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A CONTINUATION PRINCIPLE FOR FORCED OSCILLATIONS ON DIFFERENTIABLE MANIFOLDS

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In the present paper we are concerned with the existence of T-periodic solutions for the differential equation $\dot{x}(t) = f(t, x(t))$, $t \in \mathbf{R}$, where f is a continuous time dependent T-periodic tangent vector field defined on an n-dimensional differentiable manifold M possibly with boundary. We prove that if the Euler characteristic of the average vector field $w(p) = (1/T)\int_0^T f(t, p)\,dt$ is defined and nonzero and if all the possible orbits of the parametrized equation $\dot{x}(t) = \lambda f(t, x(t))$, $t \in \mathbf{R}$ and $\lambda \in (0, 1]$, lie in a compact set and do not hit the boundary of M, then the given equation admits a T-periodic solution.

0. Introduction. Let M be an n-dimensional differentiable manifold, possibly with boundary, and let $f\colon \mathbf{R} \times M \to T(M)$ be a time dependent T-periodic tangent vector field on M. In this paper we give a topological result concerning the existence of T-periodic solutions for the differential equation

$$\dot{x}(t) = f(t, x(t)), \qquad t \in \mathbf{R}. \tag{0.1}$$

Roughly speaking, we associate to f a tangent vector field $w\colon M \to T(M)$, the average wind velocity

$$w(p) = \frac{1}{T}\int_0^T f(t, p)\,dt,$$

and we show that if the Euler characteristic $\chi(w)$ of w is well defined and different from zero, then the parametrized equation

$$\dot{x}(t) = \lambda f(t, x(t)), \qquad \lambda \in \mathbf{R},\ t \in \mathbf{R} \tag*{$(0.2)_\lambda$}$$

admits, for small values of λ, T-periodic orbits which, for λ converging to zero, approach points of M where the average wind velocity vanishes. Moreover, still under the starting assumption $\chi(w) \neq 0$, we prove that, when suitable a priori estimates for the T-periodic orbits of $(0.2)_\lambda$, $\lambda \in (0, 1]$, are ensured (i.e., if these orbits lie in a compact set and do not hit the boundary of M), then (0.1) admits a T-periodic solution.

In other words, our result represents a sort of "continuation principle" for forced oscillations on differentiable manifolds which, as we shall see later, extends and unifies in a global setting many well known and apparently not related existence results.

For instance, when in particular M is an open subset of $\mathbf{R}^n$, then $\chi(w)$ is just the Brouwer degree of the map $w\colon M \to \mathbf{R}^n$, which is well defined provided that $w^{-1}(0)$ is a compact subset of M. In this case we extend a well known continuation principle for T-periodic orbits due to J. Mawhin (see e.g. [**12**], [**13**]).

If for example M is a compact boundaryless manifold, then $\chi(w)$ does not depend on w and coincides with the Euler-Poincaré characteristic $\chi(M)$ of M. In this case we obtain a classical result which ensures the existence of T-periodic solutions for the equation (0.1) provided that $\chi(M) \neq 0$ (see e.g. J. G. Borisovič and J. E. Glikhlih [**2**] and references therein).

In the case when M is a compact manifold, f does not depend on t and points outward along the boundary of M (if nonempty), then $\chi(f) = \chi(M)$. So, our result includes the famous Poincaré-Hopf theorem which asserts that if $\chi(M) \neq 0$, then f mush vanish somewhere.

Finally, let us mention the case when f is a T-periodic map from $\mathbf{R} \times \mathbf{R}^n$ into $\mathbf{R}^n$ which admits a guiding function, i.e., following Krasnosel'skij [**8**], a continuously differentiable function $G\colon \mathbf{R}^n \to \mathbf{R}$ with the property that the inner product $f(t, p) \cdot \operatorname{grad} G(p)$ is positive when the vector p has sufficiently large norm. A classical result of Krasnosel'skij [**8**] asserts that if f admits a guiding function G such that $G(p) \to +\infty$ as $\|p\| \to +\infty$, then (0.1) must have a T-periodic solution. This result can be directly deduced from Mawhin's continuation principle (see [**13**]). However we want to show that it is also included in our Theorem 2.4. In fact, observe that $\operatorname{grad} G(p) \neq 0$ when $\|p\| \to +\infty$. Thus, the condition $G(p) \to +\infty$ as $p \to +\infty$ implies that if $r > 0$ is sufficiently large, the set $M = \{p \in \mathbf{R}^n\colon G(p) \leq r\}$ is a compact n-dimensional manifold with boundary $\partial M = G^{-1}(r)$ and G has no critical points outside M. Thus, $\chi(M) = \chi(\mathbf{R}^n) = 1$ since M is a deformation retract of $\mathbf{R}^n$ (to see this just move outside M along the flow of $-\operatorname{grad} G$). On the other hand f points outward along the boundary $G^{-1}(r)$ of M and the same does its average field w. Therefore the starting assumption $\chi(w) \neq 0$ is satisfied. To deduce the existence of a T-periodic solution for $(0.2)_\lambda$ when $\lambda = 1$ observe that no T-periodic orbit of $(0.2)_\lambda$, $\lambda \neq 0$, can hit the boundary of M since, otherwise, f would be tangent to ∂M somewhere at some instant, contradicting the condition $f(t, p) \cdot \operatorname{grad} G(p) \neq 0$ for $p \in \partial M$, $t \in \mathbf{R}$.

In what follows, given a subset A of M, we will denote by $\bar{A}$ and $\operatorname{Fr} A$ the closure and the boundary of A respectively. Moreover, for any $X \subset \mathbf{R} \times M$ and $\lambda \in \mathbf{R}$, we will indicate by X_λ the slice of X at λ, i.e.

$$X_\lambda = \{p \in M\colon (\lambda, p) \in X\}.$$

1. The Euler characteristic of a vector field. Let M be an n-dimensional differentiable manifold embedded in some $\mathbf{R}^k$ $(k \geq n)$ and let $(T(M), \pi)$ denote the tangent bundle of M, π: $T(M) \to M$ being the bundle projection.

