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HORSESHOE MAPS AND INVERSE LIMITS

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Inverse limits of interval maps are used to show that certain homeomorphisms of the disk into itself factor over a standard n-fold horseshoe map on attracting sets and to give a topological classification of the attracting sets of horseshoe maps.

1. Introduction. In [S] Smale defined the horseshoe map on the two-sphere and gave a complete description of its complicated nonwandering set. Horseshoe maps occur in a wide variety of dynamical systems and an understanding of their properties is crucial in analyzing the dynamics of such systems.

Here we consider the full attracting set for the horseshoe map. This attracting set is a type of snakelike continuum called a Knaster continuum [**B**, **W**]. By realizing these continua as inverse limit spaces for a map of the interval we can, in some situations, construct a particular surjection of one of these continua onto another and, in so doing, provide a dynamical factorization of one horseshoe map over another (at least on attracting sets). This is carried out in §§2 and 3. Similar techniques are used by Block in [**B1**].

In the final section of this paper we use a result of Watkins [**W**] to give a topological classification of the attracting sets for horseshoe maps.

2. Let I be the interval $I = [0,1]$, and let D^2 be the two-dimensional disk $D^2 = I \times I \cup A \cup B$ where A and B are half disks attached to opposite sides $\{0\} \times I$ and $\{1\} \times I$ of the square $I \times I$ as in Figure 1.

Now let P: $D^2 \to I$ by

$$P(A) = \{0\}, \quad P(B) = \{1\},$$

and

$$P((x, y)) = x \quad \text{for all } (x, y) \in I \times I.$$

For each integer $n \geq 2$ we define an n-fold horseshoe map F_n: $D^2 \to D^2$ to be a homeomorphism (into) having the following properties:

(i) $F_n(P^{-1}(P(z))) \subseteq P^{-1}(P(F_n(z)))$ for all $z \in D^2$;

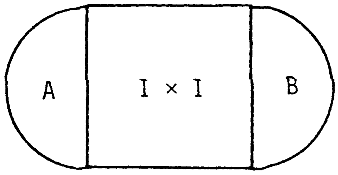


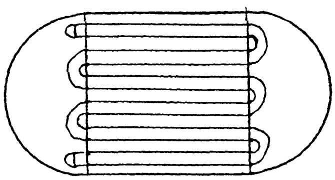
FIGURE 1

(ii) $F_n(A) \subseteq$ interior(A) and
$F_n(B) \subseteq$ interior(A) for n even,
$F_n(B) \subseteq$ interior(B) for n odd;

(iii) for all $x \in I$, $P^{-1}(x) \cap F_n(D^2)$ has exactly n components; and

(iv) diameter $(F_n^k(P^{-1}(P(z)))) \to 0$ uniformly in $z \in D^2$ as $k \to \infty$.

The attracting set for the n-fold horseshoe map F_n is the set $\Lambda_n = \bigcap_{k \geq 0} F_n^k(D^2)$. That is, given $z \in D^2$, the distance between $F_n^k(z)$ and Λ_n goes to zero as k goes to infinity.



$F_6(D^2)$

FIGURE 2

For each $n \geq 2$, F_n induces a continuous map of the interval, f_n: $I \to I$, defined by

$$f_n(x) = P\big(F_n(P^{-1}(x))\big).$$

The map f_n has the following properties:

$$f_n(0) = 0;$$

$$f_n(1) = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}; \end{cases}$$

and there exists $a_i \in I$, $0 = a_0 < a_1 < \cdots < a_{2n} < a_{2n+1} = 1$ such that f_n is strictly monotone on $[a_{2i-1}, a_{2i}]$ for $i = 1, 2, \ldots, n$ and

$$f_n([a_{2i}, a_{2i+1}]) = \begin{cases} \{0\}, & i \text{ odd}, \\ \{1\}, & i \text{ even}. \end{cases}$$

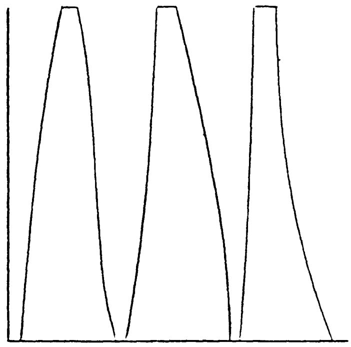


FIGURE 3

Associated with a continuous map of the interval, $f\colon I \to I$, is the inverse limit space

$$(I, f) = \{ (x_0, x_1, \dots) | x_i \in I \text{ and } f(x_{i+1}) = x_i,\ i = 0, 1, 2, \dots \}.$$

