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ON POLYNOMIAL GENERATORS IN THE ALGEBRA OF COMPLEX FUNCTIONS ON A COMPACT SPACE

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In this paper we prove that in the space of all continuous mappings of a k-dimensional compact space X into complex linear space C^n the imbeddings $F\colon X \to C^n$ with the property "any complex continuous function on $F(X)$ can be uniformly approximated by complex polynomials on C^n" form a dense subset of type G_δ, provided that $k \leq \frac{2}{3}n$.

If is known [**2**] that if the algebra of continuous complex functions $C(X)$ for a topological space X has k multiplicative generators then X has to be acyclic (with complex coefficients) in dimensions $\geq k$. In particular, $C(M^k)$ has at least $k+1$ generators for any closed orientable k-manifold M. On the other hand, it was proved in [**6**] that there exist $k+1$ polynomial generators in the algebra $C(X^k)$ for a finite k-dimensional simplicial polyhedron X^k. This means that any such function on X^k may be uniformly approximated by complex polynomials in certain specially constructed functions $f_0^*, \ldots, f_k^* \in C(X^k)$. In other words, there exists a continuous embedding $F^*\colon X^k \to \mathbf{C}^{k+1}$ of the polyhedron X^k into complex vector space $\mathbf{C}^{k+1}$ such that any continuous complex valued function on the image $F^*(X^k)$ may be approximated by complex polynomials in the coordinate functions $z_i\colon \mathbf{C}^{k+1} \to \mathbf{C}$, $0 \leq i \leq k$.

It seems that analogous results follow for any compact space X^k (not only for polyhedra). Moreover, it is quite natural to conjecture that for X^k compact the existence of polynomial approximation on $F(X^k) \subset \mathbf{C}^{k+1}$ is a "general position" phenomenon with respect to perturbations of $F\colon X^k \to \mathbf{C}^{k+1}$. Note, that this would be a complete complex analog of the classical Whitney theorems [**9**] (see also [**4**]).

In this paper we prove similar propositions for imbeddings $F\colon X^k \to \mathbf{C}^n$ satisfying the dimensional condition $k \leq \frac{2}{3}n$. In particular, for 2-dimensional compact spaces X^2 one has the following result ("complex Whitney theorem"): there are 3 multiplicative generators in the algebra $C(X^2)$, in fact, starting with any $f_1, f_2, f_3 \in C(X^2)$ one can perturb them by an arbitrarily small amount to get a set of multiplicative generators for $C(X^2)$. Note, that this is the best possible general result for $k = 2$.

Our main result is

THEOREM A. *Let $3k \leq 2n$. In the space* $\mathrm{Map}(X^k, \mathbf{C}^n)$ *of all continuous mappings of a k-dimensional compact space X^k into complex linear space $\mathbf{C}^n$ consider the mappings $F\colon X^k \to \mathbf{C}^n$ satisfying the following properties*:

1. *F is an imbedding*;
2. *any continuous function on X^k may be approximated by complex polynomials in the multiplicative generators $f_1 = z_1 \circ F, \ldots, f_n = z_n \circ F$, where $z_1, \ldots, z_n$ are complex coordinate functions on $\mathbf{C}^n$*;
3. *in particular, $F(X^k)$ is polynomially convex in $\mathbf{C}^n$.*

These mappings form a dense subset of type G_δ in $\mathrm{Map}(X^k, \mathbf{C}^n)$.

The proof of this theorem is based on the following proposition.

THEOREM B. *Let $3k \leq 2n$. In the space* $\mathrm{SL\,Map}(Y^k, \mathbf{C}^n)$ *of simplicially linear mappings of a finite k-dimensional simplicial polyhedron Y^k into $\mathbf{C}^n$ there exists an open and everywhere dense subset of imbeddings $F\colon Y^k \to \mathbf{C}^n$ such that any continuous function on the image $F(Y^k)$ may be approximated by complex polynomials over $\mathbf{C}^n$ and, consequently, $F(Y^k)$ is polynomially convex in $\mathbf{C}^n$.*

We don't know if Theorems A and B have immediate analogs for smooth *regular* imbeddings. For example, it is easy to show that there is no smooth regular imbedding $F\colon \mathbf{C}P^2 \to \mathbf{C}^6$ of complex projective space $\mathbf{C}P^2$ with the tangent bundle of $F(\mathbf{C}P^2)$ being a totally real subbundle of a trivial complex 6-dimensional bundle. On the other hand, $3 \cdot \dim \mathbf{C}P^2 \leq 2 \cdot 6$, which is perfectly consistent with the dimensional assumptions of Theorems A and B.

Prior to the proof of Theorem B we need to introduce some terminology and to prove some auxiliary propositions.

