PACIFIC JOURNAL OF MATHEMATICS
Vol. 122, No. 1, 1986

SOLVABILITY OF VARIOUS BOUNDARY VALUE PROBLEMS FOR THE EQUATION $x'' = f(t, x, x', x'') - y$

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In this paper some of the solvability results of Granas, Guenther and Lee for various homogeneous boundary value problems for the equation $x'' = f(t, x, x')$ are extended in an essentially constructive way to the equation (*): $x'' = f(t, x, x', x'')$ where f is assumed to satisfy the growth condition:

$$|f(t,x,r,q)| \leq A(t,x)r^2 + B|q| + C(t,x)$$

for r, q in R with A and C bounded functions on each compact subset of $[0,T] \times R$ and B in $(0,1)$ and some further conditions stated below. Our proofs are based on the author's continuation theorem for semilinear A-proper maps and the approach used by Granas, Guenther and Lee in obtaining the a priori bounds for the solutions of equation (*).

Introduction. In this paper we establish the existence (in some cases constructive) of a classical solution x in $C^2([0,T])$ for certain y in $C([0,T])$ for the second order ODE's of the form

$$\text{(A)} \qquad x''(t) = f(t,x,x',x'') - y(t), \qquad 0 \leq t \leq T,$$

subject to either Dirichlet, Neumann, periodic, Sturm-Liouville, or anti-periodic boundary conditions, where the continuous nonlinearity f: $[0,T] \times R^3 \to R$ is required to satisfy a new and a rather general growth condition of the form

$$\text{(B)} \quad |f(t,x,r,q)| \leq A(t,x)|r|^2 + B|q| + C(t,x) \quad \text{for } r, q \in R,$$

where A, C are functions bounded on each compact subset of $[0,T] \times R$ and B is a constant in $[0,1)$ and f is also such that $L - \lambda N$: $X_J \to C([0,T])$ is A-proper for each $\lambda \in (0,1]$, where $Lx = x''$ and $Nx = f(t,x,x',x'')$ for $x \in X_J$, with X_J a closed subspace of $C^2([0,T])$ defined below by various boundary conditions. In §3 we discuss various verifiable conditions on f which ensure the fulfillment of the latter condition on $L - \lambda N$ in many situations. The boundary conditions referred to above are, respectively,

(I) $x(0)$, $x(T) = 0$;
(II) $x'(0) = 0$, $x'(T) = 0$;
(III) $x(0) = x(T)$, $x'(0) = x'(T)$;

(IV) $-\alpha x(0) + \beta x'(0) = 0$, $ax(T) + bx'(T) = 0$,
where α, β, a, $b \geq 0$, $\alpha^2 + \beta^2 > 0$, $a^2 + b^2 > 0$, and $\alpha^2 + a^2 > 0$.

(V) $x(0) = -x(T)$, $x'(0) = -x'(T)$.

We say that a function x in $C^2([0, T])$ satisfies the problem (I) if x satisfies the differential equation (A) and the Dirichlet boundary conditions (I). Similar notation is used for the other problems.

The type of growth condition on f imposed in (B) is motivated by a recent paper of Granas, Guenther and Lee **[8]** and the earlier paepr by S. Berstein **[1]** who has shown in **[1]** that the Dirichlet problem for the equation

(C) $$x'' = f(t, x, x'), \qquad 0 \leq t \leq 1,$$

has a unique solution in $C^2([0,1])$ provided the continuous function $f(t, x, r)$ has continuous partial derivatives f_x and f_r, and satisfies:

(D) $$f_x \geq k > 0 \quad \textit{for some constant } k \in R^+\backslash\{0\}.$$

(E) $$|f(t, x, r)| \leq A(t, x)r^2 + C(t, x) \quad \textit{for } r \in R \textit{ with } A, C \textit{ as in } (\mathrm{B}).$$