We give (I, f) the topology induced by the metric

$$d((x_0, x_1, \dots), (y_0, y_1, \dots)) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}.$$

Then (I, f) is a compact connected metric space. Finally, let $\hat{f}\colon (I, f) \to (I, f)$ be the homeomorphism defined by

$$\hat{f}((x_0, x_1, \dots)) = (f(x_0), x_0, x_1, \dots).$$

Notice that $\hat{f}^{-1}$ is just the shift:

$$\hat{f}^{-1}((x_0, x_1, x_2, \dots)) = (x_1, x_2, \dots).$$

Now let F_n be an n-fold horseshoe map of the two disk with attracting set Λ_n and let f_n be the induced map of the interval.

THEOREM 1. *The function* $\hat{P}\colon \Lambda_n \to (I, f_n)$ *given by*

$$\hat{P}(z) = (P(z), P(F_n^{-1}(z)), P(F_n^{-2}(z)), \dots)$$

is a homeomorphism (*onto*) *and the diagram of homeomorphisms*

$$\begin{array}{ccc} \Lambda_n & \xrightarrow{F} & \Lambda_n \\ \hat{P}\downarrow & & \hat{P}\downarrow \\ (I, f_n) & \xrightarrow{\hat{f}_n} & (I, f_n) \end{array}$$

commutes.

Proof. Since $f_n(P(F_n^{-(i+1)}(z))) = P(F_n(F_n^{-(i+1)}(z))) = P(F_n^{-i}(z))$, we see that $\hat{P}(z)$ is indeed an element of (I, f_n).

$\hat{P}$ is clearly continuous. To see that $\hat{P}$ is one-to-one and onto, let $\mathbf{x} = (x_0, x_1, \ldots) \in (I, f_n)$ and let

$$C_k = F_n^k\left(P^{-1}(x_k)\right) \quad \text{for } k = 0, 1, 2, \ldots.$$

Then C_k is a closed, nonempty subset of D^2 for each $k \geq 0$, and since

$$F_n\left(P^{-1}(x_{k+1})\right) \subseteq P^{-1}\left(f_n(x_{k+1})\right) = P^{-1}(x_k),$$

we have $C_{k+1} \subseteq C_k$ for $k = 0, 1, 2, \ldots$. Thus $\bigcap_{k \geq 0} C_k$ is a nonempty set and if $z \in \bigcap_{k \geq 0} C_k$, then $P(z) = x_0$, $P(F_n^{-1}(z)) = x_1, \ldots$. That is, $\hat{P}(z) = \mathbf{x}$. Moreover, if $\hat{P}(z) = \mathbf{x}$ then z must be in $\bigcap_{k \geq 0} C_k$. But since diameter$(F_n^k(P^{-1}(x_k))) \to 0$ as $k \to \infty$ (condition (iv) in the definition of F_n), we have $\bigcap_{k \geq 0} C_k = \{z\}$ and $\hat{P}$ is one-to-one as well as onto.

3. Given continuous maps $F\colon \Lambda \to \Lambda$ and $G\colon \Gamma \to \Gamma$, we say that F factors over G if there is a continuous surjection $H\colon \Lambda \to \Gamma$ such that $H \circ F = G \circ H$. If H is a homeomorphism then F and G are said to be conjugate. For example, Theorem 1 states that $F_n|_{\Lambda_n}$ and $\hat{f}_n$ are conjugate.

We will see in §3 that the attracting sets for any two n-fold horseshoe maps F_n and G_n are homeomorphic. It is not necessarily the case, however, that F_n and G_n are conjugate on their attracting sets. Nonetheless, there is a "standard" n-fold horseshoe map S_n such that every other n-fold horseshoe map factors over S_n (on attracting sets). Actually every $(n + 2k)$-fold horseshoe map F_{n+2k} factors over S_n on attracting sets and, in fact, maps that in rough detail resemble an n-fold horseshoe map factor over S_n on attracting sets.

