PACIFIC JOURNAL OF MATHEMATICS
Vol. 120, No. 2, 1985

A GALOIS-CORRESPONDENCE FOR GENERAL LOCALLY COMPACT GROUPS

JOACHIM BOIDOL

We give a characterization in terms of $\hat{G}$ of those parts in the unitary dual of a locally compact group G, which correspond to closed normal subgroups of G. These are exactly the sets $S \subset \hat{G}$, which have the property that for all $\pi, \rho \in S$ the support of $\pi \otimes \bar{\rho}$ is contained in S and which are closed in a topology on $\hat{G}$, which is in general weaker than the standard topology on $\hat{G}$, and which we call the L^1-hull-kernel-topology. As an easy consequence we obtain that for $*$-regular groups G the mapping $N \to N^\perp = \{\pi \in \hat{G} | \pi_{|N} = 1|_{\mathscr{H}_\pi}\}$ is a bijection from the set of closed normal subgroups of G onto the set of closed subsets $S \subset \hat{G}$ with the property that $\pi \otimes \bar{\rho}$ has support in S for all $\pi, \rho \in S$. This generalizes and unifies results of Pontryagin, Helgason and Hauenschild, with a considerably simplified proof. Furthermore we prove that $*$-regular groups have the weak Frobenius property (TP 1), i.e. 1_G is weakly contained in $\pi \otimes \bar{\pi}$ for all unitary representations π of G, generalizing a result of E. Kaniuth.

Let G be a locally compact group with unitary dual $\hat{G}$ and let $\mathscr{N}_G$ denote the set of closed normal subgroups of G. To every $N \in \mathscr{N}_G$ corresponds a canonical subset of $\hat{G}$, namely the annihilator $N^\perp = \{\pi \in \hat{G} | \pi_{|N} = 1|_{\mathscr{H}_\pi}\}$ of N. By the Gelfand-Raikov theorem $N \to N^\perp$ is an injective mapping from $\mathscr{N}_G$ into the subsets of $\hat{G}$ and it is an important problem in harmonic analysis to describe the image of this mapping in terms of $\hat{G}$.

DEFINITION. A nontrivial subset S of $\hat{G}$ is called a *subdual* of $\hat{G}$, if for all $\pi, \rho \in S$ the tensor product $\pi \otimes \bar{\rho}$ of π and the conjugate $\bar{\rho}$ of ρ has support in S. We denote by $\mathscr{S}_G$ the set of closed subduals of $\hat{G}$.

It is clear that $N \to N^\perp$ is an injective mapping from $\mathscr{N}_G$ into $\mathscr{S}_G$. Let $[H]$ be the class of locally compact groups G, for which $N \to N^\perp$ is a surjection onto $\mathscr{S}_G$.

As a well known consequence of the duality theorem of Pontryagin one obtains that all *abelian locally compact groups* belong to $[H]$ (see for example [**6**], Chap. II, §1.7). S. Helgason proved in [**8**], Theorem 1, that all *compact groups* belong to $[H]$. It was then W. Hauenschild, who generalized and unified these results in [**7**], and proved that all *Moore groups*, i.e. all locally compact groups G, which have only finite dimensional irreducible unitary representations, belong to the class $[H]$.

On the other hand the support $\hat{G}_r$ of the left regular representation λ_G of a locally compact group G is clearly a closed subdual of $\hat{G}$. If $\hat{G}_r = N^{\perp}$ for some $N \in \mathcal{N}_G$, then $N = \{e\}$ and $\hat{G}_r = \hat{G}$. Therefore every group G, which belongs to $[H]$, has to be amenable.

We recall that the (*standard*) *topology* on $\hat{G}$ is induced by the Jacobson topology on the primitive ideal space $\mathrm{Prim}(G)$ of the group C^*-algebra $C^*(G)$ of G via the mapping $\pi \to \mathrm{kern}_{C^*(G)} \pi$. Let $\mathrm{Prim}_* L^1(G)$ denote the space of kernels in $L^1(G)$ of topologically irreducible $*$-representations of $L^1(G)$ in Hilbert spaces. $\mathrm{Prim}_* L^1(G)$ is also a topological space with the Jacobson topology and the mapping $\pi \to \mathrm{kern}_{L^1(G)} \pi$ defines a second topology on $\hat{G}$, which we call the *L^1-hull-kernel-topology*. This topology is weaker than the standard one and in general both topologies are different. Both topologies coincide if and only if the canonical continuous and surjective mapping Ψ: $\mathrm{Prim}(G) \to \mathrm{Prim}_* L^1(G)$, given by $\Psi(I) = I \cap L^1(G)$, is a homeomorphism, i.e. if G is *$*$-regular*.

DEFINITION. Let $\mathcal{S}_G^* \subset \mathcal{S}_G$ be the set of subduals of $\hat{G}$, which are closed in the L^1-hull-kernel-topology.

