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REARRANGEMENTS AND CATEGORY

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Kolmogorov stated, and Zahorski proved, that there exists an L^2-Fourier series such that some rearrangement of it diverges almost everywhere. Kac and Zygmund asked if the set of rearrangements which make this Fourier series diverge almost everywhere is first category or second category. A general theorem is proved which has as a special case that the set of rearrangements in question is residual.

1. Introduction. Let G be the group of all permutations of the positive integers, and let H be the normal subgroup consisting of all permutations which are the identity outside of a finite set. G is a topological group in the compact open topology, and H is a countable dense subgroup. In this topology a basic open set consists of all permutations which agree with a fixed permutation on a finite set of integers. One can check that this topology is metrizable with the metric

$$d(\pi, \pi') = \sum_{n \geq 1} 2^{-n}\big(d_n(\pi, \pi') + d_n(\pi^{-1}, \pi'^{-1})\big),$$

where

$$d_n(\pi, \pi') = |\pi(n) - \pi'(n)|/\big(1 + |\pi(n) - \pi'(n)|\big).$$

G with the metric $d(\cdot, \cdot)$ is a Polish space.

The main purpose of this paper is to prove the following theorem.

THEOREM 1.1. *Let (X, μ) be a regular locally compact σ-finite measure space, Z a Banach space, and f_n: $X \to Z$ a sequence of Borel measurable functions. Suppose the series $\sum_{n \geq 1} f_n(x)$ diverges μ-a.e. Then the set of π's in G so that $\sum_{n \geq 1} f_{\pi(n)}(x)$ diverges μ-a.e. is a residual set in G.*

The only precedent for this theorem seems to be a result of R. P. Agnew [**1**], who proved a similar theorem for sequences of complex numbers.

Theorem 1.1. has an immediate application to an open question about L^2-Fourier series. In 1927 Kolmogorov stated ([**4**], Theoreme III), but did not prove, that there exists an L^2-Fourier series such that some rearrangement of it diverges almost everywhere. Zahorski [**8**] sketched a proof of this fact. Zygmund ([**7**], p. 34) and Kac ([**7**], pp. 21–22) asked if the set of rearrangements which make this series diverge almost everywhere is first category or second category. Theorem 1.1 implies that the set of rearrangements in question is in fact residual.

Another main result of this paper is a category analogue of Theorem 1.1. Let X be a Polish space. A subset A of X is said to have the Baire property if there exists an open set U in X so that $A \triangle U$ is first category. The collection of subsets of X with the Baire property is a σ-algebra which includes the analytic sets in X (Kuratowski [**5**]). Let Z be any other Polish space. A function f: $X \to Z$ is said to have the Baire property if U open in Z implies that $f^{-1}(U)$ has the Baire property in X. Clearly, any Borel function f: $X \to Z$ is a function with the Baire property. The following theorem is then a category analogue of Theorem 1.1.

THEOREM 1.2. *Let X be a Polish space, Z a separable Banach space, and f_n: $X \to Z$ a sequence of functions with the Baire property. Suppose that the series $\Sigma_{n\geq 1} f_n(x)$ diverges on a residual subset of X. Then the set of π's in G so that $\Sigma_{n\geq 1} f_{\pi(n)}(x)$ diverges on a residual subset of X is itself a residual subset of G.*

Section 2 is devoted to recalling a variant of a category 0-1 law and giving simple applications of it. Theorem 1.1 is proved in §3, and Theorem 1.2 is proved in §4.

2. A Category 0-1 Law and Some Applications. The following proposition is a great aid in proving Theorem 1.1 and Theorem 1.2. A variant of half of it may be found in Gottschalk and Hedlund [**2**], but a slightly different, condensed proof is given because of its central importance for this paper.

PROPOSITION 2.1. *Let X be a Polish space, F a countable group of homeomorphisms of X with at least one dense orbit, and A an F-invariant subset of X with the Baire property. Then A is either first category or residual. If every F-orbit is dense, A^c is the union of F-invariant G_δ's, and A is residual, then $A = X$. If A is any subset of X such that A^c contains an F-invariant G_δ which contains a dense F-orbit, then A is first category.*

Proof. To prove the first statement, choose an open set U so that $A \triangle U$ is first category. If $U = \varnothing$, then $A = A \triangle U$ is first category. If $U \neq \varnothing$, let $V = F \cdot U$. Since F is a group of homeomorphisms and has at least one dense orbit, V is open and dense in X. $A \triangle V$ is contained in

$$\bigcup_{f \text{ in } F} A \triangle (f \cdot U) = \bigcup_{f \text{ in } F} f \cdot A \triangle (f \cdot U) = \bigcup_{f \text{ in } F} f \cdot (A \triangle U),$$

a first category set. Therefore, $A^c = A \triangle X$ is contained in the union of $A \triangle V$ and V^c, which is first category. So A is residual.

