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THE ANGULAR DERIVATIVE OF AN OPERATOR-VALUED ANALYTIC FUNCTION

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The classical theorem on the angular derivative of an analytic function on the half-plane $\operatorname{Re} z > 0$ is extended to operator-valued analytic functions.

1. Let Π denote the open half-plane

$$\Pi = \{z \in \mathbf{C}: \operatorname{Re} z > 0\}. \tag{1}$$

For a positive number k, let Σ_k denote the set

$$\Sigma_k = \{z \in \mathbf{C}: |\operatorname{Im} z| < k \operatorname{Re} z\}. \tag{2}$$

The following theorem in complex analysis is well-known:

Let f be a function analytic on Π such that $f(\Pi) \subset \Pi$. If

$$a = \inf_{z \in \Pi} \frac{\operatorname{Re} f(z)}{\operatorname{Re} z}, \tag{3}$$

then for any $k > 0$, we have

$$\lim_{\substack{z \to \infty \\ z \in \Sigma_k}} \frac{f(z)}{z} = \lim_{\substack{z \to \infty \\ z \in \Sigma_k}} \frac{\operatorname{Re} f(z)}{\operatorname{Re} z} = \lim_{\substack{z \to \infty \\ z \in \Sigma_k}} f'(z) = a. \tag{4}$$

The limit $\lim_{z \to \infty, z \in \Sigma_k} f'(z)$ is usually called the *angular derivative* of f at ∞. The above classical theorem is the work of several mathematicians: Julia, Nevanlinna, Wolff, Carathéodory, Landau, Valiron. For the original sources, the reader is referred to [**2**, p. 216] and [**5**, p.108]. The purpose of the present paper is to extend this classical theorem to operator-valued analytic functions [**3**, pp. 92–94].

2. Throughout this paper, $\mathscr{H}$ denotes a complex Hilbert space. By an operator we always mean a bounded linear operator on $\mathscr{H}$. The identity operator is denoted by I. For an operator A on $\mathscr{H}$, the adjoint of A is denoted by A^*; the real and imaginary parts of A are denoted by $\operatorname{Re} A$ and $\operatorname{Im} A$ respectively:

$$\operatorname{Re} A = \frac{A + A^*}{2}, \quad \operatorname{Im} A = \frac{A - A^*}{2i}.$$

For two Hermitian operators A, B on $\mathscr{H}$, we write $A \geq B$ to indicate that $A - B$ is a positive operator, i.e., $\langle (A - B)x, x\rangle \geq 0$ for all $x \in \mathscr{H}$. The strict inequality $A > B$ means that $A - B$ is positive and invertible. The classical theorem stated above can be generalized to the following result.

THEOREM. *Let F be an operator-valued analytic function on the open half-plane Π such that for each $z \in \Pi$, $F(z)$ is an operator on $\mathscr{H}$ with* $\operatorname{Re} F(z) > 0$. *Suppose there is a Hermitian operator A on $\mathscr{H}$ satisfying*

$$\frac{\operatorname{Re} F(z)}{\operatorname{Re} z} > A \quad \text{for all } z \in \Pi \tag{5}$$

and

(6) *for any $\varepsilon > 0$, there is $z_0 \in \Pi$ such that*

$$\left\| \frac{\operatorname{Re} F(z_0)}{\operatorname{Re} z_0} - A \right\| < \varepsilon.$$

Then for any $k > 0$ we have

$$\lim_{\substack{z \to \infty \\ z \in \Sigma_k}} \left\| \frac{F(z)}{z} - A \right\| = \lim_{\substack{z \to \infty \\ z \in \Sigma_k}} \left\| \frac{\operatorname{Re} F(z)}{\operatorname{Re} z} - A \right\| \tag{7}$$

$$= \lim_{\substack{z \to \infty \\ z \in \Sigma_k}} \| F'(z) - A \| = 0.$$

3. In proving our theorem, we shall need the following lemmas.

LEMMA 1. *Let F be an analytic function on Π such that for each $z \in \Pi$, $F(z)$ is an operator on $\mathscr{H}$ with* $\operatorname{Re} F(z) > 0$. *If z, $z_0 \in \Pi$ and*

$$\Psi(F(z), F(z_0)) = [\operatorname{Re} F(z_0)]^{-1/2}[F(z) - F(z_0)] \tag{8}$$

$$\times [F(z) + F(z_0)^*]^{-1}[\operatorname{Re} F(z_0)]^{1/2},$$

then

$$\Psi(F(z), F(z_0))^* \Psi(F(z), F(z_0)) \leq \left| \frac{z - z_0}{z + \bar{z}_0} \right|^2 I. \tag{9}$$

Proof. This is part (d) of Theorem 3 in [**1**].