Let $\mathscr{L}$ be any finite family of real affine subspaces $\{V_\alpha\}_{\alpha \in \mathscr{L}}$ of $\mathbf{C}^n$ with the property $V_\alpha \not\subseteq V_\beta$ for any pair $\alpha, \beta \in \mathscr{L}$, $\alpha \neq \beta$. Consider the subspace $|\mathscr{L}| = \bigcup_{\alpha \in \mathscr{L}} V_\alpha \subset \mathbf{C}^n$. In fact, it is a stratified set with the stratification induced by the multiple intersections of different spaces V_α parameterized by $\mathscr{L}$.

We say that the family $\mathscr{L}$ is *totally real* if any $V_\alpha \subset |\mathscr{L}|$, $\alpha \in \mathscr{L}$, is a totally real affine subspace of $\mathbf{C}^n$, i.e. it does not contain any complex line. Of course, if $\mathscr{L}$ is totally real, then its dimension $\dim \mathscr{L} = \max_{\alpha \in \mathscr{L}}\{\dim_{\mathbf{R}} V_\alpha\}$ is not greater than n.

We denote by $V_\alpha^{\mathbf{C}}$ the complexification of $V_\alpha \subset \mathbf{C}^n$ (which for totally real V_α is an affine subspace of real dimension 2 dim V_α). We call a totally real family $\mathscr{L}$ *weakly generic* if the following holds: $V_\beta \not\supseteq V_\alpha$ implies $V_\beta^{\mathbf{C}} \not\supseteq V_\alpha$ for any $\alpha, \beta \in \mathscr{L}$.

One can associate a new family $D\mathscr{L}$ with any (totally real) family $\mathscr{L}$. This derived family $D\mathscr{L}$ is formed by all the spaces $V_{\alpha,\beta} = V_\alpha \cap V_\beta^{\mathbf{C}}$, $V_\beta \not\supseteq V_\alpha$ and which are maximal with respect to inclusion relations. In fact, $|D\mathscr{L}|$ contains $V_\alpha \cap V_\beta$ for any pair $\alpha, \beta \in \mathscr{L}$. If $\mathscr{L}$ is weakly generic then dim $\mathscr{L} >$ dim $D\mathscr{L}$. Moreover, if $\mathscr{L}$ is totally real then $D\mathscr{L}$ also has this property.

We call a totally real family $\mathscr{L}$ *perfectly generic* if $\mathscr{L}$ and all its derived families $D\mathscr{L}, D(D\mathscr{L}), \ldots,$ are weakly generic. Note, that if $\mathscr{L}$ is perfectly generic then its $(k+1)$-derivative $D^{(k+1)}\mathscr{L} = \varnothing$, where $k = \dim \mathscr{L}$.

The following Lemma is the main step to prove Theorem B.

LEMMA 1. *Given a totally real and perfectly generic family $\mathscr{L}$ of real affine subspace of $\mathbf{C}^n$, dim $\mathscr{L} < n$, and any compact subset $K \subset |\mathscr{L}|$, then any continuous complex function on K may be uniformly approximated by complex polynomials in coordinate functions $z_1, \ldots, z_n$ on $\mathbf{C}^n$. In particular, K is polynomially convex in $\mathbf{C}^n$.*

Let $C(K)$ be the algebra of all continuous functions on K. Let $\mathscr{P}(K)$ denote the uniform closure in $C(K)$ of the subalgebra multiplicatively generated by the functions $\mathrm{Res}_K(z_i)$, $1 \leq i \leq n$. By Bishop's theorem on maximal antisymmetric subdivisions to prove that $\mathscr{P}(K) = C(K)$ it is sufficient to show that any antisymmetry set Ω for $\mathscr{P}(K)$ is a singleton [3]. Recall, that a subset $\Omega \subseteq K$ is called an antisymmetry set for $\mathscr{P}(K)$ if any function $f \in \mathscr{P}(K)$ which is real valued on Ω, in fact, is constant.

As a first step we prove that any antisymmetry set Ω is a singleton or is contained in the intersection of K with the derived family $|D\mathscr{L}|$ (providing that $\mathscr{L}$ is totally real and weakly generic). Denote by Ω_α the intersection $V_\alpha \cap \Omega$ and by $\mathring{\Omega}_\alpha$ the intersection $\mathring{V}_\alpha \cap \Omega$, where $\mathring{V}_\alpha = V_\alpha \setminus (V_\alpha \cap |D\mathscr{L}|) = V_\alpha \setminus \bigcup_{\beta \neq \alpha}(V_\alpha \cap V_\beta^{\mathbf{C}})$. Note that $\mathscr{L}$ weakly generic implies that $\mathring{V}_\alpha$ is open and everywhere dense in V_α.