Using the topological transversality theorem of Granas **[9]**, the authors of **[8]** presented a generalization of the Berstein theorem (which permits the treatment of equations such as $x'' = x^3 + x'^2 + 1$ to which the original result does not apply) and they have also shown that Berstein's result for Eq. (C) can be extended to other boundary conditions stated above. Thus, for example, it was shown in Theorems 4.1 and 6.1 in **[8]** that if (E) holds and if additionally

(F) *There is a constant* $M > 0$ *such that* $xf(t, x, 0) > 0$ *for* $|x| > M$,

then Eq. (C) has at least one solution x in $C^2([0,1])$ which satisfies one of the boundary conditions stated above. The uniqueness of a solution to Eq. (C), subject to various boundary conditions, was also treated in **[8]** under some additional conditions on $f(t, x, r)$. In particular, it was shown there that if, in addition to (E) and (F), f_x and f_r exist, are bounded, and $f_x \geq 0$, then the Dirichlet problem for Eq. (C) has a unique solution in $C^2([0,1])$. Let us add that the approach used in **[8]** to obtain the a priori estimates on the solutions x to Eq. (C) is not applicable if the inequality in (F) is reversed.

It should be noted that the problem dealing with the solvability of Eq. (C) subject to various boundary conditions of the form (I) to (V) has been studied earlier by many authors under various conditions on $f(t, x, r)$ and using different methods. For numerous references dealing with these problems see, for example, the books **[25]**, [7], **[23]**, and the recent articles in **[4]**, **[10]** and others.

The purpose of this paper is to present an extension of a number of existence and/or uniqueness results obtained in **[8]** for Eq. (C) to the more general equations (A) (which permits the treatment of equations such as $x'' = x^3 + x'^2 + 1 + \alpha \sin x'' - y$ for any fixed $\alpha \in (0,1)$ to which the reuslts in **[8]** do not apply) and to show that in some cases the solution $x \in C^2([0,T])$ to Eq. (A) can be obtained as a strong C^2-limit of the finite dimensional Galerkin type computable approximations $x_n \in \tilde{X}_n \subset X_J = \{x \in C^2([0,T]) \colon J = \text{I, II, III, IV, or V}\}$. The latter constructive results are *new* even when f is independent of x'' and the conditions on $f(t,x,r)$ ensuring the uniqueness are those imposed in **[8]** when treating Eq. (C). In our study of problems (I) to (V) (associated with Eq. (A)) we use the abstract continuation type theorem of the author **[14**, Theorem 1.1] for semilinear equations involving A-proper mappings and the approach of **[8]** for obtaining the a priori bounds for solutions x to problems (I) to (V). For the purposes of comparison, we state as an example the following extension of the existence Theorems 4.1 and 6.1 in **[8]** (see Theorem 2.1 below).

Assume that the continuous function $f(t,x,r,q)$ in Eq. (A) is such that for maps L, N: $X_J \to Y \equiv C([0,T])$, defined by $Lx = x''$ and $Nx = f(t,x,x',x'')$ for $t \in [0,T]$ and $x \in X_J$, the following conditions hold:

(i) $L - \lambda N$: $X_J \to Y$ *is A-proper for each* $\lambda \in (0,1]$ *with respect to a suitable scheme* $\Gamma_L = \{\tilde{X}_n, Y_n, Q_n\}$ *(for details see* §2).

(ii) *There are* $M > 0$ *and* $a, b \in R$ *such that* $b \leq y_m \leq y_M \leq a$ *and* $x \geq M \Rightarrow f(t,x,0,q) > a$ *if* $t \in [0,T]$ *and* $q \in R$, *while* $x \leq -M \Rightarrow f(t,x,0,q) < b$ *if* $t \in [0,T]$ *and* $q \in R$, *where* $y_m = \min y(t)$ *and* $y_M = \max y(t)$ *for* $t \in [0,T]$.

(iii) $f(t,x,r,q)$ *satisfies the growth condition* (B).