The main result of our paper will be that $\mathcal{S}_G^*$ *is the exact image of the mapping* $N \to N^{\perp}$ *for general locally compact groups*. The results of Helgason and Hauenschild will be an easy consequence. But first we need the following

PROPOSITION. *For every unitary representation of G in a Hilbert space $\mathcal{H}_\pi$ we have* $\mathrm{kern}_{L^1(G)}\pi \otimes \bar{\pi} \subset \mathrm{ker}_{L^1(G)} 1_G$.

Proof. Let $\overline{\mathcal{H}_\pi}$ be the adjoint space of $\mathcal{H}_\pi$ and denote by $\bar{\eta}$ the vector $\eta \in \mathcal{H}_\pi$ considered as element of $\overline{\mathcal{H}_\pi}$. Then $\bar{\pi}$ is the representation π considered as a representation acting in $\overline{\mathcal{H}_\pi}$. We fix a unit vector $\xi \in \mathcal{H}_\pi$ and an orthonormal basis $\{\xi_i\}_{i \in I}$ of $\mathcal{H}_\pi$. Then for all $x \in G$ we have $\langle \bar{\pi}(x)\bar{\xi}, \bar{\xi}_i \rangle = \overline{\langle \pi(x)\xi, \xi_i \rangle}$ and we obtain for all $x \in G$

$$1 = \langle \xi, \xi \rangle = \langle \pi(x)\xi, \pi(x)\xi \rangle = \sum_{i \in I} \langle \pi(x)\xi, \xi_i \rangle \langle \bar{\pi}(x)\bar{\xi}, \bar{\xi}_i \rangle .$$

Let $\mathcal{F}$ denote the family of all finite sums of the functions

$$\langle \pi(x)\xi, \xi_i \rangle \langle \bar{\pi}(x)\bar{\xi}, \bar{\xi}_i \rangle ,$$

which are matrix-coefficients of $\pi \otimes \bar{\pi}$. If $\varphi \in \mathcal{F}$ then φ is continuous and $0 \leq \varphi \leq 1$. Furthermore $1 = \sup_{\varphi \in \mathcal{F}} \varphi$.

Assume now that $f \in \operatorname{kern}_{L^1(G)} \pi \otimes \bar{\pi}$. Then $\int_G f(x)\varphi(x)\,dx = 0$ for all $\varphi \in \mathscr{F}$. Given $\varepsilon > 0$ choose a compact set $\mathscr{K} \subset G$ such that $\int_{F\setminus\mathscr{K}} |f(x)|\,dx \leq \varepsilon/2$. By Dini there exists a sequence $\{\varphi_n\}_{n\in\mathbf{N}}$ in $\mathscr{F}$ (depending on $\mathscr{K}$), such that $1 = \lim_{n\to\infty} \varphi_n$ uniformly on $\mathscr{K}$. Then

$$\begin{aligned}\left|\int_G f(x)\,dx\right| &= \lim_{n\to\infty}\left|\int_{\mathscr{K}} f(x)\varphi_n(x)\,dx + \int_{G\setminus\mathscr{K}} f(x)\,dx\right| \\ &= \lim_{n\to\infty}\left|\int_{G\setminus\mathscr{K}} f(x)\,dx - \int_{G\setminus\mathscr{K}} f(x)\varphi_n(x)\,dx\right| \\ &\leq 2\int_{G\setminus\mathscr{K}} |f(x)|dx \leq \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain $\int_G f(x)\,dx = 0$.

The following corollary generalizes a result of E. Kaniuth (see [**9**], Lemma 1):

COROLLARY 1. *Let G be a $*$-regular locally compact group. Then for every unitary representation π of G $\pi \otimes \bar{\pi}$ weakly contains the trivial representation, i.e. every $*$-regular group has the property* (TP 1) *of* [**9**].

Proof. For $*$-regular groups $\operatorname{kern}_{L^1(G)} \pi * \bar{\pi} \subset \operatorname{kern}_{L^1(G)} 1_G$ implies that 1_G is weakly contained in $\pi \otimes \bar{\pi}$.

REMARK. Corollary 1 shows that a quite big class of amenable groups has the weak Frobenius property (TP 1). This supports the conjecture that all amenable groups have the property (TP 1).

THEOREM. *For every locally compact group G the mapping $N \to N^{\perp}$ is a bijection from $\mathscr{N}_G$ onto $\mathscr{S}_G^*$.*

Proof. As we remarked above, the mapping $N \to N^{\perp}$ is an injection from $\mathscr{N}_G$ into $\mathscr{S}_G$. If $N \in \mathscr{N}_G$, then $N^{\perp}$ corresponds to the set of topological irreducible $*$-representations of $L^1(G)$, which are trivial on the kernel of the canonical homomorphims from $L^1(G)$ onto $L^1(G/N)$. Therefore $N^{\perp} \in \mathscr{S}_G^*$, and we only have to prove that every set $\mathscr{S} \in \mathscr{S}_G^*$ is of the form $N^{\perp}$ for some $N \in \mathscr{G}_G$.

First observe that by the proposition every $S \in \mathscr{S}_G^*$ has the following properties:

(i) S contains 1_G and $\pi \in S$ implies $\bar{\pi} \in S$.

