theorem 3.37]. By [**12**, Theorem 3.10 and Theorem 5.17], every finitely generated, non-singular R-module can be embedded in a free R-module. Hence, $M/Z(M)$ is projective since R is hereditary. Therefore, $M = P \oplus Z(M)$ where P is a finitely generated, projective R-module.

Another important property of modules over Noetherian rings was discovered by Jensen, see [**11**, Satz 6.2] for instance. He showed that $\Pi_I R$ is a locally projective right and left R-module if R is right and left Noetherian.

PROPOSITION 2.2. *Let R be a semi-prime, right and left Noetherian, hereditary ring. An R-module M is locally projective if and only if it is isomorphic to a submodule M' of $\Pi_I R$ such that $(\Pi_I R)/M'$ is a non-singular R-module.*

Proof. Suppose M is locally projective. Define

$$\mathfrak{M} = \{P\colon P \text{ is a finitely generated, projective summand of } M\}.$$

For every $P \in \mathfrak{M}$, choose a projection $\pi_P\colon M \to P$ which is the identity on P and define $\Phi\colon M \to \Pi_{P\in\mathfrak{M}} P$ by $\Phi(m) = (\pi_P(m))_{P\in\mathfrak{M}}$.

If $m \in \ker \Phi$, choose a finitely generated, projective direct summand P_0 of M containing m. Then, $0 = \pi_{P_0}(m) = m$, and Φ is a monomorphism.

To show that $(\Pi_{P\in\mathfrak{M}} P)/\Phi(M)$ is non-singular, let $x \in \Pi_{P\in\mathfrak{M}} P$ such that $cx \in \Phi(M)$ for some regular element c of R, say $cx = \Phi(m) = (\pi_P(m))_{P\in\mathfrak{M}}$. Write $x = (x_P)_{P\in\mathfrak{M}}$ with $x_P \in P$. Since M is locally projective, there is $P_1 \in \mathfrak{M}$ such that $m \in P_1$. Then, $m = \pi_{P_1}(m) = cx_{P_1}$ and $c\Phi(x_{P_1}) = \Phi(m)$. Hence, $c(x - \Phi(x_{P_1})) = 0$. Consequently, $x - \Phi(x_{P_1}) \in Z(M) = 0$. Therefore, $(\Pi_{P\in\mathfrak{M}} P)/\Phi(M)$ is a non-singular left R-module.

For each $P \in \mathfrak{M}$, choose Q_P such that $P \oplus Q_P$ is a finitely generated, free R-module. Then, $\Pi_{P\in\mathfrak{M}} P$ is a direct summand of $\Pi_{P\in\mathfrak{M}}(P \oplus Q_P)$, and the latter module is isomorphic to $\Pi_I R$ for some index set I. Thus, $(\Pi_{P\in\mathfrak{M}}(P \oplus Q_P))/\Phi(M)$ is non-singular.

Conversely, suppose $M \subseteq \Pi_I R$ such that $(\Pi_I R)/M$ is non-singular. If $\{m_1,\dots,m_n\} \subseteq M$, then there is a finitely generated, projective summand of P of $\Pi_I R$ containing $m_1,\dots,m_n$ by Jensen's result. Let $U = \sum_{i=1}^n Rm_i$ and choose a submodule V of P containing U such that $V/U = Z(P/U)$. Since $(\Pi_I R)/M$ is non-singular, V is contained in M.

Moreover, $Z(P/V) = Z((P/U)/(V/U)) = Z((P/U)/Z(P/U)) = 0$. Therefore, P/V is a finitely generated, non-singular R-module. Consequently, P/V is projective, and V is a summand of $\Pi_I R$. But then, V is projective summand of M containing $m_1,\dots,m_n$.

From this last result, some important corollaries can be deduced which have an interest of their own, although they will not be used in the following.

COROLLARY 2.3. *Let R be a semi-prime, right and left Noetherian, hereditary ring, and let M be a left R-module.*

(i) *If M is locally projective, then every submodule U of M with M/U non-singular is locally projective.*

(ii) *Every submodule U of* $\mathrm{Hom}_R(M, R)$ *with* $\mathrm{Hom}_R(M, R)/U$ *non-singular is a locally projective right R-module.*

Proof. (i) is obvious.

For (ii) consider a projective resolution of M, say

$$0 \to K \to \bigoplus_I R \to M \to 0.$$

Applying the functor $\mathrm{Hom}_R(-, R)$ induces a sequence

$$0 \to \mathrm{Hom}_R(M, R) \to \Pi_I R \to \mathrm{Hom}_R(K, R).$$

Since $\mathrm{Hom}_R(K, R)$ is a non-singular right R-module, $\mathrm{Hom}_R(M, R)$ is locally projective by Proposition 2.2. An application of (i) completes the proof.

For the sake of an easier notation, denote $\mathrm{Hom}_R(M, R)$ by M^*. There is a natural transformation i_M: $M \to M^{**}$ given by $i_M(m)(\varphi) = \varphi(m)$ for all $\varphi \in M^*$ and all $m \in M$.