Then each of the problems (I) to (V) has at least one solution $x \in X_J$. If x is unique for a given y in Y, then x is the strong C^2-limit of the Galerkin approximates $x_n \in \tilde{X}_n$.

Some sufficient conditions on $f(t,x,r,q)$ which ensure the uniqueness are given in §§2 and 3. We note in passing that when f in Eq. (A) is independent of x'', then the growth conditions (B) and (E) are the same and condition (i) always holds since X_J is compactly imbedded into $C^1([0,T])$. Thus, in this case our Theorem 2.1 includes the existence results of **[8]** when $a = b = 0$.

Let us add, for the purposes of comparison, that when the growth of $f(t,x,r,q)$ is linear, i.e, for $t \in [0,1]$ and $(x,r,q) \in R^3$

$$(*) \qquad |f(t,x,r,q)| \leq a + b|x| + c|r| + d|q|$$

with b, c, d sufficiently small, the solvability of the problems (II) and (III) have been studied in [**15**, **16**] under condition (i) and the assumptions:

(G) *There is* $M > 0$ *(depending on* y*) such that* $\int_0^1 \{f(t,x,x',x'') - y\}\,dt \neq 0$ *for* $x \in X_J$ *with* $\|x(t)\| \geq M$ *for* $t \in [0,1]$.

(H) *There are* $M_1 \geq M$ *and* a, $b \in R$ *such that either* (j): $a \geq b$ *and* $c \geq M_1 \Rightarrow f(t,c,0,0) \geq a$ *if* $t \in [0,1]$, *while* $c \leq -M_1 \Rightarrow f(t,c,0,0) \leq b$ *if* $t \in [0,1]$, *and* $b \leq \int_0^T y\,dt \leq a$; *(or* (jj): $a \leq b$ *and* $c \geq M_1 \Rightarrow f(t,c,0,0) \leq a$ *if* $t \in [0,1]$, *while* $c \leq -M_1 \Rightarrow f(t,c,0,0) \geq b$ *if* $t \in [0,T]$, *and* $a \leq \int_0^1 y\,dt \leq b$*)*.

Note that condition (j) of (H) is weaker than (jj) and that it is of the same type so far as the one-sided inequality is concerned at least when f is independent of x''. However, condition (jj) of (H), which is opposite in sign to that of (j), has no comparable analogue when f has a quadratic growth in r (as in (B)) and the approach of [**8**] is used to get the a priori bounds even when f is independent of x''. The above remark is particularly transparent when $a = b = 0$. We add that the method used in [**15**, **16**] required the additional condition (G) and depended on the linear growth of f. For the earlier results for problems (I) and (III) see [**5**] when f has a linear growth and [**6**, **17**] when f has a semilinear growth and some further conditions are assumed. The sublinearity of f was essential in the method used in [**6**, **17**].

Our discussion proceeds as follows. In §1, we state some relevant definitions and the abstract (essentially constructive) continuation theorem of the author [**14**, Theorem 1.1] concerning the solvability of semilinear equations (at resonance) involving A-proper mappings. This result and the approach used in [**8**] for obtaining a priori bounds are then used in §2 to establish the existence (sometimes constructive) of solutions x in $C^2([0,T])$ for certain given y in $C([0,T])$ for the problems (I) to (V). In §3 we discuss some special cases of the boundary value problems studied in §2. In particular, we study the solvability of the generalized Lienard equation

(J) $$x'' = g(x)x' + h(t,x,x',x'') - y(t), \qquad 0 \leq t \leq T,$$

under various boundary conditions. Some of our results obtained in this paper for the periodic boundary value problem for equation (J) are related to some of the results for this problem obtained in [**15**].