In order to be more specific we will work on the level of interval maps. Let s_n be a continuous map of the interval $I = [0, 1]$ with the following properties: there are points b_i, $0 = b_0 < b_1 < \cdots < b_{2n} < b_{2n+1} = 1$ such that

$$s_n\left([b_{2i}, b_{2i+1}]\right) = \begin{cases} \{0\}, & i \text{ odd} \\ \{1\}, & i \text{ even} \end{cases}$$

and s_n is linear on $[b_{2i-1}, b_{2i}]$, $i = 1, 2, \ldots$. Now let t_n be any continuous map of the interval satisfying: there exists points a_i, $0 = a_0 < a_1 < \cdots < a_{2n} < a_{2n+1} = 1$ such that

$$t_n\left([a_{2i}, a_{2i+1}]\right) = \begin{cases} \{0\}, & i \text{ odd}, \\ \{1\}, & i \text{ even}, \end{cases} \qquad i = 0, \cdots, 2n.$$

Notice that $t_n([a_{2i-1}, a_{2i}]) = [0, 1]$ for $i = 1, \ldots, n$: but we make no assumptions about how t_n behaves on $[a_{2i-1}, a_{2i}]$.

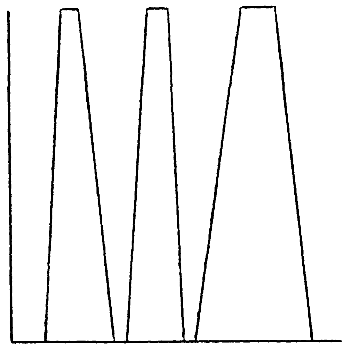


FIGURE 4
Graph of s_6

THEOREM 2. *Given s_n and t_n as above, there is a continuous surjection $h\colon I \to I$ such that $s_n \circ h = h \circ t_n$ on I.*

Proof. So that we have room for future subscripting, let $s = s_n$ and $t = t_n$.

We will construct the surjection h in steps. Assume that $n \geq 2$ is even. For odd n the construction is much the same.

Step 1. Let $\mathscr{D}_0 = \bigcup_{i=0}^{n}[a_{2i}, a_{2i+1}]$. In this step we simply define h on $\mathscr{D}_0$ such that $h|_{[a_{2i}, a_{2i+1}]}$ is an increasing homeomorphism from $[a_{2i}, a_{2i+1}]$ onto $[b_{2i}, b_{2i+1}]$ for each $i = 0, 1, \ldots, n$. Denote $h|\mathscr{D}_0$ by h_0. Then it is clear that for $x \in \mathscr{D}_0$, $s(h_0(x)) = h_0(t(x))$.

Step 2. Let $\mathscr{D}_m = t^{-m}(\mathscr{D}_0)$ for $m = 0, 1, 2, \ldots$ and let $\mathscr{D} = \bigcup_{m \geq 0} \mathscr{D}_m$. Since $t(\mathscr{D}_0) \subseteq \mathscr{D}_0$ we have $\mathscr{D}_{m+1} \supseteq \mathscr{D}_m$ for all $m \geq 0$. In this step we extend h_0 to $\mathscr{D}$. To do this first define

$$s_i^{-1}\colon I \to [b_{2i-1}, b_{2i}], \qquad i = 1, \ldots, n$$

to be the inverse of $s|_{[b_{2i-1}, b_{2i}]}$. Also, for $x \in I$ such that $t^k(x) \in I - \mathscr{D}_0$ for $k = 0, 1, \ldots, m-1$, define

$$\alpha_k = \alpha_k(x) = i \text{ provided } t^{k-1}(x) \in (a_{2i-1}, a_i) \text{ for } k = 1, \ldots, m.$$

Now let $x \in \mathscr{D}$. Then $x \in \mathscr{D}_0$ or $x \in \mathscr{D}_m - \mathscr{D}_{m-1}$ for some $m \geq 1$. If $x \in \mathscr{D}_0$ let $h(x) = h_0(x)$. If $x \in \mathscr{D}_m - \mathscr{D}_{m-1}$ for some $m \geq 1$, let

$$h(x) = s_{\alpha_1}^{-1} \circ \cdots \circ s_{\alpha_m}^{-1} \circ h_0 \circ t^m(x)$$

where $\alpha_k = \alpha_k(x)$, $k = 1, \ldots, m$. This is well defined since $x \notin \mathscr{D}_{m-1}$. We will show that h is continuous on $\mathscr{D}$ and that $s \circ h = h \circ t$ on $\mathscr{D}$.