THEOREM 2.4. *Let R be a semi-prime, right and left Noetherian, hereditary ring. A left R-module M is locally projective if and only if i_M is a monomorphism, and $M^{**}/i_M(M)$ is non-singular.*

Proof. Let $\varphi \in M^{**}$ such that $c\varphi \in i_M(M)$ for some regular element c of R, say $c\varphi = i_M(m)$. Suppose, $x \in \ker i_M$. If M is locally projective, then there is a finitely generated, projective summand U of M containing m and x. The sequence

$$0 \to U \to M \to M/U \to 0$$

splits, and consequently the sequence

$$0 \to (M/U)^* \to M^* \to U^* \to 0$$

is split-exact. Applying $*$ once more induces a commutative diagram

$$\begin{array}{ccccc} 0 & \to & U^{**} & \to & M^{**} \\ & & \uparrow i_U & & \uparrow i_M \\ 0 & \to & U & \to & M. \end{array}$$

Since U is finitely generated and projective, i_U is an isomorphism by [**14**, Theorem 5.1]. Therefore, $0 = i_M(x) = i_U(x)$ implies $x = 0$. Moreover, since $c\varphi = i_M(m)$ and $m \in U$, one has $c\varphi \in i_M(U)$. Consequently, $\varphi + U^{**}$ is an element of $Z(M^{**}/U^{**})$. Since U^{**} is a direct summand of M^{**}, the module M^{**}/U^{**} is non-singular. Therefore, there is $u \in U$ such that $\varphi = i_U(u) = i_M(u)$. Consequently, $M^{**}/i_M(M)$ is non-singular.

The converse is an immediate consequence of Corollary 2.3.

3. A-projective Abelian groups and the finite topology. An abelian group is self-small if the functor $\mathrm{Hom}(A, -)$ preserves direct sums of copies of A. For these groups A, $\mathrm{Hom}(A, -)$ induces a category equivalence between the category of A-projective abelian groups and the category of projective right $E(A)$-modules. Its inverse is given by the functor $- \otimes_{E(A)} A$ using that A is a left module over its endomorphism ring. This category equivalence was introduced by Arnold, Lady, and Murley in [**4**] and [**5**]. Using it, the author was able to prove the following result in [**1**]. Because of the point of view taken there, it was stated differently, but the proof carries over literally.

As in [**4**], $S_A(G)$ is written for the image of the natural map

$$\theta_G \colon \mathrm{Hom}(A, G) \otimes_{E(A)} A \to G$$

defined by $\theta_G(f \otimes a) = f(a)$. It is easy to show that $S_A(G) = G$ if and only if G is an epimorphic image of an A-projective group.

LEMMA 3.1. *Let A be a torsion-free, reduced, self-small abelian group which is flat as a left $E(A)$-module. If $E(A)$ is right hereditary, then*

(i) *every exact sequence*

$$0 \to B \to G \to P \to 0$$

where P is A-projective, and $S_A(G) + B = G$ splits, and

(ii) *every subgroup B of an A-projective group with $S_A(B) = B$ is A-projective.*

In this paper, interest concentrates on torsion-free reduced abelian groups A whose endomorphism ring is semi-prime, right and left Noetherian, and hereditary.

PROPOSITION 3.2. *A torsion-free, reduced abelian group A whose endomorphism ring is semi-prime, right and left Noetherian, and hereditary is self-small. Moreover, it is flat as a left $E(A)$-module, each regular element of*

$E(A)$ is a monomorphism, and conditions (i) *and* (ii) *of Lemma* 3.1 *are valid for A.*

Proof. Let φ be a regular element of $E(A)$, and $a \in \ker \varphi$. Then, $E(A)/\mathrm{ann}_A(a) \cong E(A)a \subseteq A$. Since $\varphi \in \mathrm{ann}_A(a)$, and $E(A)$ is a semi-prime, right and left Noetherian ring, $\mathrm{ann}_A(a)$ is an essential left ideal of $E(A)$. Because a semi-prime, right and left Noetherian, hereditary ring has Krull dimension 1, $E(A)/\mathrm{ann}_A(a)$ is an Artinian left $E(A)$-module by [**8**, Theorem 8.21].

Define $E(A)$-submodules U_n of $E(A)a$ by $U_n = E(A)n!a$ for all non-negative integers n. Since $U_{n+1} \subseteq U_n$ for all n, there is $n_0 < \omega$ such that $U_{n+1} = U_n$ for all $n_0 \leq n < \omega$. Here ω denotes the first infinite ordinal number. Consequently, U_{n_0} is divisible. Since A is reduced, $0 = U_{n_0} = n_0!E(A)a$. But A is torsion-free implies that $E(A)a = 0$. Therefore, φ is a monomorphism. By [**2**, Theorem 5.1], A is self-small.

Moreover, since the regular elements of $E(A)$ are monomorphisms, A is a non-singular $E(A)$-module. In particular, every finitely generated $E(A)$-submodule of A is projective by Theorem 2.1. By [**15**, Corollary 3.31], A is flat as an $E(A)$-module.

