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# A REMARK ON FIELDS WITH THE DENSE ORBITS PROPERTY

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**Let $K$ be a formally real field and $\Omega$ its order space. The automorphisms group of $K$ acts on $\Omega$, and $K$ is called D.O.P. when all the orbits are dense in $\Omega$. In this note the following is shown: The field of meromorphic function germs of a real irreducible analytic germ of dimension $> 1$ is never D.O.P.**

A formally real field $K$ has an associated order space $\Omega = \text{Spec}_R K$. The automorphisms group $\text{Aut}(K)$ of $K$ acts over $\Omega$ in an obvious way, and the space of orbits under this action measures the homogeneity of $\Omega$. Actually, $\Omega$ or $K$ are called *homogeneous* if $\text{Aut}(K)$ acts transitively. There is also a weaker homogeneity condition, the so-called "dense orbits property": $K$ is D.O.P. if all orbits are dense in $\Omega$ (cf. [**2**], [**4**]). These notions are specially considered in the geometric case, i.e., when $K$ is a field of functions. It turns out then that both are very strong rigidity conditions. Indeed, let $V$ be an irreducible algebraic variety over $\mathbf{R}$ and $K$ its field of rational functions:

(a) If $K$ is homogeneous, then $V$ is a curve, either rational or elliptic.

(b) If $\dim V = 1$ and $K$ is D.O.P., then $V$ is homogeneous.

(c) If $V$ is of general type, then $K$ is not D.O.P.

(These results are contained in [**3**], although in some slightly different form.) Thus, the open problem is to characterize D.O.P. (fields of functions of) varieties. Some partial answers are known (for instance, affine spaces are D.O.P., cf. [**2**]), but the general solution appears to be quite misterious. In this small note we show that this is certainly not the case in a local context. i.e., for analytic germs. Namely, we prove:

THEOREM. *Let $X_0$ be an irreducible analytic germ in $\mathbf{R}^n_0$ and $K$ its field of real meromorphic function germs. If* $\dim X_0 > 1$, *$K$ is not* D.O.P.

REMARK. If $\dim X_0 = 1$, then $K$ is nothing but the quotient field of $\mathbf{R}\{t\}$ and it is trivially homogeneous. The interesting question in this case appears to be whether $\Omega$ is still homogeneous under the action of $\text{Aut}(A)$, $A$ being the ring of analytic function germs in $X_0$.

**2. Proof of the theorem.** We may clearly assume that $X_0$ is normal. First we need the following

(2.1) LEMMA. *Any* $\psi \in \operatorname{Aut}(K)$ *is induced by some (unique) analytic diffeomorphism* $\psi^*$ *of* $X_0$.

*Proof.* Let $A$ stand for the ring of analytic function germs on $X_0$, so that $K = qfA$. We have to check that $\psi(A) \subset A$ and $\psi|\mathbf{R} = \mathrm{Id}_{\mathbf{R}}$. Let $h \in A$, $h \neq 0$ and $h(0) \neq 1$. Set $\psi(h) = x \in K$. For each $n \geq 0$ there is $h_n = (1-h)^{1/n} \in A$ (implicit function theorem) and, setting $y_n = \psi(h_n)$, we have: $(y_n)^n = 1 - x = y$. Now consider any discrete valuation $\nu$ of $K$ over $A$. As $\nu(y) = n\nu(y_n)$, we conclude $\nu(y) = 0$ (were $\nu(y) \neq 0$, would it be $|\nu(y)| = n|\nu(y_n)| \geq n$, all $n$). Thus, $y \in \bigcap_\nu A_\nu = A$, $A$ being normal, and $x \in A$.

This proves $\psi(A) \subset A$ and, by symmetry, $\psi(A) = A$. Hence $\psi$: $A \to A$ is an isomorphism and in particular a local homomorphism of analytic rings. But by a result in [1], this implies $\psi|\mathbf{R} = \mathrm{Id}_{\mathbf{R}}$, and the proof is complete.

EXAMPLE. If $X_0$ is not normal, 2.1 is not true any more: take $A = \mathbf{R}\{t^2, t^5\} \subset \mathbf{R}\{t\} \subset K$ and $\psi$ induced by $t \mapsto t + t^2$.

We now prove our theorem.

*Step* I. Let $X_0^*$ denote the maximum dimension locus of $X_0$. There are two analytic half-branches $c_0$ and $c_0'$, both in $X_0^*$, whose multiplicities $m$ and $m'$ are different, say $m < m'$.