For any two points $a, b \in \mathring{\Omega}_\alpha$, $\alpha \in \mathscr{L}$, we construct a polynomial $P_\alpha = P_\alpha(z_1, \ldots, z_n)$ which is real-valued on $|\mathscr{L}|$ and separates a and b. Note, that for any two points $a, b \notin V_\beta^{\mathbf{C}}$ one can find a linear polynomial $L_\beta\colon \mathbf{C}^n \to \mathbf{C}$ which is zero on $V_\beta^{\mathbf{C}}$ and such that $L_\beta(a) \neq 0 \neq L_\beta(b)$. Now take the product $Q_\alpha = \Pi_{\beta \neq \alpha} L_\beta$. The polynomial Q_α is zero on each V_β, $\beta \neq \alpha$, and $Q_\alpha(a) \neq 0 \neq Q_\alpha(b)$. Over V_α one can represent Q_α in the form $S_\alpha + iT_\alpha$ where the polynomials $S_\alpha: V_\alpha \to \mathbf{C}$, $T_\alpha: V_\alpha \to \mathbf{C}$ are real valued. Denote by $\tilde{Q}^*_\alpha: V_\alpha \to \mathbf{C}$ the polynomial $S_\alpha - iT_\alpha$. Using that V_α is totally real, one can extend $\tilde{Q}^*_\alpha$ to a polynomial $Q^*_\alpha: \mathbf{C}^n \to \mathbf{C}$ (first take the analytic extension of $\tilde{Q}^*_\alpha$ from V_α to $V_\alpha^{\mathbf{C}}$ and then use a complex linear projection $\mathbf{C}^n \twoheadrightarrow V_\alpha^{\mathbf{C}}$).

Consider the product $Q_\alpha Q^*_\alpha$ of the polynomials Q_α and Q^*_α. This complex polynomial has the following remarkable properties: (1) $Q_\alpha Q^*_\alpha|_{V_\beta} \equiv 0$ for any $\beta \neq \alpha$; (2) $Q_\alpha Q^*_\alpha$ is real valued on V_α; (3) $Q_\alpha Q^*_\alpha(a) \neq 0 \neq Q_\alpha Q^*_\alpha(b)$.

Again, using that V_α is totally real, one can construct some polynomial $G_\alpha: \mathbf{C}^n \to \mathbf{C}$ which is real-valued on V_α and such that $G_\alpha Q_\alpha Q^*_\alpha(a) \neq G_\alpha Q_\alpha Q^*_\alpha(b)$ (recall, that $Q_\alpha Q^*_\alpha$ cannot simultaneously vanish at a and b). Hence, the polynomial $P_\alpha = G_\alpha \cdot Q_\alpha \cdot Q^*_\alpha$ separates a and b. Moreover, it is real-valued on V_α and vanishes on any V_β, $\beta \neq \alpha$. Consequently, Ω_α is a singleton or $\Omega_\alpha \subset |D\mathscr{L}|$. In fact, if $\Omega_\alpha = \hat{\Omega}_\alpha$ is a singleton a, then $\Omega = \Omega_\alpha$ (note that $Q_\alpha Q^*_\alpha(a) \neq 0$ and, hence, it separates a from $|D\mathscr{L}| \cup (\bigcup_{\beta\neq\alpha} V_\beta) \subset \bigcup_{\beta\neq\alpha} V_\beta^{\mathbf{C}}$.

To complete the proof of Lemma 1 we apply inductively the same argument to the derived families $D\mathscr{L}, D^2\mathscr{L}, \dots$ and use that $\mathscr{L}$ is perfectly generic (i.e. each $D^s\mathscr{L}$, $s = 1, 2, \dots$ is weakly generic and totally real). □

As we mentioned before, $|\mathscr{L}|$ is a stratified space with the stratification induced by different intersections $V_{\hat{\alpha}} = V_{\alpha_1} \cap V_{\alpha_2} \cap \cdots \cap V_{\alpha_t}$, $\alpha_1, \alpha_2, \dots, \alpha_t \in \mathscr{L}$. In this way, starting with $\mathscr{L}$ one can produce a new family $\hat{\mathscr{L}} \supset \mathscr{L}$ of real affine subspaces parameterizing different multiple intersections. Let us say that $\mathscr{L}$ is a *generic family* if any two spaces $V_{\hat{\alpha}}$ and $V_{\hat{\beta}}^{\mathbf{C}}$ are in "general position" in $\mathbf{C}^n$ for each pair $\hat{\alpha}, \hat{\beta} \in \hat{\mathscr{L}}$, i.e. $V_{\hat{\alpha}} \cap V_{\hat{\beta}}^{\mathbf{C}}$ is of the smallest possible dimension, provided that $V_{\hat{\alpha}} \cap V_{\hat{\beta}}$ is fixed. More precisely, the spaces $W_{\hat{\alpha},\hat{\beta}} \subseteq V_\alpha$ and $V_\beta^{\mathbf{C}}$ should be in general position as real subspaces of $\mathbf{C}^n$, where $W_{\hat{\alpha},\hat{\beta}}$ denotes a subspace of $V_{\hat{\alpha}}$ which does not intersect $V_{\hat{\alpha}} \cap V_{\hat{\beta}}$ and which is of a maximal possible dimension.

