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THE BAIRE-CATEGORY METHOD IN SOME COMPACT EXTENSION PROBLEMS

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We characterize metrizable separable spaces X such that almost every, in the sense of Baire category, embedding h of X into the Hilbert cube I^ω provides a compact extension $\overline{h(X)}$ such that the remainder $\overline{h(X)} \setminus h(X)$ has certain dimensional property (for instance, is n-dimensional, countable-dimensional or "metrically weakly infinite-dimensional"). We obtain a characterization of metrizable separable spaces which have large transfinite dimension by means of compactifications. Two examples related to the results mentioned above are constructed.

1. Introduction. Consider the following two classes of separable metrizable spaces: the class (P_n) of spaces X with $\dim X \leq n$ and the class (P_ω) of countable-dimensional spaces (for terminology see §2). In this paper, for each of the classes (P), we characterize the class of the spaces X such that almost every, in sense of Baire category, homeomorphic embedding h of X into the Hilbert cube I^ω yields a compact extension $\overline{h(X)}$ of $h(X)$ whose remainder $\overline{h(X)} \setminus h(X)$ is in the class (P).

In contrast with the classical result in dimension theory that if X has a compact extension with $\dim \leq n$ (i.e. $\dim X \leq n$, by Hurewicz theorem) then almost every embedding h: $X \to I^\omega$ provides a compact extension $\overline{h(X)}$ with $\dim \overline{h(X)} \leq n$, the existence of a compactification $\tilde{Y}$ of Y whose remainder $\tilde{Y} \setminus Y$ is in the class (P), for any of the two (P) we consider, is not enough to guarantee that almost every embedding h: $Y \to I^\omega$ provides such a compactification $\overline{h(Y)}$ for $h(Y)$.

In the case of (P_n) the characterization is simple: the class consists exactly of spaces which are unions of a compact set and an n-dimensional set (§3).

To give a characterization for (P_ω), we introduce a somewhat weaker property than the weak infinite-dimensionality (in the sense of Smirnov). In particular, spaces having large transfinite dimension have this property, which yields a characterization of spaces with tr Ind, analogous to that given by Hurewicz for small transfinite dimension (§4).

We consider also (§5) spaces X such that for almost every embedding $h\colon X \to I^\omega$ the remainder $\overline{h(X)} \setminus h(X)$ is "metrically weakly infinite-dimensional"; this class has a close connection with the class related to (P_ω).

We end the paper with two examples related to the subject of this paper (§6).

I would like to thank the referee for some valuable remarks.

2. Preliminaries. Our terminology follows [2], [3], and [4]. All spaces considered in this paper are assumed to be metrizable separable. If A is a subset of X, then $B(A, \varepsilon)$ denotes an open ε-ball about A with respect to a fixed metric in X. By the dimension we understand the covering dimension dim. A space X is said to be S-weakly infinite-dimensional (weakly infinite-dimensional) if for every sequence $\{(A_1, B_1), (A_2, B_2), \dots\}$ of pairs of disjoint closed subsets of X there exists a sequence $L_1, L_2, \dots$ of closed subsets of X such that L_i is a partition between A_i and B_i (see [3], Definition 1.1.3), and $\bigcap_{i=1}^N L_i = \varnothing$ for some integer N ($\bigcap_{i=1}^\infty L_i = \varnothing$). A space is countable-dimensional, if it is a union of countably many subspaces of dimension zero. The large (small) transfinite dimension tr Ind (trind) is the transfinite extension of the classical large (small) inductive dimension Ind (ind) (see [4], Definitions 1.1 and 1.2). A family $\{(A_1, B_1), \dots, (A_n, B_n)\}$ of pairs of closed disjoint subsets of X is essential, if for every sequence $L_1, \dots, L_n$ of closed sets such that L_i is a partition between A_i and B_i we have $\bigcap_{i=1}^n L_i \neq \varnothing$.

In the sequel we will often apply, without explicitly referring to, the following lemma on partitions (see [3], Lemma 1.2.9).

LEMMA 2.1. *Let M be a subspace of a space X and let (A, B) be a pair of disjoint closed subsets of X. Then for every partition L' in the space M between $M \cap \overline{U}$ and $M \cap \overline{V}$, where U, V are open subsets of X such that $A \subset U$, $B \subset V$ and $\overline{U} \cap \overline{V} = \varnothing$, there exists a partition L in the space X between A and B which satisfies the inclusion $M \cap L \subset L'$.*

By I we denote the real interval $[-1, 1]$, I^ω denotes the Hilbert cube and let ρ be a fixed metric in I^ω. By $C(X, I^\omega)$ we denote the function space of all continuous mappings from X into I^ω endowed with a complete metric

$$d(f_1, f_2) = \sup\{\rho(f_1(x), f_2(x))\colon x \in X\}.$$

The subspace of $C(X, I^\omega)$ consisting of all embeddings of X into I^ω will be denoted by $E(X, I^\omega)$. A subset A of a space X is residual, if its complement $X \setminus A$ is a first category set (i.e. $X \setminus A$ is a union of countably many nowhere dense sets); a countable intersection of residual sets is residual in X, hence it is dense if X is complete. We will need the following three classical theorems.