If $x \in \mathscr{D}_{m+k} - \mathscr{D}_{m+k-1}$, then $t^k(x) \in \mathscr{D}_m - \mathscr{D}_{m-1}$ and

$$\begin{aligned} h(x) &= s_{\alpha_1}^{-1} \circ \cdots \circ s_{\alpha_k}^{-1} \circ s_{\alpha_{k+1}}^{-1} \circ \cdots \circ s_{\alpha_{m+k}}^{-1} \circ h_0 \circ t^m \circ t^k(x) \\ &= s_{\alpha_1}^{-1} \circ \cdots \circ s_{d\alpha_k}^{-1} \circ \left(s_{\alpha_{k+1}}^{-1} \circ \cdots \circ s_{\alpha_{m+k}}^{-1} \circ h_0 \circ t^m \right) \circ t^k(x) \\ &= s_{\alpha_1}^{-1} \circ \cdots \circ s_{\alpha_k}^{-1} \circ h \circ t^k(x) \end{aligned}$$

since $\alpha_i(t^k(x)) = \alpha_{i+k}(x)$ for $i = 1, \ldots, m$. In particular, for $x \in \mathscr{D}_{m+1} - \mathscr{D}_m$ and $m \geq 0$, we have $h(x) = s_{\alpha_1}^{-1} \circ h \circ t(x)$ so that $s \circ h(x) = s \circ s_{\alpha_1}^{-1} \circ h \circ t(x) = h \circ t(x)$. Thus $s \circ h = h \circ t$ on all of $\mathscr{D}$.

Let $x \in \mathscr{D}$. Then $t^m(x) = 0$ for some $m \geq 0$ (n even). Thus, if y is sufficiently close to x, $t^m(y) \in [0, a_1) \subseteq \mathscr{D}_0$. It follows that $\mathscr{D}$ is open.

To establish the continuity of h on $\mathscr{D}$, we will first show that h is continuous on $\mathscr{D}_1$. If $x \in \mathscr{D}_1 - \mathscr{D}_0$ then for y sufficiently close to x (and $y \in \mathscr{D}_1$) we have $y \in \mathscr{D}_1 - \mathscr{D}_0$, $\alpha_1 = \alpha_1(y) = \alpha_1(x)$, and

$$h(y) = s_{\alpha_1}^{-1} \circ h_0 \circ t(y) \simeq s_{\alpha_1}^{-1} \circ h_0 \circ t(x) = h(x).$$

If $x \in \operatorname{interior}(\mathscr{D}_0)$ then for y sufficiently close to x, $y \in \operatorname{interior}(\mathscr{D}_0)$, and

$$h(y) = h_0(y) \simeq h_0(x) = h(x).$$

If $x \in \mathscr{D}_0 - \operatorname{interior}(\mathscr{D}_0) = \{a_1, a_2, \ldots, a_{2n}\}$, say $x = a_i$, then $h(x) = h_0(x) = b_i$. Now, note that if j is odd then $s_j^{-1}(0) = b_{2j-1}$ and $s_j^{-1}(1) = b_{2j}$, and if j is even then $s_j^{-1}(0) = b_{2j}$ and $s_j^{-1}(1) = b_{2j-1}$. Let $y \in \mathscr{D}_1 - \mathscr{D}_0$ be close to x. We consider four cases.

If $i \equiv 0 \bmod 4$ then $y \in (a_{i-1}, a_i)$, $\alpha_1 = i/2$ is even, $t(a_i) = 0$ so $t(y) \simeq 0$, and

$$h(y) = s_{\alpha_1}^{-1} \circ h_0 \circ t(y) \circ s_{\alpha_1}^{-1} \circ h_0(0) \simeq s_{\alpha_1}^{-1}(0) = b_{2\alpha_1} = b_i = h(x).$$

If $i \equiv 1 \bmod 4$ then $y \in (a_i, a_{i+1})$, $\alpha_1 = (i-1)/2 + 1$ is odd, $t(a_i) = 0$ so $t(y) \simeq 0$, and

$$h(y) = s_{\alpha_1}^{-1} \circ h_0 \circ t(y) \simeq s_{\alpha_1}^{-1} \circ h_0(0) \simeq s_{\alpha_1}^{-1}(0) = b_{2\alpha_1 - 1} = b_i = h(x).$$