Denote by $\operatorname{L\,Imb}(|\mathscr{L}|, \mathbf{C}^n)$ the space of all linear imbeddings of the space $|\mathscr{L}|$ into $\mathbf{C}^n$. Here $|\mathscr{L}|$ is considered without the ambient space $\mathbf{C}^n$, but with the fixed real linear structure for each $V_{\hat{\alpha}} \subset |\mathscr{L}|$, $\hat{\alpha} \in \hat{\mathscr{L}}$. Let $A(2n, \mathbf{R})$ be the Lie group of all real affine transformations of $\mathbf{R}^{2n} \simeq \mathbf{C}^n$. This group acts naturally on $\operatorname{L\,Imb}(|\mathscr{L}|, \mathbf{C}^n)$. For any $F \in \operatorname{L\,Imb}(|\mathscr{L}|, \mathbf{C}^n)$ and $g \in A(2n, \mathbf{R})$ we denote by $g(F)$ the imbedding

$$|\mathscr{L}| \xrightarrow{F} F(|\mathscr{L}|) \xrightarrow{g} g(F(|\mathscr{L}|)) \subset \mathbf{C}^n.$$

LEMMA 2. *Let* $\dim \mathscr{L} \leq n$. *Then the linear imbeddings* $F\colon |\mathscr{L}| \to \mathbf{C}^n$ *with the property "*$F(|\mathscr{L}|)$ *is totally real and generic" form an open and everywhere dense set* $\mathscr{G}$ *in the space* $\mathrm{L\,Imb}(|\mathscr{L}|, \mathbf{C}^n)$. *Moreover, for any* $F_0 \in \mathrm{L\,Imb}(|\mathscr{L}|, \mathbf{C}^n)$ *the set* A_{F_0} *of affine transformations g with the property* $g(F_0) \in \mathscr{G}$ *form an open and everywhere dense subset of* $A(2n, \mathbf{R})$.

The properties of $F(|\mathscr{L}|)$ being totally real and generic are both general position properties. Hence, the openness of $\mathscr{G}$ in $\mathrm{L\,Imb}(|\mathscr{L}|, \mathbf{C}^n)$ or of A_{F_0} in $A(2n, \mathbf{R})$ is obvious. So, we have to prove that $\mathscr{G}$ and A_{F_0} are everywhere dense in the corresponding spaces.

For any $V_\alpha \subset F_0(|\mathscr{L}|)$, $\alpha \in \mathscr{L}$, $F_0 \in \mathrm{L\,Imb}(|\mathscr{L}|, \mathbf{C}^n)$ consider the subset $\rho_\alpha \subset A(2n, \mathbf{R})$ such that $g \in \rho_\alpha$ iff $g(V_\alpha)$ is totally real. If $\dim V_\alpha \leq n$ then one can check that ρ_α is open and everywhere dense in $A(2n, \mathbf{R})$. Consequently, $\rho_{\mathscr{L}} = \bigcap_{\alpha \in \mathscr{L}} \rho_\alpha$ is open and everywhere dense as well. Picking some $\tilde{g} \in \rho_{\mathscr{L}}$ sufficiently close to the identity one can approximate F_0 by a totally real imbedding $\tilde{F}_0 = \tilde{g}(F_0)$. Hence, for $\dim \mathscr{L} \leq n$ totally real imbeddings are everywhere dense in $\mathrm{L\,Imb}(|\mathscr{L}|, \mathbf{C}^n)$. Now take any pair of affine subspaces $V_{\hat{\alpha}}$, $V_{\hat{\beta}} \subset \tilde{F}_0(|\mathscr{L}|)$, $\hat{\alpha}$, $\hat{\beta} \in \hat{\mathscr{L}}$, such that $V_{\hat{\alpha}} \subsetneqq V_{\hat{\beta}}$. Recall, that $W_{\hat{\alpha},\hat{\beta}}$ is a subspace of $V_{\hat{\alpha}}$ of a maximal dimension such that $W_{\hat{\alpha},\hat{\beta}} \cap (V_{\hat{\alpha}} \cap V_{\hat{\beta}}) = \varnothing$. Consider the following subset $\Sigma_{\hat{\alpha},\hat{\beta}} \subset A(2n, \mathbf{R})$. An element $g \in \Sigma_{\hat{\alpha},\hat{\beta}}$ iff $g(W_{\hat{\alpha},\hat{\beta}})$ is in general position with the complex subspace $[g(V_{\hat{\beta}})]^{\mathbf{C}}$. Again, the openness of $\Sigma_{\hat{\alpha},\hat{\beta}}$ is obvious. To prove that $\Sigma_{\hat{\alpha},\hat{\beta}}$ is dense in $A(2n, \mathbf{R})$ we show that the identity transformation $e \in A(2n, \mathbf{R})$ can be approximated by some $g \in A(2n, \mathbf{R})$ with the property $g(V_{\hat{\beta}}) = V_{\hat{\beta}}$ and $g(W_{\hat{\alpha},\hat{\beta}})$ being transversal to $V_{\hat{\beta}}^{\mathbf{C}}$. Note, that by the construction, $W_{\hat{\alpha},\hat{\beta}}$ and $V_{\hat{\beta}}$ are in general position in $\mathbf{C}^n$. Take $\tilde{W}_{\hat{\alpha},\hat{\beta}} \subset \mathbf{C}^n$ sufficiently close to $W_{\hat{\alpha},\hat{\beta}}$ (so it still will be in general position with $V_{\hat{\beta}}$) and transverse to $V_{\hat{\beta}}^{\mathbf{C}}$. Now it is easy to construct a real affine transformation g mapping $W_{\hat{\alpha},\hat{\beta}}$ onto $\tilde{W}_{\hat{\alpha},\hat{\beta}}$ and identical on $V_{\hat{\beta}}$. Moreover, this g can be taken close to e. So, the subset $\Sigma_{\mathscr{L}} = \rho_{\mathscr{L}} \cap (\bigcap_{\{V_{\hat{\alpha}} \subsetneqq V_{\hat{\beta}}\}} \Sigma_{\hat{\alpha},\hat{\beta}})$ of $A(2n, \mathbf{R})$ is open and everywhere dense. This implies that totally real and generic imbeddings are open and everywhere dense in $\mathrm{L\,Imb}(|\mathscr{L}|, \mathbf{C}^n)$, provided that $\dim \mathscr{L} \leq n$.□