To prove the second statement, if one of the F-invariant G_δ's in A^c is not empty, then it is dense, and therefore A^c is residual. But this is a contradiction.

The third assertion follows from the same line of reasoning. This proves Proposition 2.1.

The following sequence of corollaries are simple applications of Proposition 2.1, and are illustrative of its power. A typical example of the situation where the hypotheses of Proposition 2.1 hold is the following: Let J be a Polish group, K a closed subgroup, $X = J/K$, and F a countable dense subgroup of J.

COROLLARY. 2.2. *Let Z be a Banach space, and $\sum_{n \geq 1} z_n$ a series in Z. Let $A = [\pi$ in $G \mid \sum_{n \geq 1} z_{\pi(n)}$ converges$]$. Then either $A = G$ or A is first category.*

Proof. Use the Cauchy criterion for convergence to check that

$$A^c = \bigcup_{k \geq 1} \bigcap_{N \geq 1} \bigcup_{\substack{m,n \\ N \leq m \leq n}} \left[\pi \text{ in } G \;\middle|\; \left\| \sum_{m \leq j \leq n} z_{\pi(j)} \right\| > \frac{1}{k} \right].$$

Check easily that this displays A^c as a countable union (over k) of H-invariant G_δ's. Proposition 2.1 then implies that A is either first category or $A = G$. This proves Corollary 2.2.

The only precedent for Corollary 2.2 seems to be a result of Agnew [**1**], who proves a similar result by different methods for the special case in which Z is the complex numbers.

Next, let Ω be a set, $\mathscr{B}$ a σ-algebra of subsets of Ω, and ν and ν' two countably additive probability measures on $\mathscr{B}$. Kakutani [**3**] has defined, via the Hellinger integral, an inner product $\rho(\nu, \nu')$ which satisfies $0 \leq \rho(\nu, \nu') \leq 1$, and also that $\rho(\nu, \nu') = 0$ if and only if ν and ν' are mutually singular. For each integer $n \geq 1$, let $\Omega_n = \Omega$, $\Omega^* = \prod_{n \geq 1} \Omega_n$, and $\mathscr{B}^* = \prod_{n \geq 1} \mathscr{B}$. Let μ_n and μ'_n $(n \geq 1)$ be two sequences of probability measures on $\mathscr{B}$, all equivalent to one another, and let $\mu = \prod_{n \geq 1} \mu_n$ and $\mu' = \prod_{n \geq 1} \mu'_n$ be the corresponding product measures on $\mathscr{B}^*$. If π is in G, let $\pi(\mu') = \prod_{n \geq 1} \mu'_{\pi(n)}$.

COROLLARY 2.3. *Let* $A = [\pi \text{ in } G \mid \mu$ *is equivalent to* $\pi(\mu')]$. *Then* A *is either first category or* $A = G$.

Proof. The main theorem of Kakutani [3] implies that

$$A^c = \bigcap_{k \geq 1} \bigcup_{N \geq 1} \left[\pi \text{ in } G \mid \prod_{1 \leq j \leq N} \rho\left(\mu_j, \mu'_{\pi(j)}\right) < \frac{1}{k} \right].$$

This shows that A^c is a G_δ in G. No factor $\rho(\mu_j, \mu'_{\pi(j)})$is 0. Hence, A^c is H-invariant. Proposition 2.1 now implies that A is either first category or $A = G$. This proves Corollary 2.3.

Another general setup where Proposition 2.1 has applications is the following one. Let Y be a Polish space and $X = \prod_{n \geq 1} Y$. G acts on X by permuting coordinates. Note that H has at least one dense orbit. For let y_j $(j \geq 1)$ be a countable dense subset of Y, and let x be any member of X so that each y_j occurs as a coordinate of x infinitely often. Check that the closure of Hx is X. In particular, take $Y_1 = \{1, -1\}$ or take Y_2 to be the unit circle, and let $A_i = [x \text{ in } X_i \mid \sum_{j \geq 1} x_j/j \text{ converges}]$ $(i = 1, 2)$.

COROLLARY 2.4. A_i *is of first category in* X_i $(i = 1, 2)$.

Proof.

$$A_i^c = \bigcup_{k \geq 1} \bigcap_{N \geq 1} \bigcup_{\substack{m,n \\ N \leq m \leq n}} \left[x \text{ in } X_i \,\middle|\, \left| \sum_{m \leq j \leq n} \frac{x_j}{j} \right| > \frac{1}{k} \right]$$

for $i = 1, 2$. This demonstrates that A_i^c is a Borel set which is a countable union (over k) of H-invariant G_δ's. A_1^c contains a dense H-orbit, for let $x = (1, -1, 1, 1, -1, 1, 1, 1, -1, \ldots)$. A_2^c also contains a dense H-orbit, for if y_j $(j \geq 1)$ is a countable dense subset of the unit circle, take x to be that element of X_2 all of whose entries consist of 1's, except for those entries which are perfect squares, in which the y_j's each occur infinitely often. Proposition 2.1 now implies that each A_i is first category. This proves Corollary 2.4.