THEOREM 2.2 (*see* [7], *Ch.* IV, §44, VI, *Theorem* 2). *For any space X the set $E(X, I^\omega)$ is a residual set in $C(X, I^\omega)$.*

THEOREM 2.3 (*see* [7], *Ch.* IV, §45, VII, *Theorem* 4', *cf. also* [9]). *If $A_1, \dots, A_k$ are closed subsets of a space X such that $A_1 \cap \dots \cap A_k = \varnothing$ then the set $\{h \in C(X, I^\omega)$: $\overline{h(A_1)} \cap \dots \cap \overline{h(A_k)} = \varnothing\}$ is a residual set in $C(X, I^\omega)$.*

THEOREM 2.4 (*see* [7], *Ch.* IV, §45, VII, *Theorem* 4'). *If A is a closed subset of a space X, then $\{h \in C(X, I^\omega)$: $\dim \overline{h(A)} \leq \dim A\}$ is a residual set in $C(X, I^\omega)$.*

3. Spaces whose almost every compact extension has remainder of $\dim \leq n$.

THEOREM 3.1. *For any space X and $n = 0, 1, \dots$ the following conditions are equivalent*:

(i) *the set $\mathscr{H} = \{h \in E(X, I^\omega)$: $\dim(\overline{h(X)} \setminus h(X)) \leq n\}$ is a residual set in $C(X, I^\omega)$*:

(ii) *the set $\mathscr{H}$ is dense in $C(X, I^\omega)$*;

(iii) *there exists a compact set $K \subset X$ such that* $\dim(X \setminus K) \leq n$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. Let us show that (ii) $\Rightarrow$ (iii). First we will prove that if the condition (ii) holds then

(1) for every family $\{(A_0, B_0), \dots, (A_n, B_n)\}$ of pairs of disjoint closed subsets of X there exist closed sets $L_0, \dots, L_n$ such that L_i is a partition between A_i and B_i for $i = 0, \dots, n$ and the set $\bigcap_{i=0}^n L_i$ is compact.

Indeed, if $\{(A_0, B_0), \dots, (A_n, B_n)\}$ is such a family then by Theorems 2.2 and 2.3 there exists $f \in E(X, I^\omega)$ such that $\overline{f(A_i)} \cap \overline{f(B_i)} = \varnothing$ for $i = 0, \dots, n$. Take $\varepsilon > 0$ satisfying $\rho(\overline{f(A_i)}, \overline{f(B_i)}) > \varepsilon$. By (ii) there exists $h \in E(X, I^\omega)$ such that $d(f, h) < \varepsilon/3$ and $\dim(\overline{h(X)} \setminus h(X)) \leq n$. We have then $\rho(h(A_i), h(B_i)) \geq \varepsilon/3$ and thus $\overline{h(A_i)} \cap \overline{h(B_i)} = \varnothing$ for $i = 0, \dots, n$. Let U_i and V_i be open subsets of $\overline{h(X)}$ containing $\overline{h(A_i)}$ and

$\overline{h(B_i)}$ respectively with $\overline{U}_i \cap \overline{V}_i = \varnothing$. Since $\dim(\overline{h(X)} \setminus h(X)) \leq n$, then for $i = 0, \ldots, n$ there exists a partition L_i' between $\overline{U}_i \cap (\overline{h(X)} \setminus h(X))$ and $\overline{V}_i \cap (\overline{h(X)} \setminus h(X))$ in the space $\overline{h(X)} \setminus h(X)$ such that $\bigcap_{i=0}^{n} L_i' = \varnothing$. By Lemma 2.1 for $i = 0, \ldots, n$ there exists a partition $\tilde{L}_i$ between $\overline{h(A_i)}$ and $\overline{h(B_i)}$ in $\overline{h(X)}$ such that $\tilde{L}_i \cap (\overline{h(X)} \setminus h(X)) \subset L_i'$. The set $\bigcap_{i=0}^{n} \tilde{L}_i$ is compact and it is contained in $h(X)$. Then the set $L_i = h^{-1}(\tilde{L}_i \cap h(X))$ is a partition between A_i and B_i in X such that the set $\bigcap_{i=0}^{n} L_i = h^{-1}(\bigcap_{i=0}^{n} \tilde{L}_i)$ is compact, which finishes the proof of (1). To end the proof of (iii) put $K = X \setminus \{x \in X\colon x$ has a neighbourhood U with $\dim U \leq n\}$: we have $\dim(X \setminus K) \leq n$. If the set K is not compact, then there exists a discrete sequence $\{x_j\}_{j=1}^{\infty}$ of points of K and a discrete family $\{W_j\}_{j=1}^{\infty}$ of open subsets of X such that $x_j \in W_j$. Since $x_j \in K$, then $\dim \overline{W}_j > n$ and thus there exists an essential family $\{(A_0^j, B_0^j), \ldots, (A_n^j, B_n^j)\}$ of pairs of closed disjoint subsets of $\overline{W}_j$. Then, for $i = 0, \ldots, n$, the sets $A_i = \bigcup_{j=1}^{\infty} A_i^j$ and $B_i = \bigcup_{j=1}^{\infty} B_i^j$ are closed disjoint subsets of X such that if L_i is an arbitrary partition between A_i and B_i, then the set $\bigcap_{i=0}^{n} L_i$ is not compact, since it intersects every set $\overline{W}_j$. This is contrary to (1).

It remains to prove that (iii) $\Rightarrow$ (i). If $A_i = X \setminus B(K, 1/i)$ for $i = 1, 2, \ldots$ then $\dim A_i \leq n$. By Theorems 2.2 and 2.4 the set $\mathscr{F} = \{h \in E(X, I^{\omega})\colon \dim \overline{h(A_i)} \leq \dim A_i$ for $i = 1, 2, \ldots\}$ is residual in $C(X, I^{\omega})$. We shall show that $\dim(\overline{h(X)} \setminus h(X)) \leq n$ for every $h \in \mathscr{F}$. By the countable sum theorem it suffices to verify that $\overline{h(X)} \setminus h(X) \subset \bigcup_{i=1}^{\infty} \overline{h(A_i)}$. This follows from the compactness of K: if $x \in \overline{h(X)} \setminus h(X)$, then there exists a neighbourhood U of x in $\overline{h(X)}$ and a number i such that $U \cap h(B(K, 1/i)) = \varnothing$; we have then $x \in \overline{h(A_i)}$.

REMARK 3.2. It is easy to see that the condition (1) described in the above proof is in fact equivalent to the conditions (i), (ii) and (iii) in Theorem 3.1. It is also easy to show by induction that X satisfies (1) if and only if the strong inductive dimension which neglects the class of compact spaces of X is not greater than n (see [1] for the exact definition of this notion).