LEMMA 3. *If* $\dim \mathscr{L} \leq \frac{2}{3}n$ *then* $\mathscr{L}$ *totally real and generic implies that* $\mathscr{L}$ *is perfectly generic. Consequently, the set of imbeddings* $F \in \mathrm{L\,Imb}(|\mathscr{L}|, \mathbf{C}^n)$ *with the property "*$\mathscr{P}(K) = C(K)$*" for any compact* $K \subset F(|\mathscr{L}|)$ *contains an open and everywhere dense subset of* $\mathrm{L\,Imb}(|\mathscr{L}|, \mathbf{C}^n)$.

If $\dim V_{\hat{\alpha}} + 2\dim V_{\hat{\beta}} \leq 2n$; $\hat{\alpha}$, $\hat{\beta} \in \hat{\mathscr{L}}$, and $\mathscr{L}$ is generic then $V_{\hat{\alpha}} \cap V_{\hat{\beta}}^{\mathbf{C}} = V_{\hat{\alpha}} \cap V_{\hat{\beta}}$ (when $V_{\hat{\alpha}} \cap V_{\hat{\beta}} \neq \varnothing$) or $V_{\hat{\alpha}} \cap V_{\hat{\beta}}^{\mathbf{C}}$ is at most a singleton (when $V_{\hat{\alpha}} \cap V_{\hat{\beta}} = \varnothing$ and $\dim V_{\hat{\alpha}} + 2\dim V_{\hat{\beta}} = 2n$) (see Fig. 1). Hence, under these dimensional assumptions $|D\mathcal{L}| = |\mathcal{L}'| \cup M$, where $|\mathcal{L}'|$ is formed by $V_{\hat{\alpha}}$, $\hat{\alpha} \in \hat{\mathcal{L}} \backslash \mathcal{L}$(i.e. $\hat{\alpha}$ is not a maximal element of $\hat{\mathcal{L}}$) and M is a finite set of points (0-dimensional subspaces) in $\mathbf{C}^n$. Note, that $\mathcal{L}$ generic implies that $\mathcal{L}' \cup M$ is a generic family too. In fact, any subfamily of a generic family is generic. So, $\mathcal{L}'$ is generic. By the construction $V_{\hat{\beta}}^{\mathbf{C}} \cap M = \varnothing$ for any $\hat{\beta} \in \hat{\mathcal{L}}'$. All the higher derivatives $D^s\mathcal{L}$, $s > 1$, will be just subfamilies of $\hat{\mathcal{L}}$ and, hence, are generic (weakly generic). So, $\mathcal{L}$ is perfectly generic and Lemmas 1 and 2 imply Lemma 3. □

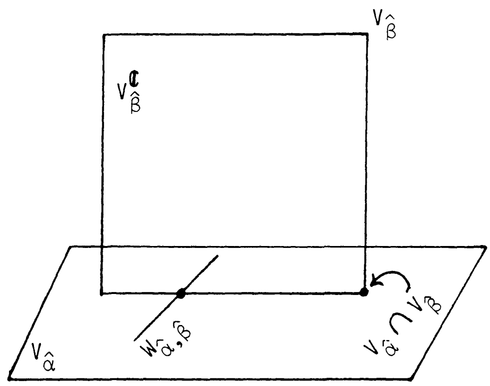

FIGURE 1

REMARK. Lemma 3 is the only place where we are using the dimensional restriction $\dim X \leq \frac{2}{3}n$. We conjecture that this lemma holds just if $\dim \mathcal{L} < n$, which would imply Theorems A and B for compact spaces or for finite polyhedra of dimensions less than n.