This corollary shows immediately the otherwise known result that there is no category analogue of the three series theorem, the strong law of large numbers, or the ergodic theorem.

3. Proof of Theorem 1.1. Define a block to be a finite sequence of consecutive positive integers. If B is a block and N is a positive integer, say that B is greater than or equal to N (written $N \leq B$) if j in B implies that $N \leq j$. It is convenient to prove the following very special case of Theorem 1.1 first.

LEMMA 3.1. *Let K be a compact Hausdorff space, Z a Banach space, f_n: $K \to Z$ a sequence of continuous functions, and $\delta > 0$. Suppose that for every x in K and positive integer N, there exists a block B so that $N \leq B$ and $\|\sum_{j \text{ in } B} f_j(x)\| > \delta$. Then the set of π in G so that $\sum_{n \geq 1} f_{\pi(n)}(x)$ diverges for every x in K is a residual subset of G.*

Proof. If B is a block, define $U(B) = [x \text{ in } K \mid \|\sum_{j \text{ in } B} f_j(x)\| > \delta]$. $U(B)$ is an open subset of K (it may well be empty). Consider

$$A = \bigcap_{N \geq 1} \bigcup_{\substack{N \leq B_1, \ldots, B_p \\ U(B_1) \cup \cdots \cup U(B_p) = K \\ U(B_i) \neq \varnothing}} \bigcap_{1 \leq i \leq p} \times \left[\pi \text{ in } G \;\middle|\; \left\| \sum_{j \text{ in } B_i} f_{\pi(j)}(x) \right\| > \delta \text{ for all } x \text{ in } U(B_i) \right].$$

This displays A as an H-invariant G_δ. A in fact is nonempty, for a simple argument using the compactness of K and the continuity of the f_n's shows that the identity of G is in A. Hence, Proposition 2.1 shows that A is residual. Check easily that A is contained in the set of π in G so that $\sum_{n \geq 1} f_{\pi(n)}(x)$ diverges for every x in K. This proves Lemma 3.1.

To prove Theorem 1.1, first note that we may assume that μ is a probability measure since μ is σ-finite. Let

$$D_q = \bigcap_{N \geq 1} \bigcup_{\substack{m,n \\ N \leq m \leq n}} \left[x \text{ in } X \;\middle|\; \left\| \sum_{m \leq j \leq n} f_j(x) \right\| > \frac{1}{q} \right],$$

q a positive integer. Each D_q is a Borel subset of X, and $\bigcup_{q \geq 1} D_q$ is the set of x in X so that $\sum_{n \geq 1} f_n(x)$ diverges. For each q, choose a compact subset K_q of D_q so that each $f_n | K_q$ is continuous and $\mu(D_q - K_q) < 1/q$, using a vector-valued version of Lusin's theorem. Lemma 3.1 implies that

$$R_q = \left[\pi \text{ in } G \;\middle|\; \sum_{n \geq 1} f_{\pi(n)}(x) \text{ diverges for every } x \text{ in } K_q \right]$$

is residual. Hence, $R = \bigcap_{q \geq 1} R_q$ is residual, and is contained in $[\pi \text{ in } G \mid \sum_{n \geq 1} f_{\pi(n)}(x)$ diverges μ-a.e.], for $\mu(\bigcup_{q \geq 1} K_q) = 1$. This proves Theorem 1.1.

4. Proof of Theorem 1.2. Easily check that for each $n \geq 1$, the mapping $(x, \pi) \to f_{\pi(n)}(x)$, $X \times G \to Z$, is a function with the Baire property. Hence,

$$B = \left[(x, \pi) \mid \sum_{n \geq 1} f_{\pi(n)}(x) \text{ diverges}\right]$$
$$= \bigcup_{k \geq 1} \bigcap_{N \geq 1} \bigcup_{\substack{m,n \\ N \leq m \leq n}} \left[(x, \pi) \mid \left\| \sum_{m \leq j \leq n} f_{\pi(j)}(x) \right\| > \frac{1}{k}\right]$$

is a subset of $X \times G$ with the Baire property. For each x in X, let $B_x^c = B^c \cap ((x) \times G)$. The hypotheses of Theorem 1.2 plus Corollary 2.2 imply that each B_x^c is a first category subset of G, except for a first category set of x's. But then B^c itself is a first category subset of $X \times G$ (Oxtoby [**6**], Theorem 15.4), and so $B_\pi^c = B^c \cap (X \times (\pi))$ is a first category subset of X, except for a first category set of π's (theorem of Kuratowski-Ulam, Oxtoby [**6**], Theorem 15.1). Hence, B_π is a residual subset of X for all except a first category set of π's. This proves Theorem 1.2.

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