Let v: $M \to T(M)$ be a smooth tangent vector field. It is well-known that if M is compact and, when the boundary ∂M of M is nonempty, if v does not vanish on ∂M, then one can associate to v an integer $\chi(v)$, the Euler characteristic of v, with the properties that $\chi(v) \neq 0$ implies $v(p) = 0$ for some $p \in M$ and that $\chi(v)$ is equal to the Euler-Poincaré characteristic $\chi(M)$ of M (if $\partial M \neq \varnothing$, this last fact holds provided v points outward along the boundary). The reader is referred for example to [7], [**14**], [**15**] for details concerning these topics.

Recently, A. J. Tromba [**16**], [**17**] extended the notion of Euler characteristic of a vector field to the noncompact (and also not necessarily finite dimensional) case. Nevertheless, his definition does not seem to fit our purposes properly. For this reason, we shall introduce below a different definition which turns out to be equivalent to Tromba's in the finite dimensional case. Our definition is merely based on the theory of fixed point index for absolute neighborhood retracts (ANR's) and will enable us to deduce in a suitable form many properties which play an essential role in proving our main results.

For completeness let us recall that since the manifold M is, in particular, an ANR, then if ϕ: $\Omega \to M$ is any continuous map defined on an open subset Ω of M and such that the set $\{p \in \Omega;\ \phi(p) = p\}$ is compact, it is possible to define the integer $\mathrm{ind}(\phi, \Omega)$—the fixed point index of ϕ in Ω—which satisfies all the classical properties (solution, excision, additivity, homotopy invariance, normalization, commutativity) of the Leray–Schauder index ([**11**]). A detailed exposition of the index theory for ANR's can be found, for instance, in [**2**], [**6**].

Given v: $M \to T(M)$ as above, let us now consider in M the differential equation

$$\dot{x}(t) = v(x(t)), \qquad t \in \mathbf{R}. \tag{1.1}$$

For sake of simplicity, from now on we will assume M without boundary (unless otherwise specified).

Denote

$$D = \{(\tau, p) \in \mathbf{R} \times M\text{: the solution of (1.1) which satisfies } x(0) = p \text{ is continuable at least to } t = \tau\}$$

and let D_τ be the slice of D at τ.

Let $\Psi_\tau\colon D_\tau \to M$ be the translation operator which associates to any $p \in D_\tau$ the value at time τ of the solution of (1.1) satisfying $x(0) = p$. An argument based on some global continuation properties of the flow (see e.g. **[10]**) shows that D is an open set (containing $\{0\} \times M$) and that Ψ_τ is a smooth (and, thus, continuous) map.

In addition, consider the equation

$$\dot{x}(t) = \tau v(x(t)), \qquad t \in \mathbf{R},\ \tau \in \mathbf{R}. \tag{1.2}$$

It is easy to see that D coincides with the set

$$\{(\tau, p) \in \mathbf{R} \times M\colon \text{the solution of (1.2) corresponding to } \tau \text{ and satisfying } x(0) = p \text{ is continuable at least to } t = 1\}$$

and that $\Psi_\tau = \Psi_1^\tau$, where Ψ_1^τ associates to $p \in D_\tau$ the value at time $t = 1$ of the solution of (1.2) satisfying $x(0) = p$.

Assume now that the set

$$Z = \{p \in M\colon v(p) = 0\}$$

is compact.

We will show that for any relatively compact open subset Ω of M containing Z there exists a positive constant $\varepsilon = \varepsilon(\Omega)$ such that $[-\varepsilon, \varepsilon] \times \overline{\Omega} \subset D$ and $\Psi_\tau(p) \neq p$ for $p \in \operatorname{Fr}\Omega$, $0 < |\tau| \leq \varepsilon$. The first assertion is obvious. As regards the latter, suppose by contradiction that there exist $\tau_j \to 0$, $\tau_j \neq 0$, and $\{p_j\}_{j \in \mathbf{N}} \subset \operatorname{Fr}\Omega$ such that $p_j = \Psi_{\tau_j}(p_j) = \Psi_1^{\tau_j}(p_j)$. Without loss of generality we may assume $p_j \to p_0 \in \operatorname{Fr}\Omega$. Denote by $x_j(\cdot)$, $j \in \mathbf{N}$, the solution of the initial value problem

$$\begin{cases} \dot{x}(t) = \tau_j v(x(t)), & t \in [0,1] \\ x(0) = p_j. \end{cases} \tag{1.3$_j$}$$

Since M is a submanifold of $\mathbf{R}^k$, the integrals $\int_0^1 v(x_j(t))\,dt$ make sense. Hence

$$0 = x_j(1) - x_j(0) = \tau_j \int_0^1 v(x_j(t))\,dt,$$

so that, as $\tau_j \neq 0$,

$$\int_0^1 v(x_j(t))\,dt = 0 \quad \text{for all } j \in \mathbf{N}. \tag{1.4}$$

On the other hand, the continuous dependence on the initial conditions of the solutions $(\tau_j(\cdot), x_j(\cdot))$ of the system

$$\begin{cases} \dot{x}(t) = \tau(t) v(x(t)), & t \in [0,1] \\ \dot{\tau}(t) = 0 \\ x(0) = p_j \\ \tau(0) = \tau_j \end{cases}$$

(which is clearly equivalent to $(1.3)_j$) and the fact that any $x_j(\cdot)$ is defined in the whole interval $[0,1]$ imply that the sequence $\{(\tau_j, x_j(\cdot))\}_{j\in\mathbf{N}}$ converges pointwise in $[0,1]$ to the (constant) solution $(0, x_0(\cdot))$ of $(1.3)_0$ corresponding to $(0, p_0)$. Moreover, it is not hard to show that if $\hat{\Omega}$ is a compact neighborhood of $\overline{\Omega}$ and if $\gamma = \max_{p\in\hat{\Omega}}|v(p)|$, then any solution of (1.2) with $x(0) \in \overline{\Omega}$ remains in $\hat{\Omega}$ for $|\tau| \leq \eta/\gamma$, where η denotes the (positive) distance between $\overline{\Omega}$ and $\operatorname{Fr}\hat{\Omega}$. Therefore, the convergence of the sequence $\{x_j(\cdot)\}_{j\in\mathbf{N}}$ is dominated and so, passing to the limit in (1.4), we obtain $0 = \int_0^1 v(x_0(t))\,dt = v(p_0)$ contradicting the emptiness of $Z \cap \operatorname{Fr}\Omega$.