Actually, the proof of [**2**, Theorem 5.1] shows more than the fact that A is self-small. On $E(A)$, a topology called the finite topology is defined by taking

$$\{\mathrm{ann}_A(x) \colon X \subseteq A \text{ finite}\}$$

as a basis of neighborhoods of 0. The proof shows that $E(A)$ is discrete in this topology. In [**5**], Arnold and Murley studied A-projective groups whose endomorphism ring is discrete in the finite topology. In the remaining part of this section, these groups are characterized in terms of their A-rank where the A-rank of an A-projective group P is defined to be the smallest cardinal number $\aleph$ such that P is an epimorphic image of $\oplus_{\aleph} A$.

Proposition 3.3. *Let A be a torsion-free, reduced abelian group such that $E(A)$ is a semi-prime, right and left Noetherian, hereditary ring. An A-projective group has finite A-rank if and only if its endomorphism ring is discrete in the finite topology.*

Proof. In a first step, it is shown that an A-projective group $P = \oplus_n A$ for $n < \omega$ has an endomorphism ring which is discrete in the finite topology. By the remarks preceding this proposition, there is a finite subset X of A such that $\mathrm{ann}_A(X) = 0$. Without loss of generality, assume $0 \in X$.

For every $i \in \{1,\ldots,n\}$, let $\delta_i\colon A \to P$ denote the embedding into the ith-coordinate, while $\pi_i\colon P \to A$ denotes the projection onto the ith-coordinate. Define

$$Y = \left\{ \sum_{i=1}^{n} \delta_i(x_i)\colon x_i \in X \text{ for } i = 1,\ldots,n \right\}.$$

Y is a finite subset of P. Suppose there is $0 \neq \varphi \in \operatorname{ann}_P(Y)$. Since $\varphi \neq 0$, there is $a_0 \in A$ and $i_0 \in \{1,\ldots,n\}$ such that $\varphi\delta_{i_0}(a_0) \neq 0$. Moreover, there is $j_0 \in \{1,\ldots,n\}$ with $\pi_{j_0}\varphi\delta_{i_0}(a_0) \neq 0$, i.e. $0 \neq \pi_{j_0}\varphi\delta_{i_0}$ is an element of $E(A)$ which annihilates X because for all $x \in X$, $\delta_{i_0}(x) \in Y$. The resulting contradiction shows that $\operatorname{ann}_P(Y) = 0$.

In the second step, assume $P \oplus Q = \oplus_n A$, and let $\pi\colon \oplus_n A \to P$ be the projection onto P with kernel Q. In order to show that $E(P)$ is discrete in the finite topology, let $Z = \pi(Y)$ where Y is defined as in the first step.

Suppose $\rho \in \operatorname{ann}_P(Z)$. Extend ρ to a map $\bar{\rho} \in E(\oplus_n A)$ by defining $\bar{\rho}(g) = \rho\pi(g)$ for all $g \in \oplus_n A$. For all $y \in Y$, $\bar{\rho}(y) = \rho\pi(y) = 0$. By the first step, $\bar{\rho}$ and hence ρ are zero.

Conversely, suppose P is a direct summand of $\oplus_I A$ whose endomorphism ring is discrete in the finite topology. Choose a finite subset X of P such that $\operatorname{ann}_P(X) = 0$. There is a finite subset J of I such that $X \subseteq \oplus_J A$. Let $\pi\colon \oplus_I A \to \oplus_{I \setminus J} A$ be the projection with kernel $\oplus_J A$. Since $S_A(\pi(P)) = \pi(P)$, $\pi(P)$ is an A-projective group. Therefore, $P = P_1 \oplus P_2$ where $P_1 = P \cap \ker \pi$. Moreover, $X \subseteq P_1$. Thus $\operatorname{ann}_P(X) \neq 0$ if $P_2 \neq 0$, a contradiction. Consequently, $P = P_1$ is a direct summand of $\oplus_J A$, and has finite A-rank.

4. Almost $\{A\}_*$-purity. As [**1**, Theorem 5.1] shows, the concept of purity is not sufficiently general for the discussion of A-projective groups. One generalization of purity was given by C. P. Walker in [**16**]. In that paper, a non-empty class Θ of abelian groups is considered, and purity with respect to Θ is defined by calling a subgroup H of an abelian group G Θ_*-pure if H is a direct summand in every subgroup B of G containing H such that $B/H \in \Theta_{G/H}$ where $\Theta_G = \{f(X)\colon X \in \Theta \text{ and } f \in \operatorname{Hom}(X, G)\}$. In this paper, only the case $\Theta = \{A\}$ is of interest.

Applying this definition to the situation given, the following difficulty arises. In view of the results of [**5**], the generalization needed shall have the property that purity implies the generalized form of purity in the case of pure subgroups of (locally) A-projective groups if $E(A) = \mathbf{Z}$ for instance. However, consider the following example.