Now we are able to prove Theorem B. Any simplicially linear mapping $F\colon Y^k \to \mathbf{C}^n$ is uniquely determined by the images $\{F(y_j)\}_j$ of the vertices $\{y_j\}_j$ of the simplicial polyhedron Y^k. If the points $\{F(y_j)\}$ are in general position over the field $\mathbf{R}$ in $\mathbf{C}^n \simeq \mathbf{R}^{2n}$, it follows from standard dimensional considerations that F is an imbedding for $k < n$. Actually, if they are in general position in $\mathbf{C}^n$ over $\mathbf{C}$, then any real affine subspace passing through arbitrary s points $\{F(y_{j_e})\}_e$, $s \leq n$, is totally real.

Let $\Delta_\alpha^s \subset Y^k$ denote an s-dimensional simplex of Y^k, where index α enumerates such simplices. For any $\Delta_\alpha^s \subset Y^k$ and $F \in \mathrm{SL\,Imb}(Y^k, \mathbf{C}^n)$ consider the real s-dimensional affine subspace $V_{\alpha,F}$ in $\mathbf{C}^n$, containing $F(\Delta_\alpha^s)$. This correspondence $\Delta_\alpha^s \rightsquigarrow V_{\alpha,F}$ defines a family of subspaces $\hat{\mathcal{L}}_F$ (the corresponding family $\mathcal{L}_F$ consists of $V_{\alpha,F}$, where Δ_α^s is not a subsimplex of any other simplex of Y^k).

Starting with any mapping $F \in \mathrm{SL\,Map}(Y^k, \mathbf{C}^n)$ one can approximate F by an imbedding $\tilde{F}(k < n)$. Note that the group $A(2n, \mathbf{R})$ acts naturally on $\mathrm{SL\,Map}(Y^k, \mathbf{C}^n)$, moreover, the subspace $\mathrm{SL\,Imb}(Y^k, \mathbf{C}^n) \subset \mathrm{SL\,Map}(Y^k, \mathbf{C}^n)$ obviously is invariant under this action. By Lemma 2 and using the continuity of the correspondence $F \rightsquigarrow |\mathscr{L}_F|$ one can approximate $\tilde{F} \in \mathrm{SL\,Imb}(Y^k, \mathbf{C}^n)$ by some imbedding $g(\tilde{F})$, $g \in A(2n, \mathbf{R})$ with the property $g(|\mathscr{L}_{\tilde{F}}|)$ is totally real and generic. By Lemma 3 such a family will be perfectly generic, provided that $3k \leq 2n$. Hence, by Lemma 1 $g(\tilde{F}(Y^k)) \subset g(|\mathscr{L}_{\tilde{F}}|)$ admits polynomial approximation.

The properties "$\mathscr{L}_F$ totally real, generic, perfectly generic" obviously are stable with respect to small perturbations of $F \in \mathrm{SL\,Imb}(Y^k, \mathbf{C}^n)$. Hence, for $k \leq \frac{2}{3}n$ the subset $\{F \in \mathrm{SL\,Imb}(Y^k, \mathbf{C}^n) | \mathscr{L}_F$ is totally real and perfectly generic$\}$ is open and everywhere dense in $\mathrm{SL\,Map}(Y^k, \mathbf{C})$, which completes the proof of Theorem B. □

Now we derive Theorem A from Theorem B.

Let X^k be any compact space. Let $\Theta_{\varepsilon,\delta}$ be the subset of $\mathrm{Map}(X^k, \mathbf{C}^n)$ defined by the following two properties: (1) the diameter of the inverse-image $F^{-1}(y)$ of any point $y \in \mathbf{C}^n$ is less than δ; (2) the functions $\bar{z}_1, \ldots, \bar{z}_n$ on $F(X^k)$, where $^-$ denotes the complex conjugation, may be approximated to within ε by complex polynomials in $z_1, \ldots, z_n$. It is readily verified that $\Theta_{\varepsilon,\delta}$ is an open set of $\mathrm{Map}(X^k, \mathbf{C}^n)$.

