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METRICALLY INVARIANT MEASURES ON LOCALLY HOMOGENEOUS SPACES AND HYPERSPACES

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We compare different invariance concepts for a Borel measure μ on a metric space. μ is called open-invariant if open isometric sets have equal measure, metrically invariant if isometric Borel sets have equal measure, and strongly invariant if any non-expansive image of A has measuure $\leq \mu(A)$. On common hyperspaces of compact and compact convex sets there are no metrically invariant measures. A locally compact metric space is called locally homogeneous if any two points have isometric neighbourhoods, the isometry transforming one point into the other. On such a space there is a unique open-invariant measure, and this measure is even strongly invariant.

1. Introduction. There are two important classes of spaces with a "natural volume function": locally compact groups with Haar measure and Riemannian manifolds with their volume form. Since in everyday life volume of sets is calculated from length measurements, we consider measures invariants with respect to a metric structure rather than a group structure or differentiable structure. We deal with Borel measures on locally compact metric spaces which are finite on compact sets and metrically invariant in the sense that

"congruent sets have equal measure".

Two sets in a metric space are congruent if there is an isometry f from one onto the other. In Euclidean R^n such f can be extended to an isometry $\bar{f}$ from the whole space onto itself but in general this is not the case (cf. Example 2).

If on a metric space (X, d) there is a unique (up to a constant factor) metrically invariant measure, it can be considered as the "natural volume function" of the space. This is the case for locally compact metric groups with left-invariant metric [**2**]. The volume form on Riemannian manifolds is metrically invariant with respect to the interior metric [**18**], and the volume form on manifolds in R^n is invariant with respect to the Euclidean metric [**10**]. However, in general it is not uniquely determined by this property (cf. Example 1).

Let us state our main result. (X, d) is called locally homogeneous if for any two points x, y in X there is an $\varepsilon = \varepsilon(x, y) > 0$ and an isometry between the ε-neighbourhoods of x and y which sends x to y.

THEOREM 1. *Let (X, d) be locally compact and locally homogeneous, and let A_0 be a compact subset of X with non-empty interior. Then there is a unique metrically invariant Borel measure λ on X with $\lambda(A_0) = 1$.*

Mycielski **[22]** proved a much more general existence theorem (see §2). However, his invariance condition for measures is weaker than ours:

"congruent open sets have equal measure".

Measures with this property will be called open-invariant. Mycielski's open invariant measures are in general not unique. Under the assumptions of Theorem 1, however, uniqueness can already be shown if only congruent ε-neighbourhoods have equal measure. In fact, our uniqueness result is quite related to theorems of Loomis **[19]**, Christensen **[6]** and Mattila **[21]**.

Thus, our essential contribution to Theorem 1 is the proof that the open-invariant measure which exists by Mycielski is actually metrically invariant. Examples in §2 will show that in general there is considerable difference between the two invariance concepts. Ulam asked whether Lebesgue product measure on $[0, 1]^\infty$ is metrically invariant with respect to metrics of the type

$$(d(x, y)) = \left(\sum a_n^2 \cdot (x_n - y_n)^2\right)^{1/2}, \qquad a = (a_1, a_2, \ldots) \in l_2.$$

Open invariance was verified by Mycielski **[23]** but metric invariance could only be proved for very fast decreasing sequences a by Fickett **[11]**. Our present paper also results from an attempt to answer this question of Ulam. We would like to express our gratitude to Professor Mycielski for a very stimulating correspondence on these topics.

It should be noted that Theorem 1 also contributes to the solution of an old problem by Banach and Ulam **[1]**: Does every compact metric space (X, d) admit a finitely additive, metrically invariant and finite Borel measure? For countable X, where a positive answer was found recently **[8]**, such measure will not be σ-additive. Our class of locally homogeneous spaces yields σ-additive invariant measures.

Although this class includes all metric groups with left-invariant metric, it is rather small. Among Riemannian manifold with interior metric it contains only those with constant curvature (theorema egregium, cf. [**17**], Theorem 12.4.2), and among C^2-curves in the Euclidean plane it contains only open line segments and circular arcs. The following example shows, however, that a strong homogeneity condition is needed for uniqueness of a metrically invariant measure.

EXAMPLE 1. The half-parabola $X = \{(t, t^2) | 0 \le t < \infty\}$ contains no two sets with more than four points which are congruent in the Euclidean sense. (The isometry would extend to an isometry $\bar{f}$ of the whole plane which transforms parabolas onto parabolas. Since five points determine a parabola, $\bar{f}$ maps X onto itself, hence $\bar{f} = \text{id}$.) Thus all non-atomic Borel measures on X are metrically invariant.

To get a unique volume function for such spaces, one has to utilize the stronger invariance concept introduced by Kolmogoroff [**16**] (cf. §6).

A very interesting fact is that even the existence of a metrically invariant measure requires a certain degree of homogeneity of the underlying space. In §3 we show that many hyperspaces of compact sets and of compact convex sets with Hausdorff metric do not admit σ-finite, metrically invariant measures due to their inhomogeneous metric structure. This generalizes a result of Boardman [**5**] and answers a question by McMullen.