LEMMA 2. *Let F be an analytic function on Π such that for each $z \in \Pi$, $F(z)$ is an operator on $\mathscr{H}$ with* $\operatorname{Re} F(z) > 0$. *If $F(z_0) = I$ for some $z_0 \in \Pi$, then*

$$\|F(z)\| \leq \frac{(|z| + |z_0|)^2}{(\operatorname{Re} z)(\operatorname{Re} z_0)} \quad \textit{for } z \in \Pi. \tag{10}$$

Proof. According to the definition (8) of Ψ, we have

$$\Psi(F(z), I) = [F(z) - I][F(z) + I]^{-1};$$

so (9) becomes

$$[F(z)^* + I]^{-1}[F(z)^* - I][F(z) - I][F(z) + I]^{-1} \leq \left|\frac{z - z_0}{z + \bar{z}_0}\right|^2 I, \tag{11}$$

for $z \in \Pi$.

Let

$$\alpha(z) = \left|\frac{z - z_0}{z + \bar{z}_0}\right|^2,$$

which is clearly < 1 for $z \in \Pi$. From (11) we have for $z \in \Pi$:

$$[F(z)^* - I][F(z) - I] \leq \alpha(z)[F(z)^* + I][F(z) + I],$$

which can be written

$$\left[F(z)^* - \frac{1 + \alpha(z)}{1 - \alpha(z)} I\right]\left[F(z) - \frac{1 + \alpha(z)}{1 - \alpha(z)} I\right] \leq \frac{4\alpha(z)}{[1 - \alpha(z)]^2} I$$

or

$$\left\|F(z) - \frac{1 + \alpha(z)}{1 - \alpha(z)} I\right\| \leq \frac{2\alpha(z)^{1/2}}{1 - \alpha(z)}.$$

Then (10) follows from

$$\begin{aligned}\|F(z)\| &\leq \left\|F(z) - \frac{1 + \alpha(z)}{1 - \alpha(z)} I\right\| + \frac{1 + \alpha(z)}{1 - \alpha(z)} \\ &\leq \frac{2\alpha(z)^{1/2}}{1 - \alpha(z)} + \frac{1 + \alpha(z)}{1 - \alpha(z)} \\ &= \frac{(|z + \bar{z}_0| + |z - z_0|)^2}{4(\operatorname{Re} z)(\operatorname{Re} z_0)} \leq \frac{(|z| + |z_0|)^2}{(\operatorname{Re} z)(\operatorname{Re} z_0)} \quad \text{for } z \in \Pi.\end{aligned}$$

4. Proof of the theorem. With the aid of Lemma 2, the proof of our theorem is an operator-analogue of Landau-Valiron's proof [**4**], [**5**, pp. 87–89] of the classical case. Consider a fixed $\varepsilon > 0$. By hypothesis, we can choose $z_0 \in \Pi$ such that

$$\left\| \frac{\operatorname{Re} F(z_0)}{\operatorname{Re} z_0} - A \right\| < \varepsilon. \tag{12}$$

Define operator-valued analytic functions E and G on Π by

$$E(z) = F(z) - Az, \tag{13}$$

$$G(z) = [\operatorname{Re} E(z_0)]^{-1/2}[E(z) - i \operatorname{Im} E(z_0)][\operatorname{Re} E(z_0)]^{-1/2}. \tag{14}$$

By (5), $\operatorname{Re} E(z) > 0$ for $z \in \Pi$. As

$$\operatorname{Re} G(z) = [\operatorname{Re} E(z_0)]^{-1/2}[\operatorname{Re} E(z)][\operatorname{Re} E(z_0)]^{-1/2}, \tag{15}$$

we have also $\operatorname{Re} G(z) > 0$ for $z \in \Pi$. Clearly $G(z_0) = I$. An application of Lemma 2 to G gives

$$\|G(z)\| \leq \frac{(|z| + |z_0|)^2}{(\operatorname{Re} z)(\operatorname{Re} z_0)} \quad \text{for } z \in \Pi. \tag{16}$$

By (13), (14) and (16), we have for $z \in \Pi$:

$$\begin{aligned}
\left\| \frac{F(z)}{z} - A \right\| &= \frac{\|E(z)\|}{|z|} \\
&= \frac{1}{|z|} \left\| [\operatorname{Re} E(z_0)]^{1/2} G(z) [\operatorname{Re} E(z_0)]^{1/2} + i \operatorname{Im} E(z_0) \right\| \\
&\leq \frac{1}{|z|} \left\| [\operatorname{Re} E(z_0)]^{1/2} \right\|^2 \|G(z)\| + \frac{\|\operatorname{Im} E(z_0)\|}{|z|} \\
&\leq \frac{\|[\operatorname{Re} E(z_0)]^{1/2}\|^2}{\operatorname{Re} z_0} \frac{(|z| + |z_0|)^2}{|z|(\operatorname{Re} z)} + \frac{\|\operatorname{Im} E(z_0)\|}{|z|}.
\end{aligned}$$