REMARK 3.3. Each of the conditions in Theorem 3.1 is equivalent to the following condition

(2) for each family $\{f_i\}_{i=0}^{n}$ of continuous functions of X into the interval I there exists $h \in E(X, I^{\omega})$ such that $\dim(\overline{h(X)} \setminus h(X)) \leq n$, and for each $i = 0, \ldots, n$ the function $f_i h^{-1}\colon h(X) \to I$ has a continuous extension $\tilde{f}_i\colon h(X) \to I$.

The implication (i) $\Rightarrow$ (2) follows from Theorem 3.4 in [**11**] stating that for every function $f\colon X \to I$ the set $\{h \in E(X, I^\omega)$: the function fh^{-1} has an extension $\tilde{f}\colon \overline{h(X)} \to I\}$ is residual in $C(X, I^\omega)$. Thus by Remark 3.2 it suffices to prove that (2) $\Rightarrow$ (1). Let $\{(A_0, B_0), \ldots, (A_n, B_n)\}$ be a sequence of pairs of disjoint closed subsets of X and $f_i\colon X \to I$ be a function such that $f_i(A_i) = 0$ and $f_i(B_i) = 1$ for $i = 0, \ldots, n$. By (2) there exists $h \in E(X, I^\omega)$ such that $\dim(\overline{h(X)} \setminus h(X)) \leq n$ and every function $f_i h^{-1}\colon h(X) \to I$ has an extension $\tilde{f}_i\colon \overline{h(X)} \to I$. For $i = 0, \ldots, n$ we have $\tilde{f}_i(\overline{h(A_i)}) = 0$ and $\tilde{f}_i(\overline{h(B_i)}) = 1$, hence $\overline{h(A_i)} \cap \overline{h(B_i)} = \varnothing$. Now it is easy to construct the required partitions L_i (compare the proof that (ii) $\Rightarrow$ (1) in Theorem 3.1).

4. Spaces whose almost every compact extension has countable-dimensional remainder.

THEOREM 4.1. *For a space X the following conditions are equivalent*:

(i) *the set $\mathscr{H} = \{h \in E(X, I^\omega)$: $\overline{h(X)} \setminus h(X)$ is countable-dimensional$\}$ is residual in $C(X, I^\omega)$*:

(ii) *X satisfies the following condition (K): X contains a compact set K such that $\dim(X \setminus U) < \infty$ for every open subset U of X which contains K*;

(iii) *the set $\mathscr{H}' = \{h \in E(X, I^\omega)$: $\overline{h(X)} \setminus h(X)$ is locally finite-dimensional$\}$ is residual in $C(X, I^\omega)$.*

Proof. Let us prove that (i) $\Rightarrow$ (ii). If the condition (K) is not satisfied, then there exists a discrete family $\{U_j\}_{j=1}^\infty$ of open subsets of X such that $\dim \overline{U}_j \geq j$ for $j = 1, 2, \ldots$, (the proof repeats the reasonings of Skljarenko [**13**]; cf. [**2**], Ch. X, §6, Lemmas 2 and 3). Let $\{(A_i^j, B_i^j)\}_{i=1}^j$ be an essential family in $\overline{U}_j$, then the sets $A_i = \bigcup_{j=i}^\infty A_i^j$ and $B_i = \bigcup_{j=i}^\infty B_i^j$ are disjoint closed subsets of X for $i = 1, 2, \ldots$. By Theorems 2.2 and 2.3 the set $\mathscr{F} = \{h \in E(X, I^\omega)$: $\overline{h(A_i)} \cap \overline{h(B_i)} = \varnothing$ for $i = 1, 2 \ldots\}$ is residual in $C(X, I^\omega)$. By (i) there exists $h \in \mathscr{F}$ such that $\overline{h(X)} \setminus h(X) = \bigcup_{i=1}^\infty Z_i$, where $\dim Z_i \leq 0$. By the separation theorem (see [**3**], Theorem 4.1.13) for each i there exists a partition L_i between the sets $\overline{h(A_i)}$ and $\overline{h(B_i)}$ such that $L_i \cap Z_i = \varnothing$. Then $L = \bigcap_{i=1}^\infty L_i$ is a compact subset of $\overline{h(X)}$ so that $h^{-1}(L)$ is a compact subset of X. Since the family $\{\overline{U}_j\}_{j=1}^\infty$ is discrete, $h^{-1}(L) \cap \overline{U}_j = \varnothing$ for $j \geq j_0$, where j_0 is some integer. Let $F = \overline{\bigcup_{j=j_0}^\infty h(U_j)}$; since $L \cap F = \varnothing$, there exists an open subset V of $\overline{h(X)}$ such that $L \subset V$ and $V \cap F = \varnothing$. By compactness of L there exists i_0 such that $\bigcap_{i=1}^{i_0} L_i \subset V$ so that $\bigcap_{i=1}^{i_0} L_i \cap h(\overline{U}_n) = \varnothing$ for $n \geq j_0$. Take

$n_0 = \max(i_0, j_0)$. Putting $K_i = h^{-1}(L_i)$, $i = 1, \ldots, i_0$, we have

$$\bigcap_{i=1}^{i_0} \left(K_i \cap \overline{U}_{n_0}\right) = \varnothing$$

and $K_i \cap \overline{U}_{n_0}$ is a partition between $A_i^{n_0}$ and $B_i^{n_0}$, which is contrary to the assumption that the family $\{(A_i^{n_0}, B_i^{n_0})\}_{i=1}^{n_0}$ is essential.