2. Open-invariant and metrically invariant measures. A Borel measure μ on a metric space (X, d) is called metrically invariant if for any pair of isometric Borel sets A, B we have $\mu(A) = \mu(B)$. This implies $\mu^*(M) = \mu^*(N)$ for any isometric subsets M, N of X. μ is called open-invariant if $\mu(A) = \mu(B)$ whenever A, B are isometric open subsets of X. Clearly, metric invariance implies open-invariance, but the converse does not hold. Even in "good" spaces an isometry f: $A \to B$ need not extend to an isometry $\bar{f}$ between open neighbourhoods of A and B.

EXAMPLE 2. Consider R^2 with maximum metric

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

The sets $A = \{(x, 0) | x \in R\}$ and $B = \{(x, x) | x \in R\}$ are isometric, an isometry is given by $f(x, 0) = (x, x)$. But there are no points $p \notin A$, $q \notin B$ such that $A \cup \{p\}$ is isometric to $B \cup \{q\}$. The reason is that B has exactly one point of shortest distance from q (for $\varepsilon = d(q, B)$ the closed square $\overline{U_\varepsilon(q)}$ intersects B in one point), but A has many points of shortest distance from p.

The following result implies that there is an open-invariant probability measure on each compact metric space. For a compact set C in a metric space (X, d) let $E(C, t)$ denote the minimum cardinality of a covering of C by sets of diameter $< t$. C is called thick in X if there is an open neighbourhood U of C in X and a constant c with $E(D, t) \leq c \cdot E(C, t)$ for all compact sets $D \subseteq U$ and $0 < t \leq 1$.

THEOREM (*Mycielski*). *Let C be a thick set in a metric space (X, d). Then there is a regular open-invariant Borel measure μ on X with $\mu(C) = 1$.*

The proof in [**22**] uses generalized limits. Halmos's existence proof for Haar measure ([**14**], §58) can also be adapted but it needs the axiom of choice, too.

Note that open-invariant measures sometimes can be rather trivial. If $X = \{1/n | n \in N\} \cup \{0\}$, or $X = [0, 1)$, the point measure δ_0 is open-invariant. Namely, two open isometric sets in X either both contain 0, or both do not contain 0.

Next, let X be a triangle with interior in the Euclidean plane. Let Y denote the boundary of the triangle and a, b, c the vertices. There are three types of points: vertices, edge points and interior points. Points of different types will never have isometric neighbourhoods in X. Thus if f: $A \to B$ is an isometry between two open subsets of X, f maps $A \cap Y$ onto $B \cap Y$, and f transforms the vertices in A onto vertices in B.

It follows that there are three open-invariant measures: the discrete measure $\delta_a + \delta_b + \delta_c$, the one-dimensional Lebesgue measure concentrated on Y and the two-dimensional Lebesgue measure on X. Only the last one is metrically invariant. Let us add that linear combinations of open-invariant measures are open-invariant, and that δ_a, δ_b and δ_c alone are open-invariant if X has three different angles.

A metrically invariant measure on an arbitrary metric space must be non-atomic, however, unless all infinite sets have measure ∞.

On very inhomogeneous metric spaces metrically invariant measures may not exist. Our next example reflects the structure we shall find in hyperspaces.

EXAMPLE 3. Let $X = \bigcup_{n=1}^{\infty} C_n$, where C_n is the n-dimensional cube $[0, 1/n]^n$. Let d be the Euclidean metric on every C_n and let $d(x, y) = \max\{1/n, 1/m\}$ for $x \in C_n$, $y \in C_m$, $m \neq n$. For every n, the Lebesgue measure λ_n on C_n induces an open-invariant λ'_n on X. All linear combinations of the λ'_n with positive coefficients are open-invariant, and there are other open-invariant measures concentrated on the edges and vertices of the C_n.

On the other hand, there is no σ-finite metrically invariant measure $\mu \neq 0$ on (X, d). If $\mu(C_n) > 0$, a subcube of C_n with side length $1/(n+1)$ will have positive measure, and C_{n+1} contains an uncountable number of pairwise disjoint isometric copies of that subcube.

We have made use of the following well-known fact.

LEMMA 1. *A measure cannot be σ-finite if there is an uncountable number of pairwise disjoint sets with positive measure.*

3. Non-existence of metrically invariant measures on hyperspaces. A situation similar to Example 3 occurs for quite familiar hyperspaces. The following theorem extends and simplifies a result of Boardman **[5]** concerning Hausdorff measures on $F([0,1])$. $F(X)$ denotes the system of all compact subsets of (X, d), equipped with the Hausdorff metric

$$d_H(A, B) = \max\left\{ \sup_{a \in A} d(a, B),\ \sup_{b \in B} d(b, A) \right\}.$$

THEOREM 2. *Let (X, d) be a locally compact separable metric space without isolated points. Then there is no σ-finite, metrically invariant Borel measure $\mu \neq 0$ on $(F(X), d_H)$.*

Proof. Let S be a countable dense set in X. For s in S, let $M_s = \{A \in F(X) | s \notin A\}$. The open sets M_s cover $F(X) - \{X\}$. Let μ be a positive metrically invariant measure on $F(X)$. Then we have $\mu(M_s) > 0$ for at least one s. (Note that $\mu(\{X\}) = 0$ in the case of compact X.) If we write M_s as the union of closed sets $M_{sn} = \{A | d(s, A) \geq 1/n\}$, we get $\mu(M_{sn}) > 0$ for one $n \in N$. Since X has no isolated points, the neighbourhood $U = U_{1/n}(s)$ in X is uncountable. For $x \in U$ let $N_x = \{A \cup \{x\} | A \in M_{sn}\}$. The natural map $f\colon M_{sn} \to N_x$ is not necessarily a d_H-isometry but it preserves all distances smaller than $\alpha = 1/n - d(x, s)$. Since $F(X)$ is a separable metric space, M_{sn} admits a countable partition into sets of diameter $< \alpha$. The invariance of μ implies then $\mu(N_x) = \mu(M_{sn}) > 0$ for all $x \in U$. Since the N_x are pairwise disjoint, μ is not σ-finite by Lemma 1.