Choose a torsion-free abelian group A of countable, infinite rank with $E(A) = \mathbf{Z}$. There is an $E(A)$-projective resolution

$$0 \to \bigotimes_I \mathbf{Z} \to \bigotimes_J \mathbf{Z} \to \mathbf{Q} \to 0.$$

Tensoring with A over $E(A)$ induces an exact sequence

$$0 \to \bigotimes_I A \to \bigotimes_J A \to \mathbf{Q} \bigotimes_{\mathbf{Z}} A \to 0.$$

Since A has countable, infinite rank, $\mathbf{Q} \otimes_{\mathbf{Z}} A \cong \oplus_\omega \mathbf{Q}$. Thus, $\oplus_I A$ is a pure subgroup of $\oplus_J A$ which is not a direct summand since A is reduced. On the other hand, there is a free subgroup F of A such that $F/F_1 \cong \oplus_\omega \mathbf{Q}$. Then, for some subgroup V of A containing F_1, $A/F_1 = V/F_1 \oplus F/F_1$. Consequently, $(\oplus_J A)/(\oplus_I A) \cong A/V \in \Theta_{\oplus_J A}$, i.e. $\oplus_I A$ is not $\{A\}_*$-pure in $\oplus_J A$.

To overcome this difficulty, the following is introduced:

DEFINITION 4.1. Let A be an abelian group. If G is an abelian group with $S_A(G) = G$, then a subgroup H of G with $S_A(H) = H$ is almost $\{A\}_*$-pure in G if for every subgroup U of G with $U \in \{A\}_G$, H is a direct summand of $H + U$.

Obviously, $(H + U)/H \in \{A\}_{G/H}$. Thus, $\{A\}_*$-purity implies almost $\{A\}_*$-purity.

LEMMA 4.2. *Let A be an abelian group. If $H \subseteq K \subseteq G$ are abelian groups with $S_A(H) = H$, $S_A(K) = K$, and $S_A(G) = G$, then H almost $\{A\}_*$-pure in K and K almost $\{A\}_*$-pure in G imply H is almost $\{A\}_*$-pure in G.*

Proof. Let $U \in \{A\}_G$. Then, $K + U = K \oplus W$ for some subgroup W of G. Let π: $K + U \to K$ denote the projection onto K with kernel W. Then, $\pi(H) = H$, and for all $h \in H$ and $u \in U$, one has $h + u = h + \pi(u) + (1 - \pi)(u)$. Therefore, $H + U \subseteq (H + \pi(U)) \oplus W$. Moreover, $\pi(U)$ is a homomorphic image of A in K. Hence, $H + \pi(U) = H \oplus V$ for some subgroup V of K. Thus, H is a direct summand of $H + U$.

Almost $\{A\}_*$-pure subgroups of an abelian group G will be of particular interest if G is a subgroup of $\Pi_I A$ in view of §5. Obviously, this groups can be described by the condition that $R_A(G) = \bigcap\{\ker f\colon f \in \operatorname{Hom}(G, A)\}$ is zero. $R_A(G)$ is the kernel of the natural map φ from G to $\operatorname{Hom}_{E(A)}(\operatorname{Hom}(G, A), A)$ defined by $\varphi(g)(f) = f(g)$.

LEMMA 4.3. *Let A be a torsion-free, reduced abelian group whose endomorphism ring is semi-prime, right and left Noetherian, and hereditary. If $U = (\oplus_n A)/V$ and $R_A(G) = 0$, then U is isomorphic to a subgroup of $\oplus_r A$ for some $r < \omega$. U is A-projective, and* $\mathrm{Hom}(A, U)$ *is a finitely generated right $E(A)$-module.*

Proof. Since U is an epimorphic image of $\oplus_n A$, $\mathrm{Hom}(U, A)$ is a submodule of the finitely generated, free left $E(A)$-module $\mathrm{Hom}(\oplus_n A, A) \cong \oplus_n E(A)$. Because $E(A)$ is Noetherian, $\mathrm{Hom}(U, A)$ is finitely generated as an $E(A)$-module, say by $f_1, \ldots, f_r$. Then, since $R_A(U) = 0$, the map $u \to (f_i(u))$ embeds U into $\oplus_r A$. The rest of the lemma follows from Lemma 3.1.

PROPOSITION 4.4. *Let A be a torsion-free, reduced abelian group whose endomorphism ring is semi-prime, right and left Noetherian, and hereditary. If G is an abelian group with $R_A(G) = 0$ and $S_A(G) = G$, then the following are equivalent for a subgroup H of G with $S_A(H) = H$.*

(i) *H is almost $\{A\}_*$-pure in G.*

(ii) $\mathrm{Hom}(A, G)/\mathrm{Hom}(A, H)$ *is a non-singular right $E(A)$-module.*

(iii) *H is a direct summand of $H + U$ for all subgroups U of G such that $U = (\oplus_n A)/V$ for some $n \in \omega$.*

Proof. (i) $\Rightarrow$ (ii): Suppose, $f \in \mathrm{Hom}(A, G)$ such that for some regular element c of $E(A)$ $fc \in \mathrm{Hom}(A, H)$. Since H is almost $\{A\}_*$-pure in G, $H + f(A) = H \oplus C$. For some g in $\mathrm{Hom}(A, H)$ and h in $\mathrm{Hom}(A, C)$, $f = g + h$. Then, $hc = 0$. Since $R_A(G) = 0$, $\mathrm{Hom}(A, G) \underset{\sim}{\subset} \Pi_I E(A)$ and is non-singular. Hence, $h = 0$. This proves (ii).