To show that (ii) $\Rightarrow$ (iii) put $F_i = X \setminus B(K, 1/i)$ for $i = 1, 2, \ldots$; we have $\dim F_i < \infty$. By Theorems 2.2 and 2.4 the set $\mathscr{F} = \{h \in E(X, I^\omega)$: $\dim \overline{h(F_i)} \leq \dim F_i$ for $i = 1, 2, \ldots\}$ is residual in $C(X, I^\omega)$. From the compactness of K it follows easily that for each $h \in \mathscr{F}$ the set $\overline{h(X)} \setminus h(X)$ is locally finite-dimensional.

The implication (iii) $\Rightarrow$ (i) is obvious, since every locally finite-dimensional space is countable-dimensional.

REMARK 4.2. Let us notice that every S-weakly infinite-dimensional space (hence every space having large transfinite dimension tr Ind) satisfies the condition (K). Indeed, let us recall that a space X is S-weakly infinite-dimensional if and only if $S(X) = X \setminus \bigcup\{U$: U is an open subspace of X and $\dim U < \infty\}$ is a compact weakly infinite-dimensional subspace of X such that for every open set $U \subset X$ which contains $S(X)$ we have $\dim(X \setminus U) < \infty$ (see [2], Ch. X, §6, Theorem 26).

By a theorem of Hurewicz (see [4], Theorem 4.15) a space X has trind if and only if there exists $h \in E(X, I^\omega)$ such that $\overline{h(X)}$ is countable-dimensional. From Theorem 4.1 we obtain a corresponding characterization of spaces having tr Ind:

COROLLARY 4.3. *For a space X the following conditions are equivalent*:

(i) *X has large transfinite dimension* tr Ind;

(ii) *the set $\mathscr{H} = \{h \in E(X, I^\omega)$: $\overline{h(X)}$ is countable-dimensional$\}$ is residual in $C(X, I^\omega)$.*

Proof. The implication (i) $\Rightarrow$ (ii) has been proved by Luxemburg [8]. We shall prove that (ii) $\Rightarrow$ (i). If the set $\mathscr{H}$ is residual, then by Theorem 4.1 the space X contains a compact set K such that $\dim(X \setminus U) < \infty$ for every open subset of X containing K. Since X is homeomorphic to a subspace of a countable-dimensional space $\overline{h(X)}$, where h is a member of $\mathscr{H}$, X is countable-dimensional. It follows that K is compact and countable-dimensional, hence K has tr Ind, which implies that X has tr Ind (see [4], Theorem 3.16).

Let us note that other characterizations of metrizable separable spaces having tr Ind, by means of compactifications, were given in [**4**], Theorem 4.10 and [**10**], Theorem 10.1(a).

COROLLARY 4.4. *For a space X the following conditions are equivalent:*

(i) *X is S-weakly infinite-dimensional;*

(ii) *the set $\mathscr{H}=\{h\in E(X,I^\omega)$: $\overline{h(X)}$ is weakly infinite-dimensional$\}$ is residual in $C(X,I^\omega)$.*

Proof. If X is S-weakly infinite-dimensional, then it satisfies (K) and by Theorem 4.1 the set $\mathscr{H}=\{h\in E(X,I^\omega)$: $\overline{h(X)}\setminus h(X)$ is countable dimensional$\}$ is residual in $C(X,I^\omega)$. To prove (ii) it suffices to show that if $h(X)$ is S-weakly infinite-dimensional and $\overline{h(X)}\setminus h(X)=\bigcup_{i=1}^{\infty}Z_i$, where $\dim Z_i\le 0$ for $i=1,2,\dots$, then $\overline{h(X)}$ is weakly infinite-dimensional. Let $\{(A_1,B_1),(A_2,B_2),\dots\}$ be a family of pairs of disjoint closed subsets of $\overline{h(X)}$. For $i=1,2,\dots$ let L_{2i} be a partition between A_{2i} and B_{2i} in $\overline{h(X)}$ such that $L_{2i}\cap Z_i=\emptyset$. Then the set $L=\bigcap_{i=1}^{\infty}L_{2i}$ is a compact subset of $h(X)$, hence it is weakly infinite-dimensional. Thus for $i=1,2,\dots$ there exists a partition L'_{2i-1} between $A_{2i-1}\cap L$ and $B_{2i-1}\cap L$ in L such that $\bigcap_{i=1}^{\infty}L'_{2i-1}=\emptyset$. Taking partitions L_{2i-1} between A_{2i-1} and B_{2i-1} in $\overline{h(X)}$ such that $L_{2i-1}\cap L\subset L'_{2i-1}$ we obtain a family $\{L_i\}_{i=1}^{\infty}$ of partitions such that $\bigcap_{i=1}^{\infty}L_i=\emptyset$.

It remains to prove that (ii) $\Rightarrow$ (i). If $\{(A_1,B_1),(A_2,B_2),\dots\}$ is a family of pairs of disjoint closed subsets of X, then by (ii) and Theorems 2.2 and 2.3 there exists $h\in E(X,I^\omega)$ such that $\overline{h(X)}$ is weakly infinite-dimensional, and $\overline{h(A_i)}\cap\overline{h(B_i)}=\emptyset$ for $i=1,2,\dots$. If L'_i is a partition between $\overline{h(A_i)}$ and $\overline{h(B_i)}$ in $\overline{h(X)}$ such that $\bigcap_{i=1}^{N}L'_i=\emptyset$, then $L_i=h^{-1}(L'_i\cap h(X))$ is a partition between A_i and B_i in X such that $\bigcap_{i=1}^{N}L_i=\emptyset$.

5. Spaces S-weakly infinite-dimensional modulo compacta.

DEFINITION 5.1. We say that a space X is S-weakly infinite-dimensional modulo compacta (shortly, S-w.i.d. modulo compacta), if for every sequence $\{(A_1,B_1),(A_2,B_2),\dots\}$ of pairs of closed disjoint subsets of X there exists a sequence $\{L_i\}_{i=1}^{\infty}$ of closed subsets of X such that L_i is a partition between A_i and B_i and for some integer N the set $\bigcap_{i=1}^{N}L_i$ is compact.