The next theorem also extends a result on Hausdorff measures, and it answers in the negative a question by McMullen (see **[13]**, problem 54). For $U \subseteq R^n$ let $K(U)$ denote the system of all compact convex subsets of U. We consider the locally compact space $(K(R^n), d_H)$, where the underlying metric d is given by a norm $\|\cdot\|$ on R^n. For the Euclidean norm, Gruber and Lettl **[12]** have shown that each isometry from the hyperspace $K(R^n)$ into itself can be written as composition of a mapping $f(A) = \bar{f}(A)$ induced by a Euclidean isometry $\bar{f}$: $R^n \to R^n$ and a mapping $g(A) = A + C = \{a + c | a \in A,\ c \in C\}$, where C is a fixed compact convex set. f is onto, but g is not surjective unless C is a singleton. Here we shall only use the fact that in the case of an arbitrary norm, g is an isometry on $K(R^n)$.

THEOREM 3. *For $n > 1$ there is no σ-finite Borel measure on $(K(R^n), d_H)$ which is invariant with respect to all isometries from the whole space into itself.*

Proof. Let $S_p = \{x | \, \|x\| < p\}$. Then $K(R^n) = \bigcup_{p=1}^{\infty} K(S_p)$. If μ is a positive measure, $\mu(K(S_p)) > 0$ for one p. It suffices to show that $K(S_{3p})$ contains an uncountable number of mutually disjoint isometric copies of $K(S_p)$. For each $x \in R^n$ with $\|x\| = 2p$ let $D_x = \{tx | 0 \leq t \leq 1\}$. We prove that the sets $K(S_p) + D_x$ are mutually disjoint. To this end let $y \neq x$ and $y \neq -x$ (the case $y = -x$ is easy) be vectors with norm $2p$, and assume there are elements A, B in $K(S_p)$ such that

(i) $$A + D_x = B + D_y.$$

Take a basis of R^n containing x and y as first and second element, and let $\alpha(z)$ denote the first coordinate of z with respect to that basis. Thus $\alpha(x) = 1$, $\alpha(y) = 0$. By compactness we find an $a^* \in A$ with $\alpha(a^*) = \min\{\alpha(a) | a \in A\}$. By (i) there is $b \in B$ and $s \in [0,1]$ with $a^* + 0 \cdot x = b + s \cdot y$ and hence $\alpha(a^*) = \alpha(b)$. Again by (i) there is $a \in A$ and $t \in [0,1]$ with $a + t \cdot x = b + 1 \cdot y$. Now $\alpha(a) + t = \alpha(b) = \alpha(a^*) \leq \alpha(a)$ implies $t = 0$. Thus $a = b + y$ which yields the contradiction $\|y\| \leq \|a\| + \|b\| < 2p$. The D_x are disjoint.

Let us note that $(K(R^1), d_H)$ is isometric to the half-plane $\{(a, b) | a \leq b\}$ with maximum metric on which Lebesgue measure is metrically invariant.

An open-invariant measure $\bar{\lambda}$ on $K(R^n)$ is induced by n-dimensional Lebesgue measure: $\bar{\lambda}(C) = \lambda_n(\{x | \{x\} \in C\})$. We omit the proof.

4. Locally homogeneous metric spaces. Having shown that metrically invariant measures do not exist on very inhomogeneous spaces, we now turn to the other extreme. A metric space (X, d) is called (metrically) homogeneous if for any two of its points x, y there is an isometry f: $X \to X$ which is onto and sends x into y. Let $U_\varepsilon(p) = \{q | d(p, q) < \varepsilon\}$. We call (X, d) locally homogeneous (abbrev. l.h.) if for any x, y there exists an $\varepsilon = \varepsilon(x, y) > 0$ and an isometry f: $U_\varepsilon(x) \to U_\varepsilon(y)$ which is onto and sends x into y. This implies $f(U_\delta(x)) = U_\delta(y)$ for $0 < \delta < \varepsilon$.

We have not tried to avoid the condition $f(x) = y$ and require that $U_\delta(x)$ and $U_\delta(y)$ are isometric for all $\delta < \varepsilon$, or only for arbitrary small δ. (The latter holds for Cantor's middle-third set with Euclidean metric.) We have chosen the definition with $f(x) = y$ since it seems technically convenient, in particular for Theorem 4 below.

If in a l.h. space there is a compact set with non-empty interior, the space must be locally compact. Hence we shall restrict our attention to locally compact spaces.