It should be added that Eq. (J) appears in various parts of mechanics, mathematical physics and other fields. The case when $J \equiv$ III and h is independent of x'', and especially when h is independent of x'', and x', has been studied by many authors. For the earlier literature and results see [**25**] and for the more recent ones where functional analytic methods are used to study these problems see [7, **23**], [4, **10**], [**11**, **12**] and others listed there. Other examples which appear in mechanics and other fields will also be considered. In particular, we treat the periodic BV Problem

$$\text{(K)} \quad \begin{cases} x'' = a(t)x^3 - b(t)x - c(t)x' - |x'|x' - y(t), & 0 \le t \le T, \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

which apears in the study of the dynamics of wires and which for some special functions $a(t)$ and $b(t)$ has been studied by Stoppelli [**26**] and others (see [3, **24**, **18**]). In this paper we extend the results of [**26**, **3**] and unlike the results in [**26** ,**3**, **24**, **18**] our results is constructive. We also show how the result in [**6**] (see also [**17**]) for the problem

$$\text{(L)} \quad \begin{cases} x'' = f_1(t, x, x') + g(t, x', x'') - y(t), & 0 \le t \le 1, \\ x(0) = x(1), \quad x'(0) = x'(1) \end{cases}$$

can be improved.

1. In this section we introduce the relevant definitions and state an abstract result of [**14**, Theorem 1.1] which we shall use and which deals with the solvability of the semilinear equations

$$(1.1) \qquad Lx - Nx = y, \qquad (x \in X, y \in Y),$$

where X, Y are real Banach spaces, L: $X \to Y$ is Fredholm of index 0 (i.e. $L \in \Phi_0(X, Y)$) with the null space $N(L)$ not necessarily $\{0\}$ and N: $X \to Y$ is a nonlinear map such that $L - \lambda N$: $X \to Y$ is A-proper for each $\lambda \in [0, 1]$ with respect to some admissible scheme Γ, which for the sake of simplicity we take to be a projective scheme.

We recall that if $\{X_n\} \subset X$ and $\{Y_n\} \subset Y$ are sequences of finite dimensional oriented spaces and Q_n: $Y \to Y_n$ is a linear projection for each $n \in Z^+$, then the scheme $\Gamma = \{X_n, Y_n, Q_n\}$ is said to be *admissible* for maps from X to Y provided that $\dim X_n = \dim Y_n$ for each n, $\operatorname{dist}(x, X_n) \equiv \inf\{\|x - v\|_X\colon v \in X_n\} \to 0$ as $n \to \infty$ for each x in X, and $Q_n y \to y$ for each y in Y, where "$\to$" denotes the strong convergence.

Now, for the convenience of the reader, we recall that for a given map $T\colon D(T) \subset X \to Y$ the equation

(1.2) $$Tx = y \qquad (x \in D(T), y \in Y),$$

is said to be *strongly* (resp. *feebly*) *approximation-solvable* (a-solvable, for short) w.r.t Γ if there exists $N_y \in Z^+$ such that the finite dimensional equation

(1.3) $$T_n(x) = Q_n y, \big(x \in D_n \equiv D \cap X_n, Q_n y \in Y_n,\ T_n = Q_n T|_{D_n}\big),$$

has a solution $x_n \in D_n$ for each $n \geq N_y$ such that $x_n \to x \in D$ in X (resp. $x_{n_j} \to x \in D$) and $Tx = y$. It was shown by the author in [**19**] that for (1.2) to be a-solvable w.r.t a given scheme Γ the operator $T\colon D \subset X \to Y$ has essentially to be *Approximation-proper* (A-proper, for short) w.r.t. Γ, where the latter notion was defined in [**19**] (see also [**20**] for historical development of the theory) by:

Definition 1.1. A map $T\colon D \subset X \to Y$ is said to be *A-proper* w.r.t Γ if $T_n\colon D_n \subset X_n \to Y_n$ is continuous for each $n \in Z^+$ and if $\{x_{n_j} | x_{n_j} \in D_{n_j}\}$ is any bounded sequence in X such that $T_{n_j}(x_{n_j}) \to g$ for some g in Y, then there is a subsequence $\{x_{n_k}\}$ of $\{x_{n_j}\}$ and $x \in D$ such that $x_{n_k} \to x$ in X and $Tx = g$.