Thus, an ε with the required properties exists and this implies that the fixed point index $\operatorname{ind}(\Psi_\tau, \Omega)$ is well-defined for $0 < |\tau| \leq \varepsilon$. Moreover, from the homotopy invariance of the index, $\operatorname{ind}(\Psi_\tau, \Omega)$ is independent of τ in each one of the intervals $(0, \varepsilon)$ and $(-\varepsilon, 0)$.

The above considerations permit us to define $\chi(v)$, *the characteristic of the vector field v in M*, as follows:

$$\chi(v) = \lim_{\tau\to 0^-} \operatorname{ind}(\Psi_\tau, \Omega).$$

The reason of the choice of negative τ's will be clear after the proof of the normalization property.

Because of the excision property of the index, the above limit does not depend on the relatively compact open set Ω, provided Ω contains Z.

Notice that in order to define $\chi(v)$ one might equivalently consider Ψ_τ^{-1}, i.e. the translation operator associated with the equation $\dot{x}(t) = -v(x(t))$, $t \in \mathbf{R}$, and set

$$\chi(v) = \lim_{\tau\to 0^+} \operatorname{ind}\left(\Psi_\tau^{-1}, \Omega\right).$$

If $\mathcal{O}$ is any open subset of M such that the set $\{p \in \mathcal{O}\colon v(p) = 0\}$ is compact, then $\mathcal{O}$ is itself a boundaryless manifold and, so, it makes sense to consider $\chi(v|\mathcal{O})$, that is the Euler characteristic of the restriction of v to $\mathcal{O}$. In the case when $\partial M \neq \varnothing$ and the set $\{p \in \mathring{M}\colon v(p) = 0\}$ is a compact subset of the interior $\mathring{M}$ of M, then we still write, for brevity, $\chi(v)$ instead of $\chi(v|\mathring{M})$.

The following properties of $\chi(v)$ are easy consequences of the given definition and depend on the analogous properties of the fixed point index.

Solution. If $\chi(v) \neq 0$, then the vector field v has a zero in M.

Excision. Let $\mathcal{O}$ be any open subset of M containing Z. Then $\chi(v) = \chi(v|\mathcal{O})$.

Additivity. Let $\mathcal{O}'$, $\mathcal{O}''$ be open subsets of M such that $Z \subset \mathcal{O}' \cup \mathcal{O}''$ and let $Z \cap \mathcal{O}'$ and $Z \cap \mathcal{O}''$ be disjoint. Then $\chi(v) = \chi(v|\mathcal{O}') + \chi(v|\mathcal{O}'')$.

Homotopy. Let $z\colon M \times [0,1] \to T(M)$ be a parametrized tangent vector field such that $\{p \in M\colon z(p,\mu) = 0 \text{ for some } \mu \in [0,1]\}$ is compact. Then $\chi(z(\cdot,\mu))$ is independent of $\mu \in [0,1]$.

From the homotopy property it follows, in particular, that if M is a compact boundaryless manifold, then $\chi(v)$ does not depend on v. If M is compact and with boundary and if v does not vanish on ∂M, then $\chi(v)$ depends only on the restriction of v to ∂M.

Topological invariance. Let $h\colon M \to N$ be a smooth diffeomorphism and let $\tilde{v}\colon N \to T(N)$ be the vector field which corresponds to v under h (i.e. for each $p \in M$, $dh_p v(p) = \tilde{v}(h(p))$). Then $\chi(\tilde{v})$ is defined and $\chi(v) = \chi(\tilde{v})$.

Proof. Since the set $Z = \{p \in M\colon v(p) = 0\}$ is compact, it follows that $\tilde{Z} = \{q \in N\colon \tilde{v}(q) = 0\}$ is compact as well. Therefore, $\chi(\tilde{v})$ is defined. Let Ψ_τ and $\tilde{\Psi}_\tau$ be the translation operators corresponding to v and $\tilde{v}$ respectively and let $\tilde{\Omega}$ be a relatively compact open subset of N containing $\tilde{Z}$. Then, $\tilde{\Psi}_\tau = h \circ \Psi_\tau \circ h^{-1}$ in $\tilde{\Omega}$ and, for τ sufficiently small, $\tilde{\Psi}_\tau(q) \neq q$ on $\operatorname{Fr}\tilde{\Omega}$. Hence from the commutativity property of the index,

$$\chi(\tilde{v}) = \operatorname{ind}(\tilde{\Psi}_\tau, \tilde{\Omega}) = \operatorname{ind}\big(\Psi_\tau \circ h^{-1} \circ h, h^{-1}(\tilde{\Omega})\big)$$

$$= \operatorname{ind}\big(\Psi_\tau, h^{-1}(\tilde{\Omega})\big) = \chi(v). \qquad \square$$

Normalization. Assume M to be compact and let $v\colon M \to T(M)$. If $\partial M = \varnothing$, then $\chi(v)$ is equal to $\chi(M)$, the Euler-Poincaré characteristic of M. If $\partial M \neq \varnothing$ and v points outward along ∂M, then again we have $\chi(v) = \chi(M)$.

Proof. In the case $\partial M = \varnothing$ clearly Ψ_τ is defined in M for all $\tau \in \mathbf{R}$. If $\partial M \neq \varnothing$ and v is outward on ∂M, then Ψ_τ is defined in M for all $\tau \leq 0$ and the set Z of the zeros of v is a compact subset of the interior of M. Therefore, in both cases, $\chi(v)$ is defined and, if Ω is an open neighborhood of Z such that $\overline{\Omega} \subset \mathring{M}$, we have $\chi(v) = \lim_{\tau \to 0^-} \operatorname{ind}(\Psi_\tau, \Omega)$.