Metric groups with left-invariant metric are examples of homogeneous metric spaces. More generally, if a uniformly equicontinuous group of one-to-one and onto mappings acts on a space (Y, d_0) (cf. [**27**], §7.3), the metric $d(x, y) = \sup_{f \in F} d_0(f(x), f(y))$ is uniformly equivalent to d_0, and every f in F is a d-isometry. So (Y, d) is homogeneous whenever the F-orbit of a point is dense in Y and Y is locally compact (cf. Lemma 2 below).

Any open subset U of a homogeneous metric space (X, d) is locally homogeneous. (Let $\varepsilon(x, y) = \min\{d(x, X - U), \ d(y, X - U)\}$.) The converse is not true.

EXAMPLE 4. Let $C = \{(x_1, x_2, x_3) | -\infty < x_3 \leq 1, (1 - x_3)^2 = x_1^2 + x_2^2\}$ be the surface of an unbounded cone, and let d be the interior metric. That is, $d(x, y)$ is the length of the shortest arc in C connecting x and y. The cone without peak $p = (0, 0, 1)$ is developable into the plane [**17**]. That is, small neighbourhoods of points in $X = C - \{p\}$ are isometric to open subsets of Euclidean R^2. This is not true for neighbourhoods of p. So X is l.h. but C is not. X is not homogeneous, only rotations around the x_3-axis are isometries of X.

Now C is the completion of X. Thus when X is imbedded in a larger l.h. space Y as an open subset then X will be closed, too. So X is not isometric to an open subset of a homogeneous space.

Local homogeneity does not only carry over to open subspaces but also to finite products. Suppose $f_i \colon U_i \to V_i$ are isometries in (X_i, d_i) for $i = 1, 2$. Then

$$f \colon U_1 \times U_2 \to V_1 \times V_2, \qquad f(x_1, x_2) = \big(f_1(x_1), f_2(x_2)\big)$$

is an isometry with respect to the maximum metric as well as to any other metric on $X_1 \times X_2$ which has the form $d(x, y) = g(d_1(x_1, y_1), d_2(x_2, y_2))$. Thus if (X_i, d_i) are l.h. then $X_1 \times X_2$ is l.h. with respect to any of the familiar metrics. Moreover, local homogeneity carries over to so-called symmetric products:

EXAMPLE 5. Let m be a positive integer, let $F_m(X)$ denote the family of m-point subsets of the metric space (X, d), and let $d_{\max}$ on X^m be defined by

$$d_{\max}(x, y) = \max\{ d(x_i, y_i) | i = 1, \dots, m \}.$$

Let

$$\tilde{X}^m = \{ (x_1, \dots, x_m) | x_i \in X, x_i \neq x_j \text{ for } i \neq j \}.$$

Then the mapping $f\colon (\tilde{X}^m, d_{\max}) \to (F_m(X), d_H)$ with $f(x_1, \dots, x_m) = \{x_1, \dots, x_m\}$ is $m!$-to-one and onto, and f is a local isometry: if $\varepsilon < \varepsilon_x = \frac{1}{2} \cdot \min\{ d(x_i, x_j) | 1 \leq i < j \leq m\}$, f maps $U_\varepsilon(x_1, \dots, x_m)$ in $\tilde{X}^m$ isometrically onto $U_\varepsilon(\{x_1, \dots, x_m\})$ in $F_m(X)$.

If (X, d) is locally homogeneous, then $(X^m, d_{\max})$, the open subset $(\tilde{X}^m, d_{\max})$ and hence $(F_m(X), d_H)$ become l.h. spaces, too.

In contrast to the results of §3 we have found a l.h. hyperspace. Unfortunately, $F_m(X)$ is only a very small part of $F(x)$. It would be interesting to know whether $F([0, 1])$—that is, the Hilbert cube—has a l.h. metric. Is one of the familiar metrics for hyperspaces of convex bodies (cf. [**28**]) l.h.? An l.h. metric on such a "space of shapes" by Theorem 1 would yield a natural measure for "shapes", and this is what people working in convex geometry [**13**] and integral geometry (cf. the first two papers in [0]) are looking for. More generally,

Problem. When does a topological space X admit a compatible l.h. metric?

Clearly it is necessary that X is l.h. in the topological sense (for any x, y there is a homeomorphism between neighbourhoods $U(x)$ and $V(y)$ which sends x into y). But is that enough? A classical result of Dantzig and van der Waerden [**7**] says that the sphere with two handles does not possess a homogeneous metric although it is homogeneous in the topological sense. However, it admits a l.h. metric, since every Riemann surface admits a l.h. Riemannian metric ([**9**], Theorem IV.8.6).

A Riemannian metric on a manifold is l.h. if and only if it has constant curvature ([**17**], Theorem 12.4.2).

The ε in the definition of l.h. space depends on x and y. But for all points of a compact set we find a common ε.