If $i \equiv 2 \bmod 4$ then $y \in (a_{i-1}, a_i)$, $\alpha_1 = i/2$ is odd, $t(a_i) = 1$ so $t(y) \simeq 1$, and

$$h(y) = s_{\alpha_1}^{-1} \circ h_0 \circ t(y) \simeq s_{\alpha_1}^{-1} \circ h_0(1) \simeq s_{\alpha_1}^{-1}(1) = b_{2\alpha_1} = b_i = h(x).$$

If $i \equiv 3 \bmod 4$ then $y \in (a_i, a_{i+1})$, $\alpha_1 = (i-1)/2 + 1$ is even, $t(a_i) = 1$ so $t(y) \simeq 1$, and

$$h(y) = s_{\alpha_1}^{-1} \circ h_0 \circ z(y) \simeq s_{\alpha_1}^{-1} \circ h_0(1) \simeq s_{\alpha_1}^{-1}(1) = b_{2\alpha_1 - 1} = b_i = h(x).$$

Thus we see that h is continuous on $\mathscr{D}_1$.

Now let $x \in \mathscr{D}_m - \mathscr{D}_{m-1}$ for some $m \geq 1$. Then $t^{m+1}(x) \in \operatorname{interior}(\mathscr{D}_0)$ so if y is sufficiently close to x then $t^{m+1}(y) \in \mathscr{D}_0$ and $y \in \mathscr{D}_{m+1}$. Also, for y close to x, $t^k(y) \in I - \mathscr{D}_0$ for $k = 0, 1, \ldots, m-1$ and $\alpha_k(y) = \alpha_k(x) = \alpha_k$ for $k = 1, \ldots, m$. Then

$$h(y) = s_{\alpha_1}^{-1} \circ \cdots \circ s_{\alpha_m}^{-1} \circ h|_{\mathscr{D}_1} \circ t^m(y)$$
$$\simeq s_{\alpha_1}^{-1} \circ \cdots \circ s_{\alpha_m}^{-1} \circ h|_{\mathscr{D}_1} \circ t^m(x) = h(x)$$

since h is continuous on $\mathscr{D}_1$. Thus, h is continuous on $\mathscr{D}$.

Step 3. We now extend h to $\overline{\mathscr{D}}$.

Let $x \in \overline{\mathscr{D}} - \mathscr{D}$. Since $x \notin \mathscr{D}$, $t^k(x) \in I - \mathscr{D}_0$ for all $k \geq 0$. Thus $\alpha_k(x)$ is defined for all $k \geq 1$ and for each $m \geq 0$ there is an $\varepsilon_m > 0$ such that $\varepsilon_m \to 0$ as $m \to \infty$ and, if $|x - y| < \varepsilon_m$, then $\alpha_k(y)$ is defined and equals $\alpha_k(x)$ for $k = 1, 2, \ldots, m$.

Now for $y, z \in \mathscr{D}$ such that $|x - y| < \varepsilon_m$ and $|x - z| < \varepsilon_m$ we have $\alpha_k(y) = \alpha_k(z) = \alpha_k$ for $k = 1, \ldots, m$ and

$$h(y) = s_{\alpha_1}^{-1} \circ \cdots \circ s_{\alpha_m}^{-1} \circ h \circ t^m(y),$$
$$h(z) = s_{\alpha_1}^{-1} \circ \cdots \circ s_{\alpha_m}^{-1} \circ h \circ t^m(z).$$

Let λ_i be the slope of $s|_{[a_{2i-1}, a_{2i}]}$ and let $\lambda = \min_{i=1,\ldots,n}(|\lambda_i|)$. Then $\lambda > 1$ and

$$\operatorname{diameter}\left(s_{\alpha_1}^{-1} \circ \cdots \circ s_{\alpha_m}^{-1}(1)\right) \leq \frac{1}{\lambda^m}.$$

Hence, letting $B_\varepsilon(x) = \{ y \in I \,|\, |x - y| < \varepsilon \}$, we see that

$$\operatorname{diameter}\left(h\left(B_{\varepsilon_m}(x) \cap \mathscr{D}\right)\right) \to 0 \quad \text{as } m \to \infty.$$

Thus h extends continuously to $\overline{\mathscr{D}}$. We see in fact that

$$h(x) = \lim_{m \to \infty} s_{\alpha_1}^{-1} \circ \cdots \circ s_{\alpha_m}^{-1}(z) \quad \text{for every } z \in I.$$

Clearly $s \circ h = h \circ t$ on $\overline{\mathscr{D}}$.