On the other hand, there exists $\varepsilon > 0$, such that $\Psi_\tau(p) \neq p$ for all $p \in M \setminus \Omega$, $-\varepsilon \leq \tau < 0$. Hence, by the excision property of the index

$$\operatorname{ind}(\Psi_\tau, \Omega) = \operatorname{ind}(\Psi_\tau, M), \qquad -\varepsilon \leq \tau < 0.$$

Moreover, by the normalization property of the index

$$\operatorname{ind}(\Psi_\tau, M) = \Lambda(\Psi_\tau)$$

where $\Lambda(\Psi_\tau)$ denotes the Lefschetz number of Ψ_τ. Now, since Ψ_τ is homotopic to Ψ_0, which is the identity over M, it follows that

$$\Lambda(\Psi_\tau) = \Lambda(\Psi_0) = \chi(M).$$

Thus, $\chi(v) = \chi(M)$ as claimed. □

In the sequel, we will make use also of the following

PROPOSITION 1.1. *Let v: $M \to T(M)$ be a smooth vector field with a compact set of zeros. Then*

$$\chi(-v) = (-1)^n \chi(v), \qquad n = \dim M.$$

Proof. By Sard's Lemma one can join v to a vector field w possessing only isolated zeros by means of a homotopy which has a compact set of zeros. Let $p_0 \in M$ be such that $w(p_0) = 0$ and let h: $\Omega \to \mathbf{R}^n$ be a smooth diffeomorphism of a bounded open neighborhood Ω of p_0 in M onto $h(\Omega)$. Because of the additivity property of the index, without loss of generality we may assume that p_0 is the only zero of w.

Thus, from the topological invariance of χ, we obtain

$$\chi(w) = \chi(\tilde{w})$$

where $\tilde{w}$: $h(\Omega) \to \mathbf{R}^n$ is given by $\tilde{w} = dh_{p_0} \circ w \circ h^{-1}$. On the other hand, since $\tilde{w}$ is a vector field of $\mathbf{R}^n$, $\chi(\tilde{w})$ coincides with the Brouwer degree $\deg(\tilde{w}, h(\Omega), 0)$ and as it is well-known by the degree theory,

$$\deg(-\tilde{w}, h(\Omega), 0) = (-1)^n \deg(\tilde{w}, h(\Omega), 0).$$ □

In order to be able to derive, as direct consequences of our main results of §2, those existence and continuation results of [2], [8], [12], [13] stated in the Introduction, we need to extend the definition of $\chi(v)$ also to the case when the vector field v is only continuous. To this end, let $\{v_j\}_{j \in \mathbf{N}}$ be a sequence of smooth vector fields uniformly approximating v on compact subsets of M.

Let Ω be a relatively compact open subset of M containing Z. It is easy to see that there exists $j_0 \in \mathbf{N}$ such that

$$v_j(p) \neq 0 \quad \text{for all } p \in \operatorname{Fr}\Omega, \qquad j > j_0.$$

More precisely, one can choose j_0 in such a way that, for any $j_1, j_2 > j_0$,

$$\mu v_{j_1}(p) + (1-\mu) v_{j_2}(p) \neq 0 \quad \text{for all } p \in \operatorname{Fr}\Omega, \qquad \mu \in [0,1].$$

Thus $\chi(v_j|\Omega)$ is well-defined for $j > j_0$ and, from the homotopy invariance,

$$\chi(v_{j_1}|\Omega) = \chi(v_{j_2}|\Omega).$$

Therefore, we can define

$$\chi(v) = \lim_{j \to \infty} \chi(v_j|\Omega).$$

The same argument shows that the given definition is independent of the choice of the approximating sequence and that all the properties listed above (solution, excision etc.) still hold.

2. Global branches and continuation principles. Let M be an n-dimensional differentiable manifold and consider in M the differential equation

$$\dot{x}(t) = \lambda f(t, x(t)), \qquad \lambda \geq 0, \, t \in \mathbf{R}, \tag{2.1}$$

where $f\colon \mathbf{R} \times M \to T(M)$ is a continuous vector field periodic of period $T > 0$ with respect to t.

A pair $(\lambda, p) \in [0, \infty) \times M$ will be called a *starting point* for the equation (2.1) if (2.1) has a T-periodic C^1 solution $x\colon \mathbf{R} \to M$ corresponding to λ and satisfying the initial condition $x(0) = p$. A point $p \in M$ will be called an *emanating point* (or a *bifurcation point*) of T-periodic orbits for (2.1) if any neighborhood of $(0, p)$ in $[0, \infty) \times M$ contains nontrivial (i.e. with $\lambda \neq 0$) starting points.

Let U be an open subset of $[0, \infty) \times M$. By a *global branch* (of starting points) in U we shall mean a noncompact subset of U which is the closure (in U) of a connected set of nontrivial starting points. Roughly speaking, a global branch is a closed connected set of starting points which is unbounded or (if bounded) must contain points which are arbitrarily close to the boundary of U in $[0, \infty) \times M$. In other words it intersects the complement of any compact subset of U.

Given a global branch C observe that any $p \in M$ such that $(0, p) \in C$ is an emanating point. In what follows we shall say that C emanates from a subset X of M if the slice $\{p \in M\colon (0, p) \in C\}$ is nonempty and contained in X.

We are interested in finding conditions which ensure the existence of global branches of starting points in U.

To this end, let us associate to equation (2.1) the autonomous vector field $w\colon M \to T(M)$ defined by

$$w(p) = \frac{1}{T} \int_0^T f(t, p)\, dt.$$

Without loss of generality we may assume (in case replacing M with its interior $\mathring{M}$) that M is a boundaryless manifold. Suppose in addition that the set

$$\{p \in U_0 \colon w(p) = 0\}$$

(recall that U_0 denotes the slice of U at $\lambda = 0$) is compact. So, the Euler characteristic of w in U_0, $\chi(w | U_0)$, is defined.

In the sequel, we shall also assume, for each $(\lambda, p) \in U$, the global continuation on $[0, T]$ of the solutions of the Cauchy problem

$$\begin{cases} \dot{x}(t) = \lambda f(t, x(t)), & \lambda \geq 0,\ t \in \mathbf{R}, \\ x(0) = p. \end{cases} \tag{2.2}$$

Clearly this property is satisfied for instance when M is a compact manifold without boundary.

Under the above assumptions we can state the following:

THEOREM 2.1. *Let M, f, U, w be as above. Assume in addition that f is smooth and that the restriction of the vector field w to U_0 has nonzero Euler characteristic.*

Then the equation (2.1) *admits a global branch of starting points in U emanating from the set $\{p \in U_0 \colon w(p) = 0\}$.*

The proof of Theorem 1 requires the following point set topology result (see [**1**], [**9**, Chapter V]).