Since

$$\frac{\|[\operatorname{Re} E(z_0)]^{1/2}\|^2}{\operatorname{Re} z_0} = \frac{\|\operatorname{Re} E(z_0)\|}{\operatorname{Re} z_0} = \left\| \frac{\operatorname{Re} F(z_0)}{\operatorname{Re} z_0} - A \right\| < \varepsilon,$$

it follows that

$$\left\| \frac{F(z)}{z} - A \right\| \leq \varepsilon \frac{(|z| + |z_0|)^2}{|z|(\operatorname{Re} z)} + \frac{\|\operatorname{Im} E(z_0)\|}{|z|} \quad \text{for } z \in \Pi. \tag{17}$$

For $z \in \Sigma_k$ we have

$$\frac{(|z| + |z_0|)^2}{|z|(\operatorname{Re} z)} = \left(1 + \left|\frac{z_0}{z}\right|\right)\frac{|z| + |z_0|}{\operatorname{Re} z} \leq \left(1 + \left|\frac{z_0}{z}\right|\right)\left(\sqrt{1 + k^2} + \frac{|z_0|}{\operatorname{Re} z}\right).$$

Therefore

$$\left\|\frac{F(z)}{z} - A\right\| \leq \varepsilon\left(1 + \left|\frac{z_0}{z}\right|\right)\left(\sqrt{1 + k^2} + \frac{|z_0|}{\operatorname{Re} z}\right) + \frac{\|\operatorname{Im} E(z_0)\|}{|z|} \tag{18}$$

holds for $z \in \Sigma_k$. The right-hand side of (18) tends to $\varepsilon\sqrt{1 + k^2}$ as $z \in \Sigma_k$ tends to ∞. Since $\varepsilon > 0$ can be arbitrarily small, this proves that

$$\lim_{\substack{z \to \infty \\ z \in \Sigma_k}} \left\|\frac{F(z)}{z} - A\right\| = 0. \tag{19}$$

Next, by (13) we have

$$\left\|\frac{\operatorname{Re} F(z)}{\operatorname{Re} z} - A\right\| = \left\|\frac{\operatorname{Re} E(z)}{\operatorname{Re} z}\right\| \leq \frac{\|E(z)\|}{\operatorname{Re} z} = \frac{|z|}{\operatorname{Re} z}\left\|\frac{F(z)}{z} - A\right\| \quad \text{for } z \in \Pi$$

and therefore

$$\left\|\frac{\operatorname{Re} F(z)}{\operatorname{Re} z} - A\right\| \leq \sqrt{1 + k^2}\left\|\frac{F(z)}{z} - A\right\| \quad \text{for } z \in \Sigma_k. \tag{20}$$

From (19) and (20), it follows that

$$\lim_{\substack{z \to \infty \\ z \in \Sigma_k}} \left\|\frac{\operatorname{Re} F(z)}{\operatorname{Re} z} - A\right\| = 0. \tag{21}$$

Given $k > 0$, choose $h > 0$ so small that for every $z \in \Sigma_k$ the circle $C_h(z) = \{w \in \mathbf{C}: |w - z| = h|z|\}$ is contained in Π. Then from Cauchy's integral formula [**3**, p. 96]

$$E'(z) = \frac{1}{2\pi i}\int_{C_h(z)} \frac{E(w)\,dw}{(w - z)^2} \quad \text{for } z \in \Sigma_k,$$

we derive

$$\|E'(z)\| \leq \frac{1}{h|z|}\operatorname*{Max}_{w \in C_h(z)} \|E(w)\| = \frac{1}{h}\operatorname*{Max}_{w \in C_h(z)} \left|\frac{w}{z}\right|\left\|\frac{F(w)}{w} - A\right\| \leq \frac{1 + h}{h}\operatorname*{Max}_{w \in C_h(z)}\left\|\frac{F(w)}{w} - A\right\| \quad \text{for } z \in \Sigma_k.$$

This together with (19) implies

$$\lim_{\substack{z\to\infty\\ z\in\Sigma_k}} \|F'(z)-A\| = \lim_{\substack{z\to\infty\\ z\in\Sigma_k}} \|E'(z)\| = 0. \tag{22}$$

The proof is complete.

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