THEOREM 4. *Let e be a point and A a compact set in a locally compact and locally homogeneous space (X, d). Then there is an $\varepsilon = \varepsilon_A > 0$, such that $U_\varepsilon(x)$ and $U_\varepsilon(e)$ are isometric for every x in A, where the isometry sends x into e.*

LEMMA 2. *Let $\alpha > 0$ be so small that $\overline{U_\alpha(e)}$ is compact. Then $D_\alpha = \{x \mid \text{there is an isometry } f\colon U_\alpha(e) \to U_\alpha(x) \text{ with } f(e) = x\}$ is closed.*

Proof of Lemma 2. This is an application of Ascoli's theorem ([**26**], p. 179). Let (x_n) be a sequence in D_α converging to $x \in X$, and let $f_n\colon U_\alpha(e) \to U_\alpha(x_n)$ be isometries onto with $f_n(e) = x_n$. For $z \in U_\alpha(e)$ choose β with $d(e, z) < \beta < \alpha$ and n_0 with $d(x_n, x_{n_0}) < \beta - d(e, z) = \beta - d(x_n, f_n(z))$ for all $n \geq n_0$. Then $\{f_n(x) \mid n \in N\}$ is relatively compact since almost all $f_n(z)$ lie in the compact set $\overline{U_\beta(x_{n_0})}$. Any family of isometries is equicontinuous, so a subsequence f_{n_k} converges pointwise to a map f. Obviously f is an isometry and $f(e) = x$, $f(U_\alpha(e)) \subseteq U_\alpha(x)$. It remains to show that f is a map onto $U_\alpha(x)$. Let $y \in U_\alpha(x)$ and $\gamma = d(y, x)$. Since f_{n_k} maps $\overline{U_\gamma(e)}$ onto $\overline{U_\gamma(x_{n_k})}$ and the convergence $f_{n_k} \to f$ is uniform on $\overline{U_\gamma(e)}$, every neighbourhood of y contains a point of $f(\overline{U_\gamma(e)})$. Thus y itself belongs to this compact set, f is onto and $x \in D_\alpha$.

Proof of Theorem 4. Apply Baire's category theorem to the sets $D_{1/m}$, $m \in N$, which cover X. There is $n \in N$, $w \in X$ and $\delta > 0$ with $U_\delta(w) \subseteq D_{1/n}$. For $\alpha \leq \eta = \frac{1}{2} \cdot \min\{\delta, 1/n\}$ every point z in $U_\alpha(e)$ has an α-neighbourhood isometric to $U_\alpha(e)$. (Take isometries $g\colon U_{2\alpha}(e) \to U_{2\alpha}(w) \subseteq D_{1/n}$ with $g(e) = w$ and $h\colon U_\alpha(e) \to U_\alpha(g(z))$ with $h(e) = g(z)$ and consider $g^{-1}h\colon U_\alpha(e) \to U_\alpha(z)$.)

For every x in A we find $\alpha_x > 0$ such that the α_x-neighbourhood of e and x are isometric with $f(e) = x$. Let $\varepsilon_x = \min\{\frac{1}{2} \cdot \alpha_x, \eta\}$. Every point of $U_{\varepsilon_x}(x)$ has an ε_x-neighbourhood isometric to $U_{\varepsilon_x}(e)$. The open cover $\{U_{\varepsilon_x}(x) \mid x \in A\}$ contains a finite subcover $\{U_{\varepsilon_{x_i}}(x_i) \mid i = 1, \ldots, p\}$. Now $\varepsilon_A = \min\{\varepsilon_{x_i} \mid i = 1, \ldots, p\}$ is the number we were looking for.

5. Haar measure on locally homogeneous spaces. Let A_0 be a compact set with non-empty interior in a locally homogeneous (hence locally compact) space X. To show the existence of an open-invariant measure λ with $\lambda(A_0) = 1$ we apply Mycielski's theorem (cf. §2). We only have to show that A_0 is thick in X. Let e be an interior point of A_0, and let $\delta < \varepsilon_{A_0}$ be chosen so that $\overline{U_\delta(e)} \subseteq A_0$. By Theorem 4, A_0 is covered by

$n = n(\delta)$ isometric copies of $U_\delta(e)$. Let U be their union. Then

$$E(D, t) \le E(U, t) \le n \cdot E(\overline{U_\delta(e)}, t) \le n \cdot E(A_0, t)$$

for compact $D \subseteq U$ and $0 < t \le 1$. A_0 is thick.

It is easy to see that our measure λ is positive on open sets and finite on compact sets. (Since $1/n(\delta) \le \lambda(U_\delta(e)) \le 1$, we have $\lambda(V) \ge 1/n(\delta)$ whenever V contains a copy of $U_\delta(e)$, and $\lambda(C) \le m$ whenever C is covered by m such copies.) λ is regular since it is a Baire measure [**14**].

Now we want to show that λ is uniquely determined by the requirement that isometric ε-neighbourhoods have equal measure and $\lambda(A_0) = 1$. Christensen [**6**] called a Borel measure on (X, d) uniform if for every $\varepsilon > 0$, all ε-neighbourhoods of points in X have the same finite measure, say $f(\varepsilon)$. He proved that on any metric space such a measure is unique. This applies to our situation only if X is (globally) homogeneous, but we could use more general and sophisticated results of Mattila ([**21**], Corollary 4.5).

We prefer a simple uniqueness argument. The following approximation theorem is proved in §7. 1_A denotes the characteristic function of A ($1_A(x) = 1$ for $x \in A$ and 0 otherwise).

THEOREM 5. *In a locally homogeneous and locally compact space let e be a point, A a compact set, V an open neighbourhood of A and $\varepsilon > 0$. Then there is an $\eta > 0$ such that for every open set $L \subseteq U_\eta(e)$ there exist integers n, k and isometries f_i: $U_\eta(e) \to V$ with*

$$\frac{1}{k} \cdot \sum_{i=1}^{n} 1_{f_i(L)} \ge 1_A \quad \textit{and} \quad \frac{1}{k} \cdot \sum_{i=1}^{n} \lambda(f_i(L)) \le \lambda(A) \cdot (1 + \varepsilon).$$

(For $\lambda(A) = 0$ write ε instead of $\lambda(A) \cdot (1 + \varepsilon)$.)