LEMMA 2.2. *Let X be a compact metric space and let A, B be nonempty disjoint subsets of X. Then either A and B are separated in X (i.e. there exists a pair (F_A, F_B) of closed disjoint subsets of X such that $F_A \supset A$, $F_B \supset B$, $F_A \cup F_B = X$), or there exists a connected subset C of $X \setminus (A \cup B)$ whose closure meets both A and B.*

Proof of Theorem 2.1. Let

$$Z = \{p \in U_0 \colon w(p) = 0\}$$

and

$$S = (\{0\} \times Z) \cup \{(\lambda, p) \in U \colon \lambda > 0,\ (\lambda, p) \text{ is a starting point of } (2.1)\}.$$

The assumption $\chi(w | U_0) \neq 0$ and the solution property of the Euler characteristic imply that Z and, thus, S are nonempty. Let us show that S is closed in U (with respect to the relative topology). Take $(\lambda_0, p_0) \in U$ and a sequence $\{(\lambda_j, p_j)\}_{j \in \mathbf{N}})$ in S converging to (λ_0, p_0). Assume first

$\lambda_0 \neq 0$ and denote by $x_j(\cdot)$, $j \in \mathbf{N}$, the solution of the Cauchy problem

$$(2.2)_j \qquad \begin{cases} \dot{x}(t) = \lambda_j f(t, x(t)), & t \in \mathbf{R} \\ x(0) = p_j. \end{cases}$$

As in §1, the sequence $\{x_j(\cdot)\}_{j\in \mathbf{N}}$ converges pointwise to $x_0(\cdot)$ in $[0, T]$. Thus, in particular,

$$(\lambda_0, x_0(T)) = \lim_{j\to\infty} (\lambda_j, x_j(T)) = \lim_{j\to\infty} (\lambda_j, p_j) = (\lambda_0, p_0),$$

so that (λ_0, p_0) is a starting point of (2.1) and, hence, it belongs to S.

In the case $\lambda_0 = 0$, we need to show that p_0 belongs to Z. This is obvious if $\lambda_j = 0$ for infinitely many j since w is continuous. Otherwise, there exists $\bar{j} \in \mathbf{N}$ such that $\lambda_j \neq 0$ for $j > \bar{j}$ and, as

$$0 = x_j(T) - x_j(0) = \lambda_j \int_0^T f(t, x_j(t))\, dt,$$

we obtain

$$\int_0^T f(t, x_j(t))\, dt = 0 \quad \text{for all } j > \bar{j}.$$

Now, the same argument already used in §1 shows that we can pass to the limit in the above equality. Thus, $\int_0^T f(t, x_0(t))\, dt = 0$ and, since (as $\lambda_0 = 0$) the solution $x_0(\cdot)$ is the constant function $x_0(t) \equiv p_0$, it follows that

$$w(p_0) = \int_0^T f(t, p_0)\, dt = 0,$$

i.e. $p_0 \in Z$.

Our aim now is to prove that if V is any relatively compact open subset of U containing $\{0\} \times Z$ and such that $\overline{V} \subset U$, then S intersects $\operatorname{Fr} V$.

To this end let us consider the equation

$$(2.3) \qquad \dot{x}(t) = \lambda[\mu f(t, x(t)) + (1 - \mu) w(x(t))], \qquad t \in \mathbf{R},\ \lambda \geq 0,\ \mu \in [0, 1]$$

and let

$$W = \{(\lambda, p) \in U\colon \text{the solution } x(\cdot, \lambda, \mu) \text{ of (2.3) satisfying } x(0) = p \text{ is defined in } [0, \mathrm{T}] \text{ for all } \mu \in [0, 1]\}.$$

Let $H_T\colon W \times [0, 1] \to M$ denote the translation operator which associates to any (λ, p, μ) the value at time T of the solution of (2.3) satisfying $x(0) = p$.

It can be shown (see e.g. **[10]**) that W is an open set (containing the compact set $\{0\} \times \overline{V}_0$) and that H_T is smooth in $W \times [0,1]$.

Let Ω be an open subset of the slice V_0 such that $Z \subset \Omega \subset \overline{\Omega} \subset V_0$ and take $\varepsilon > 0$ such that $[0,\varepsilon] \times \overline{\Omega} \subset W \cap V$. Clearly, we may also choose ε in such a way that $W_\lambda \supset \overline{V}_\lambda$ for $0 \le \lambda \le \varepsilon$. Observe that this implies, in particular, that the operator $p \mapsto H_T(\lambda, p, 1)$ is defined in $\overline{V}_\lambda$ for $0 \le \lambda \le \varepsilon$. In addition, we may assume, reducing ε if necessary, that

$$\left\{(\lambda, p, \mu) \in \overline{V} \times [0,1] \colon H_T(\lambda, p, \mu) = p, 0 < \lambda \le \varepsilon\right\}$$
$$\subset (0,\varepsilon] \times \Omega \times [0,1]$$

(the proof of this fact is similar to the one given in §1 in introducing our definition of the Euler characteristic of a vector field).

Consequently, for any $0 < \lambda \le \varepsilon$ and $\mu \in [0,1]$, the fixed point index $\operatorname{ind}(H_T(\lambda, \cdot, \mu), \Omega)$ is well-defined and independent of λ and μ. Moreover, from the excision property of the index

$$\operatorname{ind}(H_T(\lambda, \cdot, 1), \Omega) = \operatorname{ind}(H_T(\lambda, \cdot, 1), V_\lambda), \tag{2.4}$$

and, since the translation operator $\Phi_T\colon U \to M$ associated with equation (2.2) clearly coincides with the operator $(\lambda, p) \mapsto H_T(\lambda, p, 1)$ on the set $\{(\lambda, p) \in \overline{V}\colon 0 \le \lambda \le \varepsilon\}$, we may write

$$\operatorname{ind}(H_T(\lambda, \cdot, 1), V_\lambda) = \operatorname{ind}(\Phi_T(\lambda, \cdot), V_\lambda). \tag{2.5}$$