(ii) $\Rightarrow$ (iii): By Lemma 4.3. U is A-projective of finite A-rank, and $\mathrm{Hom}(A, U)$ is a finitely generated $E(A)$-module. Thus,

$$(\mathrm{Hom}(A, H) + \mathrm{Hom}(A, U))/\mathrm{Hom}(A, H) \subseteq \mathrm{Hom}(A, G)/\mathrm{Hom}(A, H)$$

is a finitely generated submodule of the nonsingular $E(A)$-module $\mathrm{Hom}(A, G)/\mathrm{Hom}(A, H)$. Therefore, it is projective, and

$$\mathrm{Hom}(A, H) + \mathrm{Hom}(A, U) = \mathrm{Hom}(A, H) \oplus M$$

for some $E(A)$-module M.

By [**1**, Lemma 6.2], the natural map θ_G from $\mathrm{Hom}(A, G) \otimes_{E(A)} G$ to G is a monomorphism. Observing that $S_A(H) = H$ and $S_A(U) = U$, one obtains

$$\begin{aligned}
H + U &= S_A(H) + S_A(U) \\
&= \theta_G\big(\mathrm{Hom}(A, H) \otimes_{E(A)} A\big) + \theta_G\big(\mathrm{Hom}(A, U) \otimes_{E(A)} A\big) \\
&= \theta_G\big((\mathrm{Hom}(A, H) + \mathrm{Hom}(A, U)) \otimes_{E(A)} A\big) \\
&= \theta_G\big(\mathrm{Hom}(A, H) \otimes_{E(A)} A\big) \oplus \theta_G\big(M \otimes_{E(A)} A\big) \\
&= H \oplus \theta_G\big(M \otimes_{E(A)} A\big).
\end{aligned}$$

This proves (iii).

(iii) implies (i) is obvious.

From this, several important corollaries can be deduced.

COROLLARY 4.5. *Let A be a torsion-free, reduced abelian group whose endomorphism ring is semi-prime, right and left Noetherian, and hereditary. If B is an almost $\{A\}_*$-pure subgroup of an A-projective group P such that $S_A(B) = B$ and B is discrete in the finite topology, then B is a direct summand of P.*

Proof. By Lemma 3.1, B is A-projective, and by Proposition 3.3, it has finite A-rank. Consequently, $\mathrm{Hom}(A, B)$ is a finitely generated right $E(A)$-module which is contained in a finitely generated direct summand Q of $\mathrm{Hom}(A, P)$. Since B is almost $\{A\}_*$-pure in P, $Q/\mathrm{Hom}(A, B)$ is a non-singular $E(A)$-module. Since Q is finitely generated, $\mathrm{Hom}(A, B)$ is a direct summand of Q and hence of $\mathrm{Hom}(A, P)$. Tensoring with A over $E(A)$ and observing that the natural maps θ_B and θ_P are isomorphisms shows that $0 \to B \to P \to P/B \to 0$ splits.

COROLLARY 4.6. *Let A be a torsion-free, reduced abelian group whose endomorphism ring is semi-prime, right and left Noetherian, and hereditary. An abelian group G is A-projective of countable A-rank if and only if it is the union of an ascending chain $\{G_n\}_{n<\omega}$ of almost $\{A\}_*$-pure A-projective subgroups of G of finite A-rank.*

Proof. Since A is self-small, $G \cong \mathrm{Hom}(A, G) \otimes_{E(A)} A$ if G is A-projective. Since $E(A)$ is right hereditary, there is a countable family $\{I_n\}_{n<\omega}$ of right ideals of $E(A)$ such that $\mathrm{Hom}(A, G) \cong \oplus_{n<\omega} I_n$. Since $E(A)$ is right Noetherian, $I_n \otimes_{E(A)} A$ is A-projective of finite A-rank.

Conversely, $G_{n+1} = G_n \oplus C_{n+1}$ for all $n < \omega$. If $C_0 = G_0$, then G is the direct sum of the C_i's. Since each C_i is A-projective of finite A-rank, G is A-projective of countable A-rank.

From the last result, it is possible to deduce the following. It was stated in **[1]** under some additional assumptions as Theorem 6.3, but its proof carries over literally to the setting presented here.

COROLLARY 4.7. *Let A be a torsion-free, reduced abelian group with a semi-prime, right and left Noetherian, hereditary endomorphism ring. If G is a subgroup of $\Pi_I A$ which is an epimorphic image of $\oplus_\omega A$, then G is A-projective.*

This section concludes with a discussion of the relation between almost $\{A\}_*$-purity and purity. In this, only pure subgroups of an abelian group G with $R_A(G) = 0$ are considered. Under the conditions of Corollary 4.7 on A, the following gives a sufficient condition on A such that a pure subgroup H of such a G with $S_A(H) = H$ always is almost $\{A\}_*$-pure.