For the uniqueness proof we need the case $L = U_\eta(e)$. Let $f_i(e) = x_i$. Then $f_i(L) = U_\eta(x_i)$. Given A, we choose V such that $\overline{V}$ is compact, and we take $\eta < \varepsilon_{\overline{V}}$. Now all $U_\eta(x_i)$ are isometric to L. Since isometric neighbourhoods should have equal measure, the right-hand inequality of Theorem 5 becomes $(n/k) \cdot \lambda(L) \le \lambda(A) \cdot (1 + \varepsilon)$. Applying λ (or more exactly, the corresponding integral) to the left inequality of Theorem 5 we get $(n/k) \cdot \lambda(L) \ge \lambda(A)$. Thus

$$\text{(i)} \qquad \frac{n}{k} \cdot (1 + \varepsilon)^{-1} \cdot \lambda(L) \le \lambda(A) \le \frac{n}{k} \cdot \lambda(L).$$

If η was chosen small enough, we have a similar estimation for A_0.

$$\text{(ii)} \qquad \frac{n'}{k'} \cdot \lambda(L) \ge \lambda(A_0) \ge \frac{n'}{k'} \cdot (1 + \varepsilon)^{-1} \cdot \lambda(L).$$

Dividing (i) by (ii) and using $\lambda(A_0) = 1$,

$$\frac{nk'}{kn'} \cdot (1+\varepsilon)^{-1} \leq \lambda(A) \leq \frac{nk'}{kn'} \cdot (1+\varepsilon).$$

If μ would be another measure with $\mu(A_0) = 1$ which assumes equal values on isometric ε-neighbourhoods, we obtain the same inequality for $\mu(A)$ instead of $\lambda(A)$. Hence

$$(1+\varepsilon)^{-2} \leq \frac{\lambda(A)}{\mu(A)} \leq (1+\varepsilon)^2,$$

and this implies $\lambda(A) = \mu(A)$ since ε can be chosen arbitrarily. So λ and μ agree on all compact and by regularity on all Borel sets.

6. Hausdorff measures and strong invariance. Hausdorff found a method to construct metrically invariant measures [**25**] Let h: $[0, \infty) \to [0, \infty)$ be a non-decreasing function with $\lim_{t \to 0} h(t) = h(0) = 0$. The h-dimensional Hausdorff measure on (X, d) is defined for Borel sets D by

$$\mu^h(D) = \lim_{t \to 0} \inf\left\{ \sum_{j=1}^{\infty} h(\delta(B_j)) \middle| \bigcup_{j=1}^{\infty} B_j \supseteq D, \delta(B_j) \leq t \right\}.$$

Here $\delta(B)$ denotes the diameter of the set B in (X, d). Since $\delta(B \cap D) \leq \delta(B)$, we can require that $B_j \subseteq D$ for all j. Now it is easy to see that Hausdorff measures are metrically invariant. And we can say more.

Following Kolmogoroff [**16**] we call a Borel measure μ on (X, d) strongly invariant if $\mu(C) \leq \mu(D)$ whenever C, D are Borel sets and C is a non-expansive image of D. That is, there is a map f: $D \twoheadrightarrow C$ with $d(f(x), f(y)) \leq d(x, y)$ for $x, y \in D$.

Obviously strong invariance implies metric invariance. Example 1 shows that the converse is not true: the measure μ on $X = \{(t, t^2) | 0 \leq t < \infty\}$ given by $\mu(C) = \lambda_1(\{t | (t, t^2) \in C\})$ is not strongly invariant since the mapping g: $[2, 3] \to [0, 2]$, $g(t) = 2t - 4$ induces a non-expansive map between the respective parts of the parabola.

LEMMA 3a. *Hausdorff measures are strongly invariant.*

Proof. If f: $D \to C$ is non-expansive and $D = \bigcup B_j$, we have $C = \bigcup f(B_j)$ and $\delta(f(B_j)) \leq \delta(B_j)$. Thus to every sum for $\mu^h(D)$ we get a corresponding sum for $\mu^h(C)$ which is smaller or equal. Consequently, $\mu^h(C) \leq \mu^h(D)$.

To show that our measure λ on a l.h. space is metrically invariant, it suffices to find some function h with $\lambda = \mu^h$. Sometimes this is easy.

EXAMPLE 6. If X is an open set in an n-dimensional normed space, the n-dimensional (with $h(t) = t^n$) Hausdorff measure fulfils $0 < \mu^n(X) < \infty$ [**25**]. Furthermore, on the space $(F_m(X), d_H)$ of m-point subsets of X the $(n \cdot m)$-dimensional Hausdorff measure fulfils

$$\mu^{n \cdot m}(F_m(X)) = \frac{1}{m!} \cdot (\mu^n(X))^m.$$

To prove this fact we use the $m!$-to-one map $f\colon (\tilde{X}^m, d_{\max}) \to (F_m(X), d_H)$ of Example 5. We partition $F_m(X)$ into $V_1, V_2, \ldots$ such that, for each i, $f^{-1}(V_i)$ consists of $m!$ isometric copies of V_i. We omit the details.