Now choose some countable monotone sequence $\{\varepsilon_i\} \to 0$, $\{\delta_i\} \to 0$. It is easy to verify that $\bigcap_i \Theta_{\varepsilon_i,\delta_i}$ is the set Θ of all imbeddings F admitting polynomial approximation on $F(X^k)$. Indeed, if we let $\delta_i \to 0$ property (1) of the sets $\Theta_{\varepsilon_i,\delta_i}$ guarantees that the limiting mapping is an imbedding. Property (2) of the sets implies that if $F \in \bigcap_i \Theta_{\varepsilon_i,\delta_i}$ then the functions $\{\bar{z}_j\}$ on the image $F(X^k)$ may be approximated to within arbitrary accuracy by polynomials in $\{z_j\}$. On the other hand, by the Weierstrass-Stone theorem any continuous function on $F(X^k)$ may be approximated by polynomials in $\{z_j, \bar{z}_j\}$; hence it may be approximated by polynomials in the variables in the variables $\{z_j\}$ alone.

To complete the proof, it remains to verify that every set $\Theta_{\varepsilon_i,\delta_i}$ is dense in $\mathrm{Map}(X^k, \mathbf{C}^n)$.

Let $F' \in \mathrm{Map}(X^k, \mathbf{C}^n)$ be an arbitrary mapping. In accordance with the classical Alexandroff construction [**1**], if $m < n$, then for any ε, $\delta > 0$ there is a mapping$F$: $X^k \to \mathbf{C}^n$ such that $F(X^k)$ is contained in a k-dimensional simplicial polyhedron Y^k simplicially-linearly imbedded in $\mathbf{C}^n$, in such a way that

(a) $\rho(F', F) < \varepsilon$, where ρ is the natural distance between mappings;
(b) $\mathrm{diam}(F^{-1}(y)) < \delta$ for any point $y \in Y^k$.

(A complete proof of this theorem can also be found in [4], Chapter V, §3).

Set $\delta = \delta_i$. By a trivial modification of this construction one can guarantee that, in addition to these two properties (a) and (b), the family of affine subspaces $\mathscr{L}_{\text{id}}$ (generated by id: $Y^k \to \mathbf{C}^n$) will be totally real and generic (just use the appropriate transformation from $A(2n, \mathbf{R})$). If $3k \leq 2n$ then, by Lemma 3, these properties are a sufficient condition for the existence of polynomial approximation on the polyhedron Y^k. The modification is as follows. By Theorem B there exists an imbedding κ: $Y^k \to \mathbf{C}^n$, arbitrarily close to the original imbedding id: $Y^k \to \mathbf{C}^n$, such that continuous functions admit polynomial approximation on $\kappa(Y^k)$. The imbedding $\kappa \in \mathrm{SL\,Map}(Y^k, \mathbf{C}^n)$ may be chosen in such a way that $\rho(F', \kappa \circ F) < \varepsilon$, while $\mathrm{diam}(F^{-1} \circ \kappa^{-1}(y)) < \delta_i$ for any $y \in \mathbf{C}^n$. Moreover, the functions $\{\bar{z}_j\}$ may be approximated on $\kappa \circ F(X^k)$ to within arbitrary accuracy by polynomials in $\{z_i\}$, i.e., $\kappa \circ F \in \Theta_{\varepsilon_i, \delta_i}$ and $\kappa \circ F$ is in the ε-neighborhood of the original mapping F'. This proves that $\Theta_{\varepsilon_i, \delta_i}$ is dense in $\mathrm{Map}(X^k, \mathbf{C}^n)$.

Recall that for any compact set K in $\mathbf{C}^n$ the space of maximal ideals of the algebra $\mathscr{P}(K)$ is precisely the polynomially convex hull of K. Therefore, if $\mathscr{P}(K)$ coincides with the algebra of all complex functions, then K is polynomially convex and this property is hereditary with respect to compact subsets of K. Thus, if $3k \leq 2n$ the polynomially convex imbeddings of a k-dimensional compact space into $\mathbf{C}^n$ form a massive set (i.e. of type G_δ). This completes the proof of Theorem A. □

It is obvious that if all continuous functions on a compact subset $K \subset \mathbf{C}^n$ admit polynomial approximation, this property is hereditary with respect to closed subsets and therefore, in particular, the intersection $K \cap \mathbf{C}^l$ of a compact subset K with any affine complex subspace also admits approximation by polynomials in $z_1, \ldots, z_n$. In particular, in the case $k = l$, it follows from the maximum modulus theorem that the set $K \cap \mathbf{C}^1$ is necessarily nowhere dense in $\mathbf{C}^1$ and has connected complement.