THEOREM 4.8. *Let A be a torsion-free, reduced abelian group such that $E(A)$ is a semi-prime, right and left Noetherian, hereditary ring. The following are equivalent*:

(i) *For every torsion-free abelian group G with $S_A(G) = G$ and $R_A(G) = 0$, the smallest pure subgroup H_* containing a subgroup H of G with $S_A(H) = H$ satisfies $S_A(H_*) = H_*$ and is almost $\{A\}_*$-pure in G.*

(ii) *If U is a subgroup of A which is isomorphic to A, then A/U is torsion.*

(iii) *If U is a subgroup of A which is isomorphic to A, then A/U_* is reduced where U_* is the smallest pure subgroup of A containing U.*

Proof. (i) $\Rightarrow$ (ii): Let U be a subgroup of A with $U \cong A$. The smallest pure subgroup U_* of A containing U is almost $\{A\}_*$-pure in A by condition (i). Therefore, $S_A(U_*) = U_*$. By Lemma 3.1, U_* is A-projective. By Corollary 4.5, $A = U_* \oplus C$ for some subgroup C of A. Then, $E(A) = \operatorname{Hom}(A, U_*) \oplus \operatorname{Hom}(A, C)$. Since $U \cong A$, $\operatorname{Hom}(A, U_*)$ contains a regular element of $E(A)$. Thus, $\operatorname{Hom}(A, U_*)$ is an essential right ideal of $E(A)$, and $\operatorname{Hom}(A, C) = 0$. Since C is a direct summand of A, this is only possible if $C = 0$. Consequently, A/U is torsion.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). In a first step, the implication (iii) $\Rightarrow$ (ii) is shown. Let U be a subgroup of A with $U \cong A$. With the notation from the preceding paragraph, $\mathrm{Hom}(A, U_*)$ is essential right ideal of $E(A)$. Since $E(A)$ has Krull dimension 1, $E(A)/\mathrm{Hom}(A, U_*)$ is an Artinian right $E(A)$-module which is torsion-free as an abelian group. Thus, $E(A)/\mathrm{Hom}(A, U_*)$ is divisible. Applying the functor $-\otimes_{E(A)} A$ induces a commutative diagram

$$\begin{array}{ccccc} E(A) \otimes_{E(A)} A & \to & (E(A)/\mathrm{Hom}(A, U_*)) \otimes_{E(A)} A & \to & 0 \\ \downarrow \theta_A & & \downarrow \theta & & \\ A & \to & A/U_* & \to & 0. \end{array}$$

Since θ_A is an isomorphism, θ is an epimorphism. In particular, A/U_* is divisible. Since it is also reduced, $A/U_* = 0$. This proves (ii).

Now, consider an abelian group G with $R_A(G) = 0$ and $S_A(G) = G$. Let H be any subgroup of G with $S_A(H) = H$. In order to show that H_* is almost $\{A\}_*$-pure in G, it is enough to show that $S_A(H_*) = H_*$ and $\mathrm{Hom}(A, G)/\mathrm{Hom}(A, H_*)$ is a non-singular $E(A)$-module.

To show that $S_A(H_*) = H_*$, let $h \in H_*$. Since $S_A(G) = G$, there is a subgroup W of G containing h which is an epimorphic image of an A-projective group of finite A-rank. Moreover, $R_A(G) = 0$ implies $R_A(W) = 0$. By Lemma 4.3, W is A-projective of finite A-rank.

Consider the exact sequence

$$0 \to \mathrm{Hom}(A, H) \to \mathrm{Hom}(A, H) + \mathrm{Hom}(A, W) \to X \to 0$$

where X is an $E(A)$-submodule of $\mathrm{Hom}(A, H + W)/\mathrm{Hom}(A, H)$. Choose a submodule Y of $\mathrm{Hom}(A, H) + \mathrm{Hom}(A, W)$ containing $\mathrm{Hom}(A, H)$ such that $Y/\mathrm{Hom}(A, H) = Z(X)$. Then the module

$$(\mathrm{Hom}(A, H) + \mathrm{Hom}(A, W))/Y \cong X/Z(X)$$

is finitely generated and non-singular. Consequently, it is projective. Applying the functor $-\otimes_{E(A)} A$ gives

$$(\mathrm{Hom}(A, H) + \mathrm{Hom}(A, W)) \otimes_{E(A)} A$$

equals $Y \otimes_{E(A)} A \oplus P$ for some A-projective group P of finite A-rank. Applying the isomorphism θ_G gives $H + W = S_A(H) + S_A(W) = \theta_G((\mathrm{Hom}(A, H) + \mathrm{Hom}(A, W)) \otimes_{E(A)} A)$ and $H \subseteq \theta_G(Y \otimes_{E(A)} A)$ since A is flat as an $E(A)$-module.