Suppose now, by contradiction, that $S \cap \operatorname{Fr} V = \varnothing$. Observe that $\{(\lambda, p) \in S\colon \lambda > 0\}$ is nothing else but $\{(\lambda, p) \in U\colon \Phi_T(\lambda, p) = p, \lambda > 0\}$ and that, $S \cap V$ being compact, there exists $\bar{\lambda} > 0$ such that $(\{\bar{\lambda}\} \times V_{\bar{\lambda}}) \cap S = \varnothing$. Therefore, from the generalized homotopy invariance property of the index,

$$\operatorname{ind}(\Phi_T(\lambda, \cdot), V_\lambda) = \operatorname{ind}(\Phi_T(\bar{\lambda}, \cdot), V_{\bar{\lambda}}) = 0. \tag{2.6}$$

Hence by (2.4), (2.5), (2.6), for $0 < \lambda \le \varepsilon$, we obtain

$$\operatorname{ind}(H_T(\lambda, \cdot, 0), \Omega) = \operatorname{ind}(H_T(\lambda, \cdot, 1), \Omega) = 0. \tag{2.7}$$

On the other hand, recalling the definition of the Euler characteristic of a vector field given in §1 and taking into account of Proposition 1.1, it follows, for $0 < \lambda \le \varepsilon$, that

$$\operatorname{ind}(H_T(\lambda, \cdot, \Omega)) = \chi(-w | U_0) = (-1)^n \chi(w | U_0). \tag{2.8}$$

Since by assumption $\chi(w | U_0) \ne 0$, the equality (2.8) contradicts (2.7). Thus $S \cap \operatorname{Fr} V \ne \varnothing$ as claimed.

We are now in a position to prove the assertion of the theorem.

According to the notation introduced above, we need to show that there exists a connected subset of $S \cap \{(\lambda, p) \in U: \lambda > 0\}$ whose closure in U is noncompact and emanates from Z.

In order to do this, let us construct the one-point compactification of U, by adjoining the point ∞.

Set $\hat{U} = U \cup \{\infty\}$. A subset $\hat{\mathcal{O}}$ of $\hat{U}$ is an open neighborhood of ∞ if $\infty \in \hat{\mathcal{O}}$ and $U \setminus \hat{\mathcal{O}}$ is compact. Clearly $\hat{U}$ is a compact Hausdorff space and $\{\infty\}$ is closed in $\hat{U}$. Consider the compact metric subset of $\hat{U}$, $\hat{S} = S \cup \{\infty\}$. Observe that $\{0\} \times Z$ and $\{\infty\}$ are disjoint because of the compactness of Z. It is not hard to see that our assertion reduces now to show that there exists a connected subset of $\hat{S} \cap \{(\lambda, p) \in \hat{U}: \lambda > 0\}$ whose closure in $\hat{S}$ intersects both $\{0\} \times Z$ and $\{\infty\}$. This will follow from Lemma 2.2, provided that $\{0\} \times Z$ and $\{\infty\}$ are not separated in $\hat{S}$. By contradiction, suppose they are separated, that is suppose there exists a pair of disjoint compact subsets of $\hat{S}$, say K and H, such that $K \supset \{0\} \times Z$, $H \supset \{\infty\}$, $K \cup H = \hat{S}$. Thus, $\hat{U}$ being Hausdorff, there exists an open neighborhood $\hat{V}$ of H in $\hat{U}$ such that $\infty \notin \mathrm{cl}_{\hat{U}} \hat{V}$ and $S \cap (\mathrm{cl}_{\hat{U}} \hat{V} \setminus \hat{V}) = \varnothing$ ($\mathrm{cl}_{\hat{U}}$ denotes the closure in $\hat{U}$).

Clearly, the set $V = \hat{V} \cap U$ is a relatively compact open subset of U containing $\{0\} \times Z$. Therefore, by the first part of the proof, the set $S \cap \mathrm{Fr}\, V$, which coincides with $\hat{S} \cap (\mathrm{cl}_{\hat{U}} \hat{V} \setminus \hat{V})$ is nonempty. A contradiction. Hence $\{0\} \times Z$ and $\{\infty\}$ are not separated in $\hat{S}$ as required to achieve the proof of the theorem. □

Theorem 2.1 above is a particular case of the following more general result.

THEOREM 2.2. *Let M, f, w be as in Theorem 2.1. Consider in M the equation*

$$\dot{x}(t) = \lambda f(t, x(t)), \qquad t \in \mathbf{R},\ \lambda \in [\lambda_1, \lambda_2], \tag{2.9}$$

with $\lambda_1 \neq \lambda_2$ and $-\infty \leq \lambda_1 \leq 0 \leq \lambda_2 \leq \infty$. Let U be an open subset of $[\lambda_1, \lambda_2] \times M$ and assume that the Euler characteristic of the restriction of w to U_0 is defined and nonzero.

Then the equation (2.9) *admits in $U \cap ((\lambda_1, \lambda_2) \times M)$ a connected branch of starting points whose closure C (in U) has the following properties*:

(i) *the set $\{p \in U_0: (0, p) \in C\}$ is nonempty and contained in $\{p \in U_0: w(p) = 0\}$*;

(ii) *if $\lambda_2 > 0$ [respectively $\lambda_1 < 0$] the set $C^+ = \{(\lambda, p) \in C: \lambda \geq 0\}$ [resp. $C^- = \{(\lambda, p) \in C: \lambda \leq 0\}$] is either noncompact or intersects U_{λ_2} [resp. U_{λ_1}].*

The proof of this result is in the line of that of Theorem 2.1 but requires a little more technicalities. Thus, we omit the details. We only point out that in the case when $\lambda_1 < 0$ and $\lambda_2 > 0$ one may need to construct a two-point compactification of U in order to deduce assertion (ii). An analogous construction has been considered in [5] where, however, the manifold M was compact.

Our aim now is to remove the smoothness assumption on the vector field f. To this end, observe first that the global continuation on $[0, T]$ of the solutions of problem (2.2) is equivalent to the fact that, for any $(\lambda, p) \in U$, the set attainable in the time interval $[0, T]$ by the solutions of (2.2) is compact. Hence, taking into account this, it is not hard to show that the global continuation property implies the following:

(c) *For any compact subset $K \subset U$ the set*

$$A([0,T],K) = \{p \in M\colon \textit{there exists a solution } x(\cdot) \textit{ of } (2.1) \\ \textit{with } (\lambda, x(0)) \in K \textit{ and } x(\tau) = p \textit{ for some } \tau \in [0,T]\}$$

is compact.