Indeed, after a standard application of local parametrization the problem reduces to find $c_0$ with arbitrarily large $m$ in some given non-empty open semianalytic germ $W_0 \subset \mathbf{R}_0^d$ and, after a new linear projection, $d = 2$ (notice that transversal projections do not increase multiplicities). Also, we can suppose

$$W_0 = \{(x, y) \in \mathbf{R}^2 \colon x > 0,\ h(x^{1/p}) < y < g(x^{1/p})\}$$

for some $p \geq 1$, $h, g \in \mathbf{R}\{t\}$. As $W_0 \neq \varnothing$ we can write:

$$h = h_0(t) + t^r h_1(t), \quad g = g_0(t) + t^r g_1(t),$$

where $h_0(t)$, $g_0(t)$ are polynomials in $\mathbf{R}[t]$ of degree $< r$ and $h_0 < g_0$ for small $t$. Then choose $f_0 \in \mathbf{R}[t]$, of degree $< r$, such that $h_0 < f_0 < g_0$ for small $t$. It follows that for *any Puiseux* series $f_1(t^{1/q})$ the curve germ

$$t \mapsto (t^{pq}, f_0(t^q) + t^{qr} f_1(t)), \qquad t > 0$$

is contained in $W_0$, and if $f_1$ and $q$ are well chosen, the multiplicity of this curve germ is arbitrarily large. This completes Step I.

*Step* II. There is an open semianalytic germ $U_0 \subset \mathbf{R}^n_0$, such that $c'_0 \subset U_0$, and any curve germ $c^*_0 \subset U_0$ has multiplicity $m^* \geq m'$.

For, like in Step I, it is enough to consider the case $n = 2$, and $c'_0$ given by: $y = h(x^{1/m'})$, $x > 0$, where $h \in \mathbf{R}\{t\}$. We write $h$ in the following way:

$$h = a_1 t^{p_1} + \cdots + a_r t^{p_r} + a t^p + t^{p+1} h_1(t), \qquad 1 \leq p_1 < \cdots < p_r < p,$$

and $(p_1, \ldots, p_r, m') = 1$. Then the $U_0$ we sought is

$$\{x > 0,\ h^-(x^{1/m'}) < y < h^+(x^{1/m'})\},$$

where

$$h^-(t) = a_1 t^{p_1} + \cdots + a_r t^{p_r} + (a - \varepsilon) t^p,$$

$$h^+(t) = a_1 t^{p_1} + \cdots + a_r t^{p_r} + (a + \varepsilon) t^p$$

(pick any $\varepsilon > 0$). Indeed, if $c^*_0 \subset U_0$ is given by $y = f(x^{1/m^*})$, $x > 0$, we have

$$h^-(x^{1/m'}) < f(x^{1/m^*}) < h^+(x^{1/m'}) \quad (\text{for small } x > 0)$$

and so

$$f(x^{1/m^*}) = a_1 x^{p_1/m'} + \cdots + a_r x^{p_r/m'} + f_1(x^{1/m^*}).$$

But $m^*$ is the lowest common denominator of all exponents in this development, so $m^* \geq m'$, $m'$ being the l.c.d. of the first $r$ exponents. Thus we are done.

*Step* III: *end of the proof*. By the separation lemma in **[6]**, there is an analytic function germ $h$ such that $c'_0 \subset \{h > 0\} \subset \{h \geq 0\} \setminus \{0\} \subset U_0$. Let $\alpha'$ be any ordering centered at $c'_0$ (such an $\alpha'$ does exist, as $c'_0 \subset X^*_0$, cf. **[5]**) and consider the open neighborhood $H = H(h)$ of $\alpha'$ in $\Omega = \operatorname{Spec}_R K$. We claim that $H$ contains no ordering $\alpha^*$ isomorphic to any ordering $\alpha$ centered at $c_0 \subset X^*_0$, what finishes the proof (again because $c_0 \subset X^*_0$). Indeed, assume the contrary, i.e. there is $\psi \in \operatorname{Aut}(K)$ such that $\psi\alpha \in H$. Then by Lemma 2.1, we get an analytic half-branch $\psi^{*-1} c_0 = c^*_0$. As $h$ must be positive in $\psi\alpha$, $c^*_0 \subset \{h \geq 0\} \setminus \{0\} \subset U_0$ and $\operatorname{mult} c^*_0 = m^* \geq m'$, because of the choice of $U_0$ (Step II). But $c^*_0$ is analytically diffeomorphic to $c_0$, hence $m^* = m < m'$, contradiction.

REMARK. One technical point in the proof above is to be sure $c_0^*$ is within $U_0$ and not merely in the boundary. It is here where the separation lemma is useful. Yet there is another possibility: using formal half-branches (cf. [5]). But then one needs an approximation lemma for these half-branches (loc. cit.) and the proof becomes more involved than it actually is.

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