COROLLARY. *Let X^k be a k-dimensional compact space. If $3k \leq 2n$, the imbeddings $F \in \mathrm{Map}(X^k, \mathbf{C}^n)$ such that the intersection of $F(X^k)$ with any complex straight line $\mathbf{C}^1 \subset \mathbf{C}^n$ is nowhere dense in $\mathbf{C}^1$ and the complement of the intersection is connected in $\mathbf{C}^1$ form a dense subset of type G_δ.* □

Let M^k be a PL-manifold. Then starting with an arbitrary locally flat PL-imbedding F_0: $M^k \to \mathbf{C}^n$ $(k < n)$ it is possible to find an element $g \in A(2n, \mathbf{R})$ such that $g(F_0)(M^k)$ will generate a totally real and generic family of affine subspaces and, hence, for $k \leq \frac{2}{3}n$ one has polynomial approximation on $g(F_0)(M^k)$. Moreover, $g(F_0)(M^k)$ is again locally flat. Thus, by Theorem B for $k \leq \frac{2}{3}n$ there exists a PL-imbedding F of M^k in $\mathbf{C}^n$ with $F(M^k)$ *having a nice normal PL-bundle* and admitting polynomial approximation (hence, $F(M^k)$ is polynomially convex in $\mathbf{C}^n$). In particular, the tangent bundle to $F(M^k)$ is formed by "totally real" blocks.

Considering smooth or real-analytic manifolds M^k, it would be natural to try to prove "smooth or analytic" analogs of Theorems A and B. But it seems quite unlikely that such propositions can be established. As a matter of fact, for $k \geq \frac{2}{3}n$ there exist profound topological obstacles to the existence of totally real and regular imbedding, i.e., imbeddings $F: M^k \to \mathbf{C}^n$ such that dF is nondegenerate and $dF(T_xM^k)$ is totally real for any tangent space T_xM^k of M^k, $x \in M^k$.

One can find a very good discussion of similar and more delicate analytic phenomena in [7] and [**8**] §§17, 18 bascially, for the case $k \geq n$.

As an example, let us consider regular imbeddings $F: \mathbf{C}P^k \to \mathbf{C}^n$ of complex projective space $\mathbf{C}P^k$. Let τ be a tangent bundle of $F(\mathbf{C}P^k)$ and assume that it is a totally real subbundle of the complex tangent bundle to $\mathbf{C}^n$. Hence, its complexification $\tau^{\mathbf{C}}$ is isomorphic to $\tau \oplus J\tau$, where the infinitesimal operator J is induced by multiplication of vectors by the imaginary unit i. Let ν be the bundle complementary to $\tau \oplus J(\tau)$, i.e., $\tau \oplus J(\tau) \oplus \nu = \tau(\mathbf{C}^n)|F(\mathbf{C}P^k)$ is the trivial bundle. Since $\tau_x \oplus J(\tau_x)$ is a complex subspace of $\mathbf{C}^n$, we may assume that ν is a complex bundle of complex dimension $n - 2k$. The Chern class $c(\tau^{\mathbf{C}})$ of $\tau^{\mathbf{C}}$ is equal to

$$\sum_{i=0}^{k} c_i(\tau) \times \sum_{i=0}^{k} (-1)^i c_i(\tau) \quad \text{or} \quad (1 - h^2)^{k+1},$$

where $h \in H^2(\mathbf{C}P^k; Z)$ is a standard generator and $(1 - h^2)^{k+1}$ is considered as an element of the ring $\mathbf{Z}[h]/\{h^{k+1} = 0\}$ [**5**]. Since $\tau^{\mathbf{C}} \oplus \nu$ is trivial, it follows that $c(\nu) \cdot c(\tau^{\mathbf{C}}) = 1$. The element $c(\tau^{\mathbf{C}})$ is invertible in the ring $\mathbf{Z}[h]/\{h^{k+1} = 0\}$. As a representative of the inverse element, we take the polynomial $[\sum_{i=0}^{[k/2]} h^{2i}]^{k+1}$, where $[k/2]$ is the integral part of $k/2$. After factorization modulo $h^{k+1} = 0$ we obtain a certain polynomial $\sum_{i=0}^{[k/2]} \alpha_i h^{2i}$, where $\{\alpha_i\}$ are different from zero. Therefore $c(\nu) = \sum_{i=0}^{[k/2]} \alpha_i h^{2i}$ and, since $\alpha_i \neq 0$, the complex dimension $n - 2k$ of ν cannot be less than $2 \cdot [k/2]$. Thus, when $n < 2k + 2[k/2]$, there exist no totally real immersions of $\mathbf{C}P^k$ into $\mathbf{C}^n$. In fact, the Euler class of the normal bundle of oriented submanifolds in $\mathbf{R}^{2n}$ should be trivial [**5**], which ruins the possibility for regular totally real imbeddings $\mathbf{C}P^k \hookrightarrow \mathbf{C}^{3k}$, k even. Since $\dim_{\mathbf{R}} \mathbf{C}P^k = 2k$, it follows that the "allowed" dimensions n satisfy the conditions $3 \dim_{\mathbf{R}}(\mathbf{C}P^k) < 2n$, which should be compared with the dimensional condition that figures in Theorems A and B.

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