(ii) for all π, $\rho \in S$ the support of $\pi * \rho$ is in$S$.

Furthermore since $S^{\perp} = \{x \in G | \pi(x) = 1|_{\mathscr{H}_\pi} \text{ for all } \pi \in S\}$ is a closed normal subgroup of G, we can consider S as a subdual of $(G/S^{\perp})\hat{}$, which separates the points of $G/S^{\perp}$ and is closed in the L^1-hull-kernel-topology.

It is therefore sufficient to prove that a set $S \in \mathscr{S}_G^*$, which separates the points of G, is equal to $\hat{G}$.

Let S be such a set and let $\mathscr{P}$ be the set of unitary representations of G, which have support in S. Since S is closed in $\hat{G}$, S and $\mathscr{P}$ are weakly equivalent sets of representations of G. Since S has properties (i) and (ii) above, $\mathscr{P}$ contains the trivial representation and is closed under the tensor product and under conjugation. It follows by a Stone-Weierstraß argument (see **[1]**, Theorem) that $\mathscr{P}$ is L^1-separating, i.e. if $f \in L^1(G)$ and $\pi(f) = 0$ for all $\pi \in \mathscr{P}$, then $f = 0$. But then also S is L^1-separating, i.e. its kernel in $L^1(G)$ is the trivial ideal $\{0\}$. Since S is closed in the L^1-hull-kernel-topology, we obtain $S = \hat{G}$.

COROLLARY 2. *A locally compact group belongs to the class* $[H]$ *if and only if* $\mathscr{S}_G = \mathscr{S}_G^*$. *Especially every* $*$*-regular locally compact group belongs to* $[H]$ *and every locally compact group in* $[H]$ *is amenable.*

REMARK. Let G be a locally compact group, such that all quotients G/N are C^*-*unique*, i.e. $L^1(G/N)$ has a unique C^*-norm (see **[5]**). The same arguments as in the proof of the theorem give that G belongs to $[H]$. We do not know whether this class of groups is really bigger than the class of $*$-regular groups.

The following is known about $*$-regular groups:

(A) *Every* $*$*-regular group is amenable* (*see* **[2]**).

(B) *All groups G with polynomially growing Haar measure are* $*$*-regular* (*see* **[2]**).

(C) *All semidirect product* $G = H \ltimes N$ *with abelian* H *and* N *are* $*$*-regular* (*see* **[4]**).

(D) *A connected group G is* $*$*-regular if and only if all* $I \in \mathrm{Prim}(G)$ *are polynomially induced* (*see* **[3]**).

It follows from the classification of Moore groups given by C. C. Moore in **[10]**, that all *Moore groups* have polynomial growth and so are $*$-regular by (B). Therefore the result of W. Hauenschild is an immediate consequence of the Corollary 2 and (B). It should be noted that the proofs of the results of Pontryagin, Helgason and Hauenschild depend explicitly or implicitly on the fact that the groups under consideration are $*$-regular. Besides this they make use of central theorems as the Pontryagin duality theorem, the Peter-Weyl theorem or structure theorems for Moore groups, which are specific for these classes of groups.

Recently E. Kaniuth proved by quite different methods that a big class of amenable groups, including the *almost connected amenable groups*, belong to $[H]$. (Cf. E. Kaniuth, *Weak containment and tensor products of group representations*. II, Math. Ann., **270** (1985), 1–15.) There seems to be some hope that the class $[H]$ coincides with the class of amenable groups.

REFERENCES

[1] B. A. Barnes, *A note on separating families of representations*, Proc. Amer. Math. Soc., **87** (1983), 95–98.

[2] J. Boidol, H. Leptin, J. Schürmann and D. Vahle, *Räume primitiver Ideale von Gruppenalgebren*, Math. Ann., **236** (1978), 1–13.

[3] J. Boidol, *Connected groups with polynomially induced dual*, J. reine angew. Math., **331** (1982), 32–46.

[4] ______, *$*$-regularity of some classes of solvable gropus*, Math. Ann. **261** (1982), 477–481.

[5] ______, *Group algebras with a unique C^*-norm*, Functional Anal., **56** (1984), 220–232.

[6] N. Bourbaki, *Théories Spectrales*, Paris: Hermann 1967.

[7] W. Hauenschild, *Subhypergroups and normal subgroups*, Math. Ann., **256** (1981), 1–18.

[8] S. Helgason, *Lacunary Fourier series on noncommutative groups*, Proc. Amer. Math. Soc., **9** (1958), 782–790.

[9] E. Kaniuth, *Weak containment and tensor products of group representations*, Math. Z., **180** (1982), 107–117.

[10] C. C. Moore, *Groups with finite dimensional irreducible representations*, Trans. Amer. Math. Soc., **166** (1972), 401–410.

Received March 30, 1984 and in revised form August 28, 1984.

UNIVERSITÄT BIELEFELD
UNIVERSITÄTSSTRAßE
4800 BIELEFELD 1
W. GERMANY