Step 4. We now extend h to all of I.

Let $x \in I - \overline{\mathscr{D}}$ and let U be the component of $I - \overline{\mathscr{D}}$ containing x. Say $U = (y, z)$. Since $\mathscr{D}$ is itself open, y and z are in $\overline{\mathscr{D}} - \mathscr{D}$. Now, it must be the case that $\alpha_k(y) = \alpha_k(z)$ for all $k \geq 1$ (these numbers are defined since $y, z \notin \mathscr{D}$); otherwise there would be a $k \geq 0$ such that $t^k([y, z]) \supseteq [a_{2i}, a_{2i+1}]$ for some i in which case $U \cap \mathscr{D}_k \neq 0$. But $U \subseteq I - \overline{\mathscr{D}} \subseteq I - \mathscr{D}_k$.

Thus we see that

$$h(y) = \lim_{m\to\infty}\left(s^{-1}_{\alpha_1(y)}\circ\cdots\circ s^{-1}_{\alpha_m(y)}(w)\right)$$
$$= \lim_{m\to\infty}\left(s^{-1}_{\alpha_1(z)}\circ\cdots\circ s^{-1}_{\alpha_m(z)}(w)\right) = h(z)$$

for any $w \in I$ (see Step 3). Now define

$$h(x) = \lim_{m\to\infty}\left(s^{-1}_{\alpha_1(x)}\circ\cdots\circ s^{-1}_{\alpha_m(x)}(x)\right)$$

for $x \in I - \overline{\mathscr{D}}$. Then h is continuous on all of I. It is a simple matter to check that $s \circ h = h \circ t$ on all of I.

COROLLARY 3. *Suppose that s_n and t_n are as in Theorem 2 except that, in addition, t_n is linear on $[a_{2i-1}, a_{2i}]$ for $i = 1, 2, \ldots, n$. Then there is a homeomorphism h: $I \to I$ such that $s_n \circ h = h \circ t_n$ on all of I. That is, s_n and t_n are conjugate on I.*

Proof. The surjection h constructed in Theorem 2 is a homeomorphism in this case. In fact, if one constructs a continuous surjection k: $I \to I$ as in the theorem, beginning with k_0: $\bigcup_{i=0}^{n}[b_{2i}, b_{2i+1}] \to \bigcup_{i=0}^{n}[a_{2i}, a_{2i+1}]$ given by $k_0 = h_0^{-1}$ and reversing the roles of s and t, then $k = h^{-1}$.

Now suppose that S_n: $D^2 \to D^2$ in an n-fold horseshoe map with the property that the induced map on the interval, s_n: $I \to I$, is linear on $[b_{2i-1}, b_{2i}]$, $i = 1, 2, \ldots, n$ (as in the earlier definition of s_n). Let T: $D^2 \to D^2$ be any homeomorphism satisfying: there exists a projection Φ: $D^2 \to I$ such that $T(\Phi^{-1}(\Phi(z))) \subseteq \Phi^{-1}(\Phi(T(z)))$ for all $z \in D^2$; diameter$(T^k(\Phi^{-1}(\Phi(z)))) \to 0$ uniformly in $z \in D^2$ as $k \to \infty$; and the induced map of the interval t: $I \to I$ has the property that

$$t([a_{2i}, a_{2i+1}]) = \begin{cases} \{0\} & \text{for } i \text{ even} \\ \{1\} & \text{for } i \text{ odd} \end{cases}$$

for some $0 = a_0 < a_1 < \cdots < a_{2n+1} = 1$.

Let $\Lambda_n = \bigcap_{k\geq 0} S_n^k(D^2)$ and $\Gamma = \bigcap_{k\geq 0} T^k(D^2)$.

COROLLARY 4. *There is a continuous surjection H: $\Gamma \to \Lambda_n$ such that the diagram*

$$\begin{array}{ccc} \Gamma & \xrightarrow{T} & \Gamma \\ H\downarrow & & H\downarrow \\ \Lambda_n & \xrightarrow{S_n} & \Lambda_n \end{array}$$

commutes. That is, T on Γ factors over S_n on Λ_n.