We will now state the "continuous" version of Theorem 2.1.

THEOREM 2.3. *Let $f\colon \mathbf{R} \times M \to T(M)$ be a T-periodic in t continuous vector field. Assume that $\chi(w|U_0) \neq 0$. Then the same conclusion as in Theorem 2.1 holds.*

Proof. Let Z and S be as in Theorem 2.1. With the aid of Ascoli's theorem it is not hard to see that property (c) guarantees that the set S is still closed. So, the assertion will follow again from the point set topology argument used in Theorem 2.1 if one can show that S intersects the boundary of any relatively compact open subset V of U such that $\{0\} \times Z \subset V \subset \overline{V} \subset U$. In order to prove this, consider a sequence $\{f_j\}_{j \in \mathbf{N}}$ of T-periodic smooth vector fields uniformly approximating f on compact subsets of $\mathbf{R} \times M$. For each $j \in \mathbf{N}$, define

$$w_j(p) = \frac{1}{T}\int_0^T f_j(t,p)\,dt$$

and

$$S_j = \big(\{0\} \times \{p \in U_0\colon w_j(p) = 0\}\big) \cup \{(\lambda, p) \in U\colon \lambda > 0, (\lambda, p) \\ \text{is a starting point of } \dot{x}(t) = \lambda f_j(t, x(t))\}.$$

Clearly, the sequence $\{w_j\}_{j \in \mathbf{N}}$ uniformly approximates the vector field w associated with f and $\chi(w_j | V_0) \neq 0$ for $j > \bar{j}$.

Therefore, by Theorem 2.1, any S_j $(j > \bar{j})$ contains a global branch of starting points of $\dot{x}(t) = \lambda f_j(t, x(t))$, $t \in \mathbf{R}$, emanating from $\{p \in U_0: w_j(p) = 0\}$.

Thus, in particular, for each j there exists $(\lambda_j, p_j) \in S_j \cap \operatorname{Fr} V$, $\lambda_j \neq 0$. Without loss of generality we may assume

$$(\lambda_j, p_j) \to (\lambda_0, p_0) \in \operatorname{Fr} V.$$

We will prove that (λ_0, p_0) belongs also to S. Let $x_j(\cdot)$ be the solution of

$$\begin{cases} \dot{x}(t) = \lambda_j f_j(t, x(t)), & t \in \mathbf{R} \\ x(0) = p_j. \end{cases}$$

By using again property (c), it can be shown that the sequence $\{x_j(\cdot)\}_{j \in \mathbf{N}}$ admits a subsequence uniformly converging on the interval $[0, T]$ to a solution $x_0(\cdot)$ of the problem

$$\begin{cases} \dot{x}(t) = \lambda_0 f(t, x(t)), & t \in \mathbf{R} \\ x(0) = p_0 \end{cases}$$

(a proof of this fact can be obtained as in [**4**, Theorem 2.4]). Since $x_j(T) = p_j$, this implies $x_0(T) = p_0$ and, if $\lambda_0 = 0$, $w(p_0) = 0$. Thus $(\lambda_0, p_0) \in S$ so that $S \cap \operatorname{Fr} V \neq \varnothing$ as required. □

If M is a compact manifold, then the Euler characteristic of M, $\chi(M)$, is defined and the following consequence of Theorems 2.2 and 2.3 holds.

COROLLARY 2.1. ([**5**]). *Let M be compact with $\chi(M) \neq 0$ and let $f: \mathbf{R} \times M \to T(M)$ be a T-periodic continuous vector field which, if $\partial M \neq \varnothing$, points inward (or outward) along ∂M for each $t \in \mathbf{R}$. Then the equation*

$$(2.10) \qquad \dot{x}(t) = \lambda f(t, x(t)), \qquad t \in \mathbf{R}, \lambda \in \mathbf{R}$$

admits a connected branch of starting points intersecting $\{0\} \times M$ in a nonempty subset of $\{0\} \times \{p \in M: \int_0^T f(t, p)\, dt = 0\}$ and whose projection on $\mathbf{R}$ is onto.

Proof. Let $w: M \to T(M)$ be defined by $w(p) = (1/T)\int_0^T f(t, p)\, dt$. We prove the corollary when $\partial M \neq \varnothing$ (the other case is easier). Since M is compact and $w(p) \neq 0$ for all $p \in \partial M$ (w is, in fact, outward or inward on ∂M), the Euler characteristic of the vector field w is well-defined. Assume f to be inward. So, the global continuation on $[0, T]$ of the solutions of (2.2) is ensured. Moreover, by the normalization property and Proposition 1.1, we obtain that $\chi(w) = (-1)^n \chi(-w) = (-1)^n \chi(M)$. Thus, $\chi(w) \neq 0$.

Therefore, Theorems 2.2 and 2.3 apply yielding the existence of a closed connected branch $C \subset \mathbf{R} \times \mathring{M}$ of starting points of (2.10) intersecting $\{0\} \times M$ in a subset of $\{(0, p)\colon w(p) = 0\}$ and such that $C^{+} = \{(\lambda, p) \in C\colon \lambda \geq 0\}$ and $C^{-} = \{(\lambda, p) \in C\colon \lambda \leq 0\}$ are noncompact (closed) subsets of $\mathbf{R} \times \mathring{M}$.

Now, since by the assumption on f, no starting point of (2.10) may belong to ∂M, the sets C^{+} and C^{-} turn out to be in fact noncompact closed subsets of $\mathbf{R} \times M$. Consequently, C is unbounded with respect to positive and negative values of λ as claimed (recall that M is assumed to be compact).

Observe finally that the case f outward on ∂M can be reduced to the previous one by defining the inward vector field $\tilde{f}(t, p) = -f(-t, p)$ and by noting that if $y(\cdot)$ is a solution of $\dot{y}(t) = \lambda \tilde{f}(t, y(t))$ corresponding to a fixed $\lambda \in \mathbf{R}$, then $x(t) = y(-t)$ is a solution of $\dot{x}(t) = \lambda f(t, x(t))$. □

Let us point out that in Theorems 2.1–2.3 and, implicitly, in Corollary 2.1, we have assumed that the solutions of the Cauchy problem (2.2) are defined in the whole interval $[0, T]$. It is not hard to convince oneself that, in order to deduce the existence of global branches of starting points this hypothesis seems to be reasonable. However, in the case when one is concerned with merely existence results, the global continuation of the solutions is usually replaced by suitable a priori bounds on the T-periodic orbits of the system (see e.g. **[8]**, **[12]**, **[13]**). This viewpoint will be illustrated by Theorem 2.4 below which makes precise the continuation principle for forced oscillations roughly described in the Introduction of the present paper.