PROPOSITION 5.2. (i) *Every space X which is a union of a compact subspace and an S-weakly infinite-dimensional subspace is S-w.i.d. modulo compacta*;

(ii) *If a space X is S-w.i.d. modulo compacta, then it has property (K).*

Proof. The proof of (i) is standard: if $X = K \cup Y$, where K is compact, Y is S-weakly infinite-dimensional, and $\{(A_1, B_1), (A_2, B_2), \dots\}$ is a family of pairs of closed disjoint subsets of X, then we take open subsets U_i and V_i of X such that $A_i \subset U_i$, $B_i \subset V_i$, and $\overline{U}_i \cap \overline{V}_i = \emptyset$. There exist partitions L_i' between $\overline{U}_i \cap Y$ and $\overline{V}_i \cap Y$ such that $\bigcap_{i=1}^{N} L_i' = \emptyset$ for some integer N. Take partitions L_i between A_i and B_i such that $L_i \cap Y \subset L_i'$; then the set $\bigcap_{i=1}^{N} L_i \subset K$ is compact.

The proof of (ii) follows easily from the fact that if the condition (K) is not satisfied, then there exists a discrete family $\{U_j\}_{j=1}^{\infty}$ of open subsets of X such that $\dim \overline{U}_j \geq j$ (compare the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 4.1).

DEFINITION 5.3. We say that a subspace $X \subset I^{\omega}$ is metrically S-weakly infinite-dimensional (shortly, μ-S-w.i.d.) if for every sequence $\{(A_1, B_1), (A_2, B_2), \dots\}$ of pairs of disjoint closed subsets of X such that $\rho(A_i, B_i) > 0$ there exists a partition L_i between A_i and B_i, $i = 1, 2, \dots$, such that $\bigcap_{i=1}^{N} L_i = \emptyset$ for some integer N (recall that ρ is a fixed metric in I^{ω}).

THEOREM 5.4. *For a space X the following conditions are equivalent*:

(i) *the set $\mathscr{H} = \{h \in E(X, I^{\omega}): \overline{h(X)} \setminus h(X)$ is μ-S-w.i.d.$\}$ is residual in $C(X, I^{\omega})$*;

(ii) *X is S-weakly infinite-dimensional modulo compacta.*

Proof. To show that (i) $\Rightarrow$ (ii) let us take a sequence $\{(A_1, B_1), (A_2, B_2), \dots\}$ of pairs of disjoint closed subsets of X. By (i) and by Theorems 2.2 and 2.3 there exists $h \in E(X, I^{\omega})$ such that $\overline{h(A_i)} \cap \overline{h(B_i)} = \emptyset$, for $i = 1, 2, \dots$, and the set $\overline{h(X)} \setminus h(X)$ is μ-S-w.i.d. Take open sets U_i and V_i in $\overline{h(X)}$ containing $\overline{h(A_i)}$ and $\overline{h(B_i)}$ respectively such that $\overline{U}_i \cap \overline{V}_i = \emptyset$. Since $\rho(\overline{U}_i, \overline{V}_i) > 0$, then there exist partitions L_i' between $\overline{U}_i \cap (\overline{h(X)} \setminus h(X))$ and $\overline{V}_i \cap (\overline{h(X)} \setminus h(X))$ such that $\bigcap_{i=1}^{N} L_i' = \emptyset$ for some integer N. Finally, take partitions $\tilde{L}_i$ between $\overline{h(A_i)}$ and $\overline{h(B_i)}$ with $\tilde{L}_i \cap (\overline{h(X)} \setminus h(X)) \subset L_i'$; then $L_i = h^{-1}(\tilde{L}_i \cap h(X))$ are required partitions between A_i and B_i.

The first step in the proof that (ii) $\Rightarrow$ (i) is to show that if $\mathscr{A} = \{(A_1, B_1), \dots, (A_N, B_N)\}$ is a family of pairs of closed disjoint subsets of X such that

(3) for $i = 1, \dots, N$ there exists a partition L_i between A_i and B_i such that the set $\bigcap_{i=1}^{N} L_i$ is compact,

then

(4) the set $\mathscr{H}_{\mathscr{A}} = \{h \in E(X, I^\omega)$: for each $i = 1, \dots, N$, $\overline{h(A_i)} \cap \overline{h(B_i)} = \varnothing$ and there exists a partition K_i between $\overline{h(A_i)}$ and $\overline{h(B_i)}$ such that $\bigcap_{i=1}^{N} K_i \setminus h(X) = \varnothing\}$ is residual in $C(X, I^\omega)$.

Indeed, for $i = 1, \dots, N$, let E_i and F_i be closed subsets of X such that $L_i = E_i \cap F_i$, $A_i \subset E_i$, $B_i \subset F_i$, $A_i \cap F_i = \varnothing$, $B_i \cap E_i = \varnothing$ and $E_i \cup F_i = X$. Then the set $C = \bigcap_{i=1}^{N} L_i = \bigcap_{i=1}^{N} E_i \cap \bigcap_{i=1}^{N} F_i$ is a compact subset of X. Let $V_k = B(C, 1/k)$ for $k = 1, 2, \dots,$; we have $(\bigcap_{i=1}^{N} E_i \cap \bigcap_{i=1}^{N} F_i) \setminus V_k = \varnothing$. Hence for each k and a family $\{E_i \setminus V_k$: $i = 1, \dots, N\} \cup \{F_i \setminus V_k$: $i = 1, \dots, N\}$ of closed subsets of X we have $\bigcap_{i=1}^{N} (E_i \setminus V_k) \cap (F_i \setminus V_i) = \varnothing$. By Theorems 2.2 and 2.3 the sets $\mathscr{H}_1 = \{h \in E(X, I^\omega)$: $\bigcap_{i=1}^{N} \overline{h(E_i \setminus V_k)} \cap \bigcap_{i=1}^{N} \overline{h(F_i \setminus V_i)} = \varnothing$ for $k = 1, 2 \dots\}$ and $\mathscr{H}_2 = \{h \in E(X, I^\omega)$: $\overline{h(A_i)} \cap \overline{h(F_i)} = \varnothing$ and $\overline{h(E_i)} \cap \overline{h(B_i)} = \varnothing$ for $i = 1, 2 \dots\}$ are residual in $C(X, I^\omega)$. We will show that $\mathscr{H}_{\mathscr{A}} \supseteq \mathscr{H}_1 \cap \mathscr{H}_2$. For $i = 1, \dots, N$ let $K_i = \overline{h(E_i)} \cap \overline{h(F_i)}$; since $h \in \mathscr{H}_2$, then the set K_i is a partition between $\overline{h(A_i)}$ and $\overline{h(B_i)}$ in $\overline{h(X)}$. Let us verify that