Moreover, $\theta_G(Y \otimes_{E(A)} A)/H$ is torsion. To show this, let $x = \sum_{i=1}^n f_i \otimes a_i$ be in $Y \otimes_{E(A)} A$ with $f_i \in Y$ and $a_i \in A$. For every $i \in \{1, \dots, n\}$, there is a regular element c_i in $E(A)$ such that $f_i c_i \in \mathrm{Hom}(A, H)$ since $Y/\mathrm{Hom}(A, H) = Z(X)$. By condition (ii), $A/c_i(A)$ is torsion. Hence, there is $0 \neq m \in \mathbf{Z}$ and $b_1, \dots, b_n \in A$ such that $ma_i = c_i(b_i)$. Thus

$$m\theta_G(x) = m\sum_{i=1}^n f_i(a_i) = \sum_{i=1}^n f_i(ma_i) = \sum_{i=1}^n f_i c_i(b_i) \in H.$$

Therefore,

$$\theta_G\big(Y \otimes_{E(A)} A\big) = S_A\Big(\theta_G\big(Y \otimes_{E(A)} A\big)\Big) \subseteq S_A(H_*).$$

Furthermore, $h + \theta_G(Y \otimes_{E(A)} A)$ is a torsion element of the A-projective group $(H + W)/\theta_G(Y \otimes_{E(A)} A)$. Since A is torsion-free, $h \in \theta(Y \otimes_{E(A)} A)$. Therefore, $H_* = S_A(H_*)$.

Finally, to show that $\mathrm{Hom}(A, G)/\mathrm{Hom}(A, H_*)$ is a non-singular $E(A)$-module, let $f \in \mathrm{Hom}(A, G)$ such that $fc \in \mathrm{Hom}(A, H_*)$ for some regular element c of $E(A)$. Since $A/c(A)$ is torsion, one has that for all $a \in A$ there is $a' \in A$ and $0 \neq n \in \mathbf{Z}$ with $na = c(a')$. Thus, $nf(a) = f(na) = fc(a')$ is an element of $nG \cap H_* = nH_*$. Hence, $f(a) \in H_*$. This proves that $\mathrm{Hom}(A, G)/\mathrm{Hom}(A, H_*)$ is a non-singular $E(A)$-module.

COROLLARY 4.9. *Let A be a torsion-free, reduced abelian group whose endomorphism ring is semi-prime, right and left Noetherian, and hereditary. Suppose that A/U is torsion for all subgroups U of A isomorphic to A. If G is an abelian group with $R_A(G) = 0$ and $S_A(G) = G$, then every pure subgroup H of G with $S_A(H) = H$ is almost $\{A\}_*$-pure in G.*

Finally, it shall be remarked that every torsion-free, reduced group A whose endomorphism ring $E(A)$ is a principal ideal domain and satisfies $E(A)/I$ is torsion for every non-zero ideal I satisfies the hypotheses of Corollary 4.9. Besides the examples from Arnold's and Murley's paper, every torsion-free, reduced group A of finite rank with $E(A)$ right (left) hereditary has this property by [**13**, Theorem 2.3]

5. Locally A-projective Abelian groups. Let A be a torsion-free, reduced abelian group. An abelian group G is locally A-projective if every finite subset of G is contained in an A-projective direct summand of G. Arnold and Murley showed in [5] that the category of locally A-projective groups is equivalent to the category of locally projective $E(A)$-modules in the case that $E(A)$ is discrete in the finite topology. The equivalence is given by the functors $\mathrm{Hom}(A, -)$ and $- \otimes_{E(A)} A$. If $E(A)$ is a semi-prime, right and left Noetherian, hereditary ring, then every finite subset of G is contained in an A-projective summand of finite A-rank of G. If G is locally A-projective, then $S_A(G) = G$ and $R_A(G) = 0$. However, the converse does not hold since there are subgroups of $\Pi_I \mathbf{Z}$ that are not locally free. Applying the results on locally projective modules from §1, it is possible to prove

THEOREM 5.1. *Let A be a torsion-free, reduced abelian group whose endomorphism ring is semi-prime, right and left Noetherian, and hereditary.*

The following are equivalent for an abelian group G.

(i) *G is locally A-projective.*

(ii) *$S_A(G) = G$, and G is isomorphic to an almost $\{A\}_*$-pure subgroup of $S_A(\Pi_I A)$ for some index set I.*

Proof. (i) $\Rightarrow$ (ii): Since G is locally A-projective and $E(A)$ is discrete in the finite topology, $\mathrm{Hom}(A, G)$ is a locally projective right $E(A)$-module by [**5**, Theorem III]. By Proposition 2.2, there is an index-set I such that $\mathrm{Hom}(A, G)$ is isomorphic to a submodule U of $\Pi_I E(A)$ in such a way that $(\Pi_I E(A))/U$ is non-singular. Applying the functor $- \otimes_{E(A)} A$ induces a short exact sequence

$$0 \to U \otimes_{E(A)} A \to (\Pi_I E(A)) \otimes_{E(A)} A$$

from which the diagram

$$\begin{array}{ccccc} 0 & \to & \mathrm{Hom}(A, U \otimes_{E(A)} A) & \to & \mathrm{Hom}(A, (\Pi_I E(A)) \otimes_{E(A)} A) \\ & & \uparrow \Phi_U & & \uparrow \Phi_{\Pi_I E(A)} \\ 0 & \to & U & \to & \Pi_I E(A) \end{array}$$

is obtained. Here, Φ_M: $M \to \mathrm{Hom}(A, M \otimes_{E(A)} A)$ is defined by $\Phi_M(m)(a) = m \otimes a$. By [**5**, Lemma 4.5], Φ_M is an isomorphism if M is locally projective. Especially, this holds for U and $\Pi_I E(A)$. Consequently, the fact that $(\Pi_I E(A))/U$ is non-singular implies the same for the quotient in the first row.