For the general l.h. space it will be better to work with a modified Hausdorff measure ν^h [**2**].

$$\nu^h(D) = \lim_{t \to 0} \inf\left\{ \sum_{j=1}^{\infty} c_j \cdot h\big(\delta(B_j)\big) \Big| 1_D \le \sum_{j=1}^{\infty} c_j \cdot 1_{B_j}, c_j \ge 0, \delta(B_j) \le t \right\}.$$

LEMMA 3b. *For each h and each (X, d), ν^h is strongly invariant.*

The proof is similar to that of Lemma 3a.

LEMMA 4. *Let A_0 be compact with non-empty interior in the l.h. space (X, d), and let λ be the unique open-invariant measure with $\lambda(A_0) = 1$. Then $\lambda = \nu^h$, where*

$$h(t) = \sup\{ \lambda(B) | B \text{ closed}, \delta(B) \le t, B \cap A_0 \ne \varnothing \}.$$

Proof. We show $\nu^h(A_0) = \lambda(A_0)$ and apply the uniqueness argument. If $1_{A_0} \le \sum_{j=1}^{\infty} c_j \cdot 1_{B_j}$ with $B_j \cap A_0 \ne \varnothing$, it follows that

$$\lambda(A_0) = \int 1_{A_0}\, d\lambda \le \int \sum c_j \cdot 1_{B_j}\, d\lambda = \sum c_j \cdot \lambda(B_j)$$
$$\le \sum c_j \cdot h\big(\delta(B_j)\big).$$

So all sums in $\nu^h(A_0)$ and hence $\nu^h(A_0)$ itself are $\ge \lambda(A_0)$.

To show the reverse inequality, let $\varepsilon > 0$ and choose $t \le \min\{\eta(\varepsilon), \varepsilon, \varepsilon_{A_0}\}$ (ε_{A_0} from Theorem 4 and $\eta(\varepsilon)$ from Theorem 5 with $A = A_0$ and $V = X$), such that h is continuous in t. We take an open set $L \subseteq U_t(e)$ with $\delta(L) \le t$ and $h(t) \le (1+\varepsilon)^2 \lambda(L)$. (Let $t' < t$ with $h(t) \le (1+\varepsilon) \cdot h(t')$, let C be a set with $C \cap A_0 \ne \varnothing$, $\delta(C) \le t'$ and $h(t') \le (1+\varepsilon) \cdot \lambda(C)$, and let $B = \{x | d(x, C) < \frac{1}{2}(t - t')\}$. Since $\delta(B) \le t \le \varepsilon_{A_0}$, there is an open set L in $U_t(e)$ isometric to B and therefore assuming the same δ- and λ-values.)

The covering $(1/k) \cdot \Sigma 1_{f_i(L)} \geq 1_{A_0}$ from Theorem 5 fulfils $\delta(f_i(L)) < \varepsilon$ and

$$\sum \frac{1}{k} \cdot h(\delta(f_i(L))) \leq \frac{n}{k} \cdot h(t) \leq (1+\varepsilon)^2 \cdot \frac{n}{k} \cdot \lambda(L)$$
$$\leq (1+\varepsilon)^3 \cdot \lambda(A_0).$$

Since this holds for every ε, it follows that $\nu^h(A_0) \leq \lambda(A_0)$.

Collecting the results of §§5 and 6, we see that we have not only proved Theorem 1 but a stronger statement:

THEOREM 1′. *Let A_0 be a compact set with non-empty interior in a locally homogeneous space (X, d). Then there is a unique Borel measure λ on X with $\lambda(A_0) = 1$ which assumes equal values on isometric ε-neighbourhoods. λ is regular, positive on open sets, finite on compact sets and—most notably—strongly invariant.*

7. Proof of the approximation theorem. It remains to show Theorem 5. We derive it from a similar theorem which does not involve any measure λ.

THEOREM 6. *In a locally homogeneous and locally compact space (X, d) let e be a point, A a compact set and W a neighbourhood of A. Then there is an $\eta > 0$ such that for all open sets K, L with $\overline{K} \subseteq L \subseteq U_\eta(e)$ there exist positive integers n, k and isometries f_i: $U_\eta(e) \to W$ with*

$$\sum_{i=1}^{n} 1_{f_i(L)} \geq k \cdot 1_A \quad \textit{and} \quad \sum_{i=1}^{n} 1_{f_i(K)} \leq k \cdot 1_W.$$

Theorems 5 and 6 can be considered as modifications of a classical theorem of H. Cartan (cf. [2], [24]). Concerning the above-mentioned problem of Ulam we note that these theorems do not hold for the Hilbert cube with certain maximum metrics ([3], Example 1).