$$\bigcap_{i=1}^{N} K_i \setminus h(X) = \varnothing.$$

Suppose, to the contrary, that there exists a point $x \in (\bigcap_{i=1}^{N} \overline{h(E_i)} \cap \bigcap_{i=1}^{N} \overline{h(F_i)}) \setminus h(X)$. Since $\{x\}$ and $h(C)$ are disjoint compact subsets of $\overline{h(X)}$, there exists a neighbourhood V of the point x in $\overline{h(X)}$ such that $\overline{V} \cap h(C) = \varnothing$. Take an integer k such that $V_k \cap h^{-1}(V) = \varnothing$. For each i we have

$$x \in \overline{h(E_i) \cap V} \cap \overline{h(F_i) \cap V} \subset \overline{h(E_i \setminus V_k)} \cap \overline{h(F_i \setminus V_k)}$$

which is a contradiction, since $h \in \mathscr{H}_1$. This ends the proof of (4).

Now suppose that X is S-w.i.d. modulo compacta. Let $\mathscr{D} = \{(C_1, D_1), (C_2, D_2), \dots\}$ be a family of pairs of disjoint closed subsets of I^ω such that for every pair (A, B) of disjoint closed subsets of I^ω there exists an i such that $A \subset C_i$ and $B \subset D_i$. Let $\mathscr{U}_1$ be a maximal family of

disjoint open subsets of $C(X, I^\omega)$ of diameter < 1. If $\mathcal{U}_{i-1}$ is defined, then for each $U \in \mathcal{U}_{i-1}$ let $\mathcal{U}_{i,U}$ be a maximal family of disjoint open subsets of U of diameter $< 1/i$ and $\mathcal{U}_i = \bigcup_{U \in \mathcal{U}_{i-1}} \mathcal{U}_{i,U}$.

Let us fix an integer i. For each $U \in \mathcal{U}_i$ let us choose $f_U \in U$ and consider the family

$$\mathcal{D}_U = \{(f_U^{-1}(C_1), f_U^{-1}(D_1)), (f_U^{-1}(C_2), f_U^{-1}(D_2)), \dots\}$$

of pairs of disjoint closed subsets of X. Let $\mathcal{K}(U)$ be the family of all finite subsequences $\mathcal{A} = \{(A_1, B_1), \dots, (A_N, B_N)\}$ of the family $\mathcal{D}_U$ for which the condition (3) is satisfied. By (4), the set $\mathcal{H}_\mathcal{A}$ is residual in $C(X, I^\omega)$ for every $\mathcal{A} \in \mathcal{K}(U)$. Since $|\mathcal{K}(U)| \leq \chi_0$, then the set $\mathcal{H}(U) = \bigcap_{\mathcal{A} \in \mathcal{K}(U)} \mathcal{H}_\mathcal{A}$ is also residual in $C(X, I^\omega)$. Since the family $\mathcal{U}_i$ consists of disjoint and open subsets of $C(X, I^\omega)$ and $\overline{U\mathcal{U}_i} = C(X, I^\omega)$, then the set

$$\mathcal{H}_i = \bigcup_{U \in \mathcal{U}_i} (\mathcal{H}(U) \cap U)$$

is residual in $C(X, I^\omega)$. Finally, the set $\mathcal{H}' = \bigcap_{i=1}^\infty \mathcal{H}_i$ is residual in $C(X, I^\omega)$.

It remains to prove that $\mathcal{H}' \subset \mathcal{H}$, i.e. that for every $h \in \mathcal{H}'$ the set $\overline{h(X)} \setminus h(X)$ is μ-S-w.i.d. Let $\{(A_1, B_1), (A_2, B_2), \dots\}$ be a family of disjoint closed subsets of $\overline{h(X)} \setminus h(X)$ such that $\rho(A_i, B_i) > 0$ for $i = 1, 2, \dots$. Then for each i the closures $\overline{A_i}$ and $\overline{B_i}$ of A_i and B_i in I^ω are disjoint and thus there exists an integer $j(i)$ and a real number $\varepsilon_i > 0$ such that $B(A_i, \varepsilon_i) \subset C_{j(i)}$ and $B(B_i, \varepsilon_i) \subset D_{j(i)}$. For each i take $\eta_i > 0$ such that the sets $C_i^* = B(C_{j(i)}, \eta_i)$ and $D_i^* = B(D_{j(i)}, \eta_i)$ are disjoint. Consider the family $\{(h^{-1}(C_1^*), h^{-1}(D_1^*)), (h^{-1}(C_2^*), h^{-1}(D_2^*)), \dots\}$. Since X is S-w.i.d. modulo compacta, there exists an integer N and partitions L_i between $h^{-1}(C_i^*)$ and $h^{-1}(D_i^*)$, for $i = 1, \dots, N$, such that the set $\bigcap_{i=1}^N L_i$ is compact. Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_N, \eta_1, \dots, \eta_N\}$ and let m be a natural number such that $1/m < \epsilon$. There exists exactly one $U \in \mathcal{U}_m$ such that $h \in U$. Observe that $f_U^{-1}(C_{j(i)}) \subset h^{-1}(C_1^*)$ and $f_U^{-1}(D_{j(i)}) \subset h^{-1}(D_i^*)$. Indeed, if $x \in f_U^{-1}(C_{j(i)})$, then since $d(h, f_U) < 1/m < \varepsilon$ we have $\rho(h(x), f_U(x)) < \varepsilon$ and thus $h(x) \in B(C_{j(i)}, \varepsilon) \subset B(C_{j(i)}, \eta_i) \subset C_i^*$ i.e. $x \in h^{-1}(C_i^*)$.