Since $\mathrm{Hom}(A, \Pi_I A) = \mathrm{Hom}(A, S_A(\Pi_I A)) \cong \Pi_I E(A)$, one obtains $(\Pi_I E(A)) \otimes_{E(A)} A \cong S_A(\Pi_I A)$. Let $G' \subseteq S_A(\Pi_I A)$ be the image of $U \otimes_{E(A)} A$ under this isomorphism. Then, G' is almost $\{A\}_*$-pure in $S_A(\Pi_I A)$. Finally, [**5**, Theorem III] implies $G \cong \mathrm{Hom}(A, G) \otimes_{E(A)} A \cong U \otimes_{E(A)} A \cong G'$. This proves (ii).

(ii) $\Rightarrow$ (i): $G \subseteq S_A(\Pi_I A)$ implies $R_A(G) = 0$. Because of $S_A(G) = G$, the natural map θ_G: $\mathrm{Hom}(A, G) \otimes_{E(A)} A \to G$ is an isomorphism by [**1**, Lemma 6.2]. By Proposition 4.4, the $E(A)$-module $\mathrm{Hom}(A, S_A))/\mathrm{Hom}(A, G)$ is non-singular.

Finally, since $\mathrm{Hom}(A, S_A(\Pi_I A)) \cong \Pi_I E(A)$, Proposition 2.2 implies that $\mathrm{Hom}(A, G)$ is a locally projective right $E(A)$-module. By [**5**, Theorem III], $G \cong \mathrm{Hom}(A, G) \otimes_{E(A)} A$ is locally A-projective.

COROLLARY 5.2. *Let A be a torsion-free, reduced abelian group which has a semi-prime, right and left Noetherian, hereditary endomorphism ring.*

(i) *If B is an almost $\{A\}_*$-pure subgroup of a locally A-projective group G, then B is locally A-projective. Moreover, if $E(B)$ is discrete in the finite topology, then B is an A-projective direct summand of finite A-rank of G.*

(ii) *A locally A-projective group G which is an epimorphic image of* $\oplus_\omega A$ *is A-projective.*

Proof. In view of Lemma 4.2 and Theorem 5.1, the first part of (i) is obvious.

If $E(B)$ is discrete in the finite topology, then there is a finite subset X of B with $\operatorname{ann}_B(X) = 0$. There is an A-projective summand C of finite A-rank of B containing X, say $B = C \oplus D$. If $D \neq 0$, then $\operatorname{ann}_B(X) \neq 0$, a contradiction. Therefore, B is an A-projective group of finite A-rank.

Furthermore, $\operatorname{Hom}(A, B)$ is a finitely generated $E(A)$-module, and it is contained in a finitely generated, projective summand Q of $\operatorname{Hom}(A, G)$. Since $Q/\operatorname{Hom}(A, B)$ is non-singular, $\operatorname{Hom}(A, B)$ is a direct summand of Q and hence of $\operatorname{Hom}(A, G)$. Applying the functor $-\otimes_{E(A)} A$ completes the proof.

(ii) is an immediate consequence of Corollary 4.7.

Combining the results of §5 with the last result in §4 gives the following

COROLLARY 5.3. *Let A be a torsion-free, reduced abelian group whose endomorphism ring is semi-prime, right and left Noetherian, and hereditary. Suppose that* A/U *is torsion for every subgroup* $U \cong A$ *of A.*

(i) *If B is a pure subgroup of an A-projective group P such that* $S_A(B) = B$, *and* $E(B)$ *is discrete in the finite topology, then B is an A-projective summand of finite A-rank of G.*

(ii) *A group G is locally A-projective if and only if* $S_A(G) = G$, *and G is isomorphic to a pure subgroup of* $S_A(\Pi_I A)$ *for some index set I.*

(iii) *If B is a pure subgroup of a locally A-projective group such that* $S_A(B) = B$, *then B is locally A-projective. Moreover, if* $E(B)$ *is discrete in the finite topology, then B is an A-projective summand of finite A-rank of G.*

Proof. By Corollary 4.9, it is possible to replace in the statement of Corollary 5.3 purity by almost $\{A\}_*$-purity. The results follow from Corollary 4.5, Theorem 5.1, and Corollary 5.2.

Corollary 5.3 coincides exactly with the formulation of Arnold's and Murley's result [5, Corollary IV]. As it has been remarked following Corollary 4.5, it contains not only the previously mentioned result but also the case of finite rank, torsion-free, reduced abelian groups whose endomorphisms ring is right or left hereditary.

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AUBURN UNIVERSITY
AUBURN, AL 36849