Proof. Theorems 1 and 2 provide the following commutative diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{T} & \Gamma \\ \hat{\Phi}\downarrow & & \hat{\Phi}\downarrow \\ (I,t) & \xrightarrow{\hat{t}} & (I,t) \\ \hat{h}\downarrow & & \hat{h}\downarrow \\ (I,s_n) & \xrightarrow{\hat{s}_n} & (I,s_n) \\ \hat{P}\uparrow & & \hat{P}\uparrow \\ \Lambda_n & \xrightarrow{S_n} & \Lambda_n \end{array}$$

$\hat{\Phi}$ and $\hat{P}$ are homeomorphisms and $\hat{h}$: $(I,t) \to (I,s_n)$ is the continuous surjection given by

$$\hat{h}((x_0,x_1,\dots)) = (h(x_0),h(x_1),\dots)$$

where h is given by Theorem 2. Let $H = \hat{P}^{-1} \circ \hat{h} \circ \hat{\Phi}$.

Thus we see that if $m \geq n$ and m and n have the same parity, then the m-fold horseshoe map F_m factors over the standard n-fold horseshoe map S_n on their attracting sets.

In the next section we give a topological classification of the spaces Λ_n.

4. We first define a standard model for the spaces Λ_n.

Let g_n: $I \to I$, $n \geq 2$, be defined by

$$g_n\left(\frac{i}{n}\right) = \begin{cases} 0, & i \text{ even}, \\ 1, & i \text{ odd}, \end{cases} \qquad i = 0,\dots,n,$$

and g_n is linear on $[i/n,(i+1)/n]$, $i = 0,\dots,n-1$. Let

$$(I,g_n) = \{ (x_0,x_1,\dots) | x_i \in I,\ g_n(x_{i+1}) = x_i,\ i = 0,1,\dots \}$$

be the inverse limit space. Another description of the space (I,g_n) is as follows. Let $\Sigma_n = \{(\alpha_1,\alpha_2,\dots)|\alpha_i \in \{1,\dots,n\}\}$ and give Σ_n the topology induced by the metric

$$d((\alpha_1,\alpha_2,\dots),(\beta_1,\beta_2,\dots)) = \sum_{i=1}^{\infty} \frac{|\alpha_i - \beta_i|}{2^i}.$$

Let $I \times \Sigma_n$ have the product topology and define P_n: $I \times \Sigma_n \to (I,g_n)$ by $P_n(x,\boldsymbol{\alpha}) = \mathbf{x}$ where $\mathbf{x} = (x_0,x_1,\dots)$ is the unique element of (I,g_n) such that $x_0 = x$ and $x_i \in [\alpha_i/n,(\alpha_i+1)/n]$.

P_n is at most two-to-one and if we define the equivalence relation $\sim$ by $(x, \boldsymbol{\alpha}) \sim (y, \boldsymbol{\beta})$ if and only if $P_n((x, \boldsymbol{\alpha})) = P_n((y, \boldsymbol{\beta}))$ then $(I \times \Sigma_n)/\sim$ with the quotient topology is homeomorphic with (I, g_n).

More specifically, in $I \times \Sigma_n$ we identify $(x, \boldsymbol{\alpha})$ with $(y, \boldsymbol{\beta})$ if and only if one of the following applies.

(1) If $\alpha_1 = 2j - 1$ and $\beta_1 = 2j$ for some j and $\alpha_i = \beta_i$ for $i > 1$ then $(1, \boldsymbol{\alpha}) \sim (1, \boldsymbol{\beta})$.

(2) If n is odd and $\alpha_i = \beta_i = n$ for $i = 1, \ldots, m$; $\alpha_{m+1} = 2j - 1$, $\beta_{m+1} = 2j$ for some j; and $\alpha_i = \beta_i$ for $i > m + 1$; then $(1, \boldsymbol{\alpha}) \sim (1, \boldsymbol{\beta})$.

(3) If $\alpha_i = \beta_i = 1$ or $i = 1, 2, \ldots, m$; $\alpha_{m+1} = 2j$ and $\beta_{m+1} = 2j$ for some j; and $\alpha_i = \beta_i$ for $i > m + 1$; then $(0, \boldsymbol{\alpha}) \sim (0, \boldsymbol{\beta})$.