Thus, for $i = 1, \dots, N$, the set L_i is a partition between the sets $f_U^{-1}(C_{j(i)})$ and $f_U^{-1}(D_{j(i)})$ and the set $\bigcap_{i=1}^N L_i$ is compact. Hence the family

$$\mathcal{A} = \{(f_U^{-1}(C_{j(1)}), f_U^{-1}(D_{j(1)})), \dots, (f_U^{-1}(C_{j(N)}), f_U^{-1}(D_{j(N)}))\} \in \mathcal{K}(U).$$

Since $h \in \mathscr{H}(U)$, then $h \in \mathscr{H}_{\mathscr{A}}$ and thus there exist partitions K_i between $\overline{h\left(f_U^{-1}\left(C_{j(i)}\right)\right)}$ and $\overline{h\left(f_U^{-1}\left(D_{j(i)}\right)\right)}$ such that $\bigcap_{i=1}^{N} K_i \setminus h(X) = \varnothing$. Observe that $A_i \subset hf_U^{-1}(C_{j(i)})$ and $B_i \subset hf_U^{-1}(D_{j(i)})$. Indeed, we shall show that $h^{-1}(A_i) \subset f_U^{-1}(C_{j(i)})$ and $h^{-1}(B_i) \subset f_U^{-1}(D_{j(i)})$. Let $a \in h^{-1}(A_i)$; then $\rho(f_U(a), h(a)) < \varepsilon \leq \varepsilon_i$ and thus $f_U(a) \in B(A_i, \varepsilon_i) \subset C_{j(i)}$, i.e. $a \in f_U^{-1}(C_{j(i)})$. It follows that $K_i' = K_i \cap (\overline{h(X)} \setminus h(X))$ is a partition between A_i and B_i in $\overline{h(X)} \setminus h(X)$ for $i = 1, \ldots, N$ and $\bigcap_{i=1}^{N} K_i' = \varnothing$, which ends the proof.

6. Examples and remarks.

EXAMPLE 6.1. A space X satisfying the condition (K) which is not S-weakly infinite-dimensional modulo compacta.

Let $Z = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ be a subspace of the real line and let $E_n = \{\{x_j\} \in I^\omega\colon -1 < x_j < 1$ for $j = 1, 2, \ldots, n$ and $x_j = 0$ for $j = n+1, n+2, \ldots,\}$, for $n = 1, 2, \ldots$. Let

$$X = (I^\omega \times \{0\}) \cup \bigcup_{n=1}^{\infty} \left(E_n \times \left\{\frac{1}{n}\right\}\right)$$

be a subspace of the Cartesian product $I^\omega \times Z$. Put $A_i = \{(\{x_j\}, 0) \in I^\omega \times \{0\}\colon x_i = -1\}$ and $B_i = \{(\{x_j\}, 0) \in I^\omega \times \{0\}\colon x_i = 1\}$; A_i and B_i are compact disjoint subsets of the space X. Let L_i be an arbitrary partition between A_i and B_i. We shall show that for every integer N the set $\bigcap_{i=1}^{N} L_i$ is not compact (this implies that X is not S-w.i.d. modulo compacta; some other properties of X follow from Remark 6.3 below— see Corollary below). Let $I_n = \{\{x_j\} \in I^\omega\colon -1 \leq x_j \leq 1$ for $j = 1, \ldots, n$ and $x_j = 0$ for $j = n+1, n+2, \ldots\}$. Put

$$X' = (I^\omega \times \{0\}) \cup \bigcup_{n=1}^{\infty} \left(I_n \times \left\{\frac{1}{n}\right\}\right) \subset I^\omega \times Z.$$

For each i take a partition L_i' between A_i and B_i in X' such that $L_i' \cap X \subset L_i$. For every i there exists integer $n_i \geq 1$ such that for any $n \geq n_i$ the set $L_i' \cap (I_n \times \{1/n\})$ is a partition between the sets $\{(\{x_j\}, 1/n)\colon \{x_j\} \in I_n, x_i = -1\}$ and $\{(\{x_j\}, 1/n)\colon \{x_j\} \in I_n, x_i = 1\}$ in the space $I_n \times \{1/n\}$. Let N be an arbitrary integer and let $n \geq \max\{n_1, \ldots, n_N, N+1\}$. Then for $i = 1, \ldots, N$ the set $L_i' \cap (I_{N+1} \times \{1/n\})$ is a partition between the opposite faces $\{(\{x_j\}, 1/n) \in I_{N+1} \times \{1/n\}\colon x_i = -1\}$ and $\{(\{x_j\}, 1/n) \in I_{N+1} \times \{1/n\}\colon x_i = 1\}$ of the cube $I_{N+1} \times \{1/n\}$. Hence the set $\bigcap_{i=1}^{N} L_i' \cap (I_{N+1} \times \{1/n\})$ contains a continuum C meeting the faces $\{(\{x_j\}, 1/n) \in I_{N+1} \times \{1/n\}\colon x_{N+1} = -1\}$ and $\{(\{x_j\}, 1/n) \in I_{N+1} \times \{1/n\}\colon x_{N+1} = 1\}$ (see [12], Lemma 5.2). For

$m = 1, 2\ldots$ take

$$a_m \in \bigcap_{i=1}^{N} L_i' \cap \left\{\left(\{x_j\}, \frac{1}{n}\right) \in I_{N+1} \times \left\{\frac{1}{n}\right\}: x_{N+1} = 1 - \frac{1}{m}\right\}.$$

Then the sequence $\{a_m\}_{m=1}^{\infty}$ is contained in $\bigcap_{i=1}^{N} L_i$ and has no accumulation point in X. Thus the set $\bigcap_{i=1}^{N} L_i$ is not compact.