THEOREM 2.4. *Let M be possibly with boundary, $f\colon \mathbf{R} \times M \to T(M)$ a T-periodic continuous vector field and $w\colon M \to T(M)$ the average vector field associated with f. Assume that*:

(i) *the set $\{p \in M\colon w(p) = 0\}$ is a compact subset of the interior $\mathring{M}$ of M*;

(ii) *the Euler characteristic $\chi(w)$ is nonzero*;

(iii) *all the possible T-periodic orbits $x(\cdot)$ of the parametrized equation*

$$\dot{x}(t) = \lambda f(t, x(t)), \qquad \lambda \in (0, 1],\ t \in \mathbf{R}, \tag{2.11}$$

lie in a compact subset of $\mathring{M}$.

Then the equation

$$\dot{x}(t) = f(t, x(t)), \qquad t \in \mathbf{R},$$

has a T-periodic solution.

Proof. Assume first that f is smooth. By (i) and (iii), there exists a relatively compact open subsets Ω of $\mathring{M}$ such that $\overline{\Omega} \subset \mathring{M}$, the set $\{p \in M: w(p) = 0\}$ is contained in Ω and, if x is a T-periodic solution of (2.11), then $x(t) \notin \operatorname{Fr} \Omega$ for all $t \in \mathbf{R}$. Let ϕ: $M \to [0, 1]$ be a smooth function with compact support contained in $\mathring{M}$ and such that $\phi(p) = 1$ for each $p \in \overline{\Omega}$. Define $\hat{f}$: $\mathbf{R} \times M \to T(M)$, by $\hat{f}(t, p) = \phi(p) f(t, p)$. Clearly, for each $(\lambda, p) \in [0, \infty) \times M$, the solution of the Cauchy problem

$$(2.12) \qquad \begin{cases} \dot{x}(t) = \lambda \hat{f}(t, x(t)), & t \in \mathbf{R} \\ x(0) = p \end{cases}$$

is globally continuable on $[0, \infty)$. Moreover, if $\hat{w}$: $M \to T(M)$ is the average vector field given by $\hat{w}(p) = (1/T) \int_0^T \hat{f}(t, p)\, dt$, we obviously have $\hat{w} = \phi w$. So, by (ii) and the localization property, it follows that $\chi(\hat{w}|\Omega) = \chi(w|\Omega) = \chi(w) \neq 0$. Hence Theorem 2.1 applies to $\hat{f}$ and $\hat{w}$ yielding the existence in $U = [0, 1) \times \Omega$ of a global branch C of starting points (λ, p) of the equation

$$\dot{x}(t) = \lambda \hat{f}(t, x(t))$$

emanating from the set $\{p \in \Omega: \hat{w}(p) = 0\}$. Consider now the following subset of C:

$$C_1 = \{(\lambda, p) \in C: \text{the (periodic) solution } x(\cdot) \text{ of (2.12) corresponding to } (\lambda, p) \text{ is such that } x(t) \in \Omega \text{ for all } t\}.$$

It is clearly $C_1 \neq \varnothing$ since C emanates from the nonempty set $\{p \in \Omega: \hat{w}(p) = 0\}$. Moreover, it is easy to see that C_1 is open and closed in C. Thus, C being connected, we have $C_1 = C$. This implies that C turns out to be in fact a global branch of starting points of equation (2.11). Observe now that, since C is non-compact and closed in $U = [0, 1) \times \Omega$, the closure in $[0, \infty) \times M$ of C must intersect the boundary of U. Moreover, by the continuous dependence of the solutions on the initial conditions, it follows easily that any $(\lambda, p) \in \overline{C} \cap \operatorname{Fr} U$ is a starting point of (2.11) and, if $\lambda = 0$, is such that $w(p) = 0$. On the other hand, by the assumption on Ω, we have $\lambda < 1$. Thus, there exists a starting point of (2.11) of the form $(1, p)$, i.e. the equation $\dot{x}(t) = f(t, x(t))$, $t \in \mathbf{R}$, has a T-periodic orbit (which lies in Ω).

Let us go back to the case when f is only continuous. Take Ω as above and let $\{f_j\}_{j \in \mathbf{N}}$ be a sequence of T-periodic smooth vector fields uniformly approximating f on $\mathbf{R} \times \overline{\Omega}$. It is not hard to see that, for j sufficiently large, the f_j's and the associated w_j's satisfy the assumptions (i), (ii), (iii) of this theorem in the boundaryless manifold Ω. Hence, by the first part of the proof, any equation $\dot{x}(t) = f_j(t, x(t))$ has at least a T-periodic orbit x_j lying in Ω. To complete the proof, it suffices now only to observe that, by Ascoli's theorem, the sequence $\{x_j\}_{j \in \mathbf{N}}$ has a subsequence uniformly converging to a (T-periodic) solution of the equation $\dot{x}(t) = f(t, x(t))$, $t \in \mathbf{R}$. □

REMARK. Results analogous to the previous ones are still valid for equations of the form

$$\dot{x}(t) = \lambda(g(\lambda, t, x(t))), \qquad \lambda \in \mathbf{R},\ t \in \mathbf{R}$$

with $g\colon \mathbf{R} \times \mathbf{R} \times M \to T(M)$ a T-periodic in t continuous vector field, provided that in all statements one replaces the vector field w by $z\colon M \to T(M)$ defined by:

$$z(p) = \int_0^T g(0, t, p)\, dt.$$

Observe for instance that any smooth vector field $(\lambda, t, p) \mapsto h(\lambda, t, p)$ such that $h(0, t, p) = 0$ for all $(t, p) \in \mathbf{R} \times M$ can be written in the form $\lambda g(\lambda, t, p)$ with

$$g(\lambda, t, p) = \int_0^1 \frac{\partial f}{\partial \lambda}(s\lambda, t, p)\, ds.$$

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