(4) If n is even and $\alpha_i = \beta_i = 1$ for $i = 1, \ldots, m$; $\alpha_{m+1} = \beta_{m+1} = n$; $\alpha_{m+2} = 2j - 1$ and $\beta_{m+2} = 2j$ for some j; and $\alpha_i = \beta_i$ for $i > m + 2$; then $(0, \boldsymbol{\alpha}) \sim (0, \boldsymbol{\beta})$.

Now let f_n be the interval map corresponding to an n-fold horseshoe map F_n.

THEOREM 5. *(I, f_n) is homeomorphic with (I, g_n).*

Proof. Let $J = [\frac{1}{4}, \frac{3}{4}] \subseteq I$ and let Σ_n be as above. Define Q_n: $J \times \Sigma_n \to (I, f_n)$ by $Q_n(x, \boldsymbol{\alpha}) = \mathbf{x}$ where $\mathbf{x} = (x_0, x_1, \ldots)$ is the unique element of (I, f_n) such that $x_0 = 2(x - \frac{1}{4})$ and $x_i \in [a_{2\alpha_i - 1}, a_{2\alpha_i}]$. Here $0 = a_0 < a_1 < \cdots < a_{2n+1} = 1$ as in the description of f_n in §1. Then Q_n is a homeomorphism of $J \times \Sigma_n$ onto its image. $(I, f_n) - Q_n(J \times \Sigma_n)$ is a collection of arcs. We extend Q_n to $(I \times \Sigma_n)/\sim$ by mapping the arcs in $((I \times \Sigma_n)/\sim) - (J \times \Sigma_n)$ onto the corresponding arcs in $(I, f_n) - Q_n(J \times \Sigma_n)$.

For example, let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be as in Case (1) above in the description of the identifications made in $(I \times \Sigma_n)/\sim$. Then $Q_n((\frac{3}{4}, \boldsymbol{\alpha})) = (1, a_{4j-2}, \ldots) = \mathbf{x}$ and $Q_n((\frac{3}{4}, \boldsymbol{\beta})) = (1, a_{4j-1}, \ldots) = \mathbf{y}$ for some j. Then

$$A_{\boldsymbol{\alpha}} = \left\{ \mathbf{t} = (1, t_1, t_2, \ldots) \in (I, f_n) \mid a_{4j-2} \le t_1 \le a_{4j-1}, \right.$$
$$\left. t_i \in \left[a_{2\alpha_i - 1}, a_{2\alpha_i} \right] \right\}$$

is an arc joining $\mathbf{x}$ to $\mathbf{y}$ in (I, f_n). Now let Q_n map the arc

$$([\tfrac{3}{4}, 1] \times \{\boldsymbol{\alpha}\} \cup [\tfrac{3}{4}, 1] \times \{\boldsymbol{\beta}\})/\sim$$

in $(I \times \Sigma_n)/\sim$ onto $A_{\boldsymbol{\alpha}}$ by

$$Q_n((s, \boldsymbol{\alpha})) = \mathbf{t} = (1, t_1, t_2, \ldots) \in A_{\boldsymbol{\alpha}}$$

where

$$t_1 = (a_{4j-1} - a_{4j-2})(2s - \tfrac{3}{2}) + a_{4j-2};$$

and

$$Q_n((s, \boldsymbol{\beta})) = \mathbf{t} = (1, t_1, t_2, \ldots) \in A_{\boldsymbol{\alpha}}$$

where

$$t_1 = (a_{4j-2} - a_{4j-1})(2s - \tfrac{3}{2}) + a_{4j-1}.$$

If we extend Q_n over all the arcs in $((I \times \Sigma_n)/\sim) - (J \times \Sigma_n)$ in this way (depending on Case 1, 2, 3 or 4), the result is a homeomorphism of $(I \times \Sigma_n)/\sim$ onto (I, f_n).

The spaces (I, g_n) are classified in the following theorem of Watkins **[W]**.

THEOREM. *(I, g_n) is homeomorphic wiht (I, g_m) if and only if n and m have exactly the same prime factors.*

COROLLARY. *Λ_n is homeomorphic with Λ_m if and only if n and m have exactly the same prime factors.*

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