EXAMPLE 6.2. A space X such that X has $\operatorname{tr}\operatorname{Ind}$ and yet X has no compactification with S-weakly infinite-dimensional remainder.

Before we proceed to the construction let us recall that the deficiency of a space X, $\operatorname{def} X$, is defined as $\min\{\dim(\tilde{X} \setminus X)$: $\tilde{X}$ is a compactification of $X\}$ (see [**5**]).

Let $X = \oplus_{n=1}^{\infty} X_n \cup \{p\}$, where X_n is a separable space such that $\operatorname{def} X_n \geq n$ and the base of neighbourhoods of p consists of the sets of the form $\bigcup_{n=k}^{\infty} X_n$, where $k = 1, 2, \ldots$. For example, we can take $X_n = I^{n+1} \setminus \{(x_1, \ldots, x_n)$: $x_{n+1} = 1$ and $-1 < x_i < 1$ for $i = 1, \ldots, n\}$ (see [**6**], Ch. VI, Research problem D); notice that inequality $\operatorname{def} X_n \geq n$ follows from the fact that if L_i is a partition between the compact sets $A_i = \{(x_i, \ldots, x_{n+1}) \in I^{n+1}$: $x_i = -1\}$ and $B_i = \{(x_1, \ldots, x_{n+1}) \in I^{n+1}$: $x_i = 1\}$ in X_n, for $i = 1, \ldots, n$, then the set $\bigcap_{i=1}^{n} L_i$ is not compact (compare Remark 6.3 below). Suppose that the space X has a compactification $\tilde{X}$ such that the remainder $\tilde{X} \setminus X$ is S-weakly infinite-dimensional. Then $\tilde{X} \setminus X$ contains a compact set K such that $\dim(\tilde{X} \setminus X \setminus W) < \infty$ for every open set W in $\tilde{X} \setminus X$ containing K. Let U and V be neighbourhoods of the set K and the point x respectively in the space $\tilde{X}$ such that $\overline{U} \cap \overline{V} = \varnothing$. Then the set $(\tilde{X} \setminus X) \setminus U$ has a finite dimension, hence the set $V \subset X \cup (\tilde{X} \setminus X) \setminus U$ has $\operatorname{def} V < \infty$. This is a contradiction, since V contains almost all X_n's and $\operatorname{def} X_n \geq n$.

REMARK 6.3. In the sequel by a compactness degree of a space X ($\operatorname{cmp} X$) we understand the transfinite invariant defined as follows: $\operatorname{cmp} X = -1$ if X is compact and $\operatorname{cmp} X \leq \alpha$ if X has a base $\mathscr{B}$ of open sets such that $\operatorname{cmp} \operatorname{Fr} U < \alpha$ for every $U \in \mathscr{B}$ (see [**5**] and [**14**]).

PROPOSITION. *Suppose that a space X satisfies one of the following conditions*:

(i) *there is an embedding h: $X \to I^{\omega}$ such that the set $\overline{h(X)} \setminus h(X)$ is μ-S-weakly infinite-dimensional,*

(ii) *the transfinite compactness degree* $\operatorname{cmp} X$ *is defined.*

Then

(5) *for every sequence* $\{(A_1, B_1), (A_2, B_2), \dots\}$ *of pairs of disjoint compact subsets of* X *there exist partitions* L_i *between* A_i *and* B_i *such that for some integer* N *the set* $\bigcap_{i=1}^{N} L_i$ *is compact.*

Proof. In the case when X satisfies (i) the proof is similar to the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 5.4. Now suppose that $\operatorname{cmp} X$ is defined. We apply the transfinite induction with respect to $\alpha = \operatorname{cmp} X$. If $\alpha = -1$, then X is compact and the condition (5) is obviously satisfied. Now assume that the proposition is proved if $\operatorname{cmp} X < \alpha$ and let X be a space such that $\operatorname{cmp} X \leq \alpha$. Let $\{(A_1, B_1), (A_2, B_2), \dots\}$ be a sequence of pairs of disjoint compact subsets of X. Since A_i is compact and $\operatorname{cmp} X \leq \alpha$, there exist open subsets $U_1, \dots, U_k$ of X such that $A_1 \subset \bigcup_{i=1}^{k} U_i$ and for each $i = 1, \dots, k$ we have $\overline{U_i} \cap B_1 = \emptyset$ and $\operatorname{cmp} \operatorname{Fr} U_i \leq \alpha'$ for some $\alpha' < \alpha$. By the inductive assumption each $\operatorname{Fr} U_i$ satisfies (5), hence the set $\bigcup_{i=1}^{k} \operatorname{Fr} U_i$ satisfies (5) since the property (5) is finitely additive. It is easy to see that the set $L_1 = \operatorname{Fr} U \subset \bigcup_{i=1}^{k} \operatorname{Fr} U_i$ is a partition between A_1 and B_1 and satisfies (5). For $i \geq 2$ there exists a partition L_i' between $A_i \cap L_1$ and $B_i \cap L_1$ in L_1 such that the set $\bigcap_{i=2}^{N} L_i'$ is compact for some integer N. Now if we take partitions L_i between A_i and B_i in X such that $L_i \cap L_1 \subset L_i'$, then the set $\bigcap_{i=1}^{N} L_i$ is compact.

COROLLARY. *For a space* X *defined in Example* 6.1 *the transfinite compactness degree is not defined and there is no embedding* $h\colon X \to I^{\omega}$ *such that the set* $\overline{h(X)} \setminus h(X)$ *is* μ*-S-w.i.d. (since* X *does not satisfy* (5)).

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