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THE SET OF CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTIONS: A CORRECTION

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Let M be the set of all continuous real-valued functions defined on the closed unit interval $[0,1]$ which do not have a finite derivative anywhere. The purpose of this note is to complete the argument that M is a coanalytic set and is not a Borel set by correcting an error in the last paragraph of an earlier paper on this topic.

Let C be the space of all real-valued continuous functions defined on $[0,1]$. We consider C to be provided with the topology of uniform convergence. Let M be the subset of C consisting of all functions which do not have a finite derivative at any point of $[0,1]$. In [**1**], the author demonstrated that M forms a coanalytic subset of C. The purpose of [**1**] was to show that M is not a Borel set. To this end it was shown that there is a countable dense subset $Y = \{y_n\}_{n=1}^{\infty}$ of $[0,1]$ such that

$$D(Y) \equiv \{f \in C \mid f \text{ has a finite derivative at some point of } [0,1] \setminus Y\}$$

forms an analytic subset of C which is not a Borel set. It is then claimed in the last paragraph of [**1**] that it follows that M is not a Borel set since

$$(*) \qquad D(Y) = (C \setminus M) \setminus \bigcup_{n=1}^{\infty} D_n,$$

where $D_n = \{f \in C \mid f \text{ has a finite derivative at } y_n\}$. The argument was that since each set D_n is an $F_{\sigma\delta}$ subset of C, if M were a Borel set, then because of equation $(*)$, $D(Y)$ would be a Borel set. However, $(*)$ is false. I thank David Preiss and Alexander Kechris for pointing out this elementary error. We complete the argument as follows.

Let us assume Y is a subset of $(0,1)$. (Only minor modifications are needed if 0 or 1 belong to Y.) Let $H = \{(f, \langle\varepsilon_n\rangle) \in C \times \{0,1\}^{\mathbf{N}} \mid \forall n\,(f$ has a finite derivative at y_n if and only if $\varepsilon_n = 1)\}$. It can be checked that H is a Borel set and thus, there is a Borel measurable map $h\colon C \to \{0,1\}^{\mathbf{N}}$ such that $H = \mathrm{Graph}(h)$. Define $g\colon \{0,1\}^{\mathbf{N}} \to C$ by setting $g(\langle\varepsilon_n\rangle) = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k} g_k$, where, for each k, $g_k(x) = |x - y_k|$. The continuous map g has the property that $g(\langle\varepsilon_n\rangle)$ has a finite derivative at x if and only if $x \notin \{y_k \mid \varepsilon_k = 1\}$. Let T be the continuous map of $C \times \{0,1\}^{\mathbf{N}}$ into C defined by $T((f, \langle\varepsilon_n\rangle)) = f + g(\langle\varepsilon_n\rangle)$. We have $(f, h(f)) \in T^{-1}(M) \sqcap H$

if and only if $f \in C \setminus D(Y)$. If M were a Borel subset of C, then $C \setminus D(Y) = \text{proj}_c(T^{-1}(M) \sqcap H)$ would be a Borel subset of C. This contradiction establishes the theorem.

Reference

[1] R. D. Mauldin, *The set of continuous nowhere differentiable functions*, Pacific J. Math., **83** (1979), 199–205.

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