To derive Theorem 5 from Theorem 6, let ε, A and V be given, and take $\gamma > 1$ with $\gamma^2 < 1 + \varepsilon$. Choose open sets W, K with $A \subseteq W \subseteq V$, $\lambda(W) \leq \gamma \cdot \lambda(A)$, $\overline{K} \subseteq L$, $\lambda(L) \leq \gamma \cdot \lambda(K)$. Now open isometric sets have equal measure, and λ can be applied to the second inequality of Theorem 6:

$$\frac{n}{k} \cdot \lambda(L) \leq \gamma \cdot \frac{n}{k} \cdot \lambda(K) = \gamma \cdot \sum_{i=1}^{n} \frac{1}{k} \cdot \lambda(f_i(K))$$
$$\leq \gamma \cdot \lambda(W) \leq (1+\varepsilon) \cdot \lambda(A).$$

If we wanted to prove Theorem 5 only for $L = U_\eta(e)$ (the case used in the uniqueness proof), we could choose K as some $U_\alpha(e)$, and we need only assume that isometric α-neighbourhoods have equal measure. The proof of Theorem 6 becomes also much easier for this case, it is the same as for groups [**4**]. The difficulty of our situation is that there is no uniquely determined "translate xL of the set L". For this reason we have to work with spaces of isometries.

Proof of Theorem 6. We can assume $\overline{W}$ is compact. Let $\eta < \frac{1}{2} \cdot \varepsilon_{\overline{W}}$ such that $U_{5\eta}(A) \subseteq W$. We consider

$$F = \{f \mid f\colon U_\eta(e) \to W, f \text{ isometry}\}$$

with the metric

$$\bar{d}(f, g) = \sup_{x \in U_\eta(e)} d(f(x), g(x)).$$

$(F, \bar{d})$ is precompact. (Given $\alpha > 0$ take $U_\alpha(x_i)$, $i = 1, \dots, p$, covering $U_\eta(e)$, and $U_\alpha(y_j)$, $j = 1, \dots, q$, covering $\overline{W}$ and show that not more than q^p functions in F can have pairwise $\bar{d}$-distance $> 4\alpha$.)

Choose $\beta < \eta$ with $U_{2\beta}(K) \subseteq L$ and let $C = \{f \in F \mid f(U_\eta(e)) \cap A \neq \varnothing\}$. Let n denote the minimal number of β-neighbourhoods with respect to $\bar{d}$ necessary to cover C, and let $\mathscr{C} = \{V_\beta(f_i) \mid i = 1, \dots, n\}$ be a minimal covering of C. Then $V_\beta(f_i) \cap C \neq \varnothing$ which implies $f_i(U_\eta(e)) \subseteq U_{3\eta}(A)$ so that all $V_\beta(f_i)$ are subsets of F. Since we assume $\overline{K} \subseteq L \subseteq U(e)$, we have proved that

$$\sum_{i=1}^{n} 1_{f_i(K)}(x) = 0 \quad \text{for } x \notin U_{3\eta}(A).$$

For $x \in U_{3\eta}(A)$ define $F_x = \{f \in F \mid x \in f(K)\}$. Let k denote the minimal number of β-neighbourhoods with respect to $\bar{d}$ needed to cover $V_\beta(F_x)$. In fact k does not depend on x as we will show later. We have

$$\sum_{i=1}^{n} 1_{f_i(K)}(x) = \operatorname{card}\{\, i \mid x \in f_i(K)\} = \operatorname{card}\{\, i \mid f_i \in F_x\} \le k.$$

Namely, if $k < \operatorname{card}\{i \mid f_i \in F_x\}$, we could replace all the corresponding $V_\beta(f_i)$ in $\mathscr{C}$ by a k-element covering of $V_\beta(F_x)$ Since $f_i \in F_x$ implies $V_\beta(f_i) \subseteq V_\beta(F_x)$, this would give another covering $\mathscr{C}'$ of C, contradicting the minimality of $\mathscr{C}$. With this, the second inequality of Theorem 6 is verified.

We continue with points in $U_{3\eta}(A)$ defining $G_x = \{g \in F \mid x \in g(L)\}$. Clearly $F_x \subseteq G_x \subseteq F$. For any two points x, y the sets G_x and G_y are $\bar{d}$-isometric: There is a d-isometry $h\colon U_{2\eta}(x) \to U_{2\eta}(y)$ with $h(x) = y$, so $\bar{h}(g) = h \cdot g$ defines a $\bar{d}$-isometry $\bar{h}\colon G_x \to G_y$.

Let us show $V_{2\beta}(F_x) \subseteq G_x$. Then $\bar{h}$ maps $V_\beta(F_x)$ onto $V_\beta(F_y)$, and the k defined above does not depend on x. We have

$$V_{2\beta}(F_x) \subseteq \{ g \in F \mid \text{there is } f \in F \text{ with } f^{-1}(x) = z \in K \\ \text{and } d(f(z), g(z)) < 2\beta \}.$$

No

$$d(x, g(z)) = d(g^{-1}(x), z) < 2\beta$$

implies

$$g^{-1}(x) \in U_{2\beta}(K) \subseteq L$$

and

$$V_{2\beta}(F_x) \subseteq \{ g \in F \mid g^{-1}(x) \in L \} = G_x.$$

Finally let $x \in A$. Then $V_\beta(F_x) \subseteq G_x \subseteq C$, and $V_\beta(F_x)$ will be covered by the $V_\beta(f_i)$ in $\mathscr{C}$. For every i used in this covering, $V_\beta(f_i) \cap V_\beta(F_x) \neq \varnothing$ and hence $f_i \in V_{2\beta}(F_x) \subseteq G_x$ and $x \in f_i(L)$. This proves the first inequality of Theorem 6:

$$\sum_{i=1}^{n} 1_{f_i(L)}(x) = \text{card}\{ i \mid x \in f_i(L) \} \geq k.$$

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