It is by now well known that the class of A-proper maps is quite general and useful. It includes compact, ball-condensing and P_γ-compact vector fields, maps of type (S) and (Ks) as well as operators of strongly monotone and accretive type and their perturbations by compact and even by k-ball-condensing maps for small $k > 0$. Thus, the study of A-proper maps provides a unified approach to the study of these special classes of maps. It should be noted that the theory of A-proper maps is applicable to certain differential equations to which no other abstract theory applies. Moreover, the A-properness is intimately connected with the constructive solvability of abstract and differential equations.

For subsequent use we shall state here the following special case of [**14**, Theorem 1.1] whose proof is essentially based on the properties of the generalized degree for A-proper mapings developed by Browder and Petryshyn [**2**]. To state this result, which is used here to establish the a-solvability of problems (I) to (V) (and, in particular, the existence of solutions in X_J), we first note that, since $L \in \Phi_0(X, Y)$, there exist closed subspaces $X_1 \subset X$ and $Y_2 \subset Y$ such that $X = N(L) \oplus X_1$, $Y = Y_2 \oplus R(L)$, and $\dim N(L) = \dim Y_2$. In what follows, we let Q be the linear projection of Y onto Y_2 and assume that there exists a continuous bilinear form $[\cdot, \cdot]$ on $Y \times X$ mapping (y, x) into $[y, x]$ such that

(1.4) $$y \in R(L) \text{ if and only if } [y, x] = 0 \quad \forall x \in N(L).$$

THEOREM A [14]. *Let $L \in \Phi_0(X, Y)$ and suppose there exists a bounded open set $G \subset X$ with $0 \in G$ such that*

(a) *$L - \lambda N$: $\overline{G} \to Y$ is A-proper w.r.t. Γ for each $\lambda \in [0, 1]$ with $N(\overline{G})$ bounded,*

(b) *$Lx \neq \lambda Nx - \lambda y$ for $x \in \partial G$ and $\lambda \in (0, 1]$.*

(c) *$QNx - Qy \neq 0$ $\forall x \in N(L) \cap \partial G$.*

(d) *Either* (d1): $[QNx - Qy, x] \geq 0$ (*or* (d2): $[QNx - Qy, x] \leq 0$) $\forall x \in N(L) \cap \partial G$.

Then Eq. (1.1) *is feebly a-solvable w.r.t. Γ and, in particular,* (1.1) *has a solution $x \in G$. If x is the unique solution in G, then* (1.1) *is strongly a-solvable* (*i.e., the Galerkin method applies to* (1.1)).

REMARK 1.1. In Theorem 2.1 it is assumed that L is A-proper. However, it was shown by the author in [17] that if $L \in \Phi_0(X, Y)$ and in [21] when L is unbounded, then one can always construct an admissible scheme Γ_L (depending on L) such that L is A-proper w.r.t. Γ_L. Indeed, if $L \in \Phi_0(X, Y)$ then one can construct a compact map C: $X \to Y_2$ such that $K = L - C$ is a homeomorphism of X onto Y and choosing $\{\tilde{X}_n\} \subset X$ such that $Y_n = K(X_n)$ for $n \in Z^+$ one shows that $\Gamma_L \equiv \{\tilde{X}_n, Y_n, Q_n\}$ is admissible and L is A-proper w.r.t. Γ_L. Thus it is not necessary to assume that L is A-proper if we choose the scheme Γ more judiciously; consequently, it suffices to assume in (a) that $L - \lambda N$ is only A-proper w.r.t. Γ_L for each $\lambda \in (0, 1]$. It was shown in [21] that the latter is the case if, for example, N is k-ball constructive with $k > 0$ sufficiently small and, in particular, where N is compact (i.e., 0-ball-contractive).

REMARK 1.2. If $N(L) = \{0\}$, then L: $X \to Y$ is a linear homeomorphism and in this case (b) holds for all $\lambda \in [0, 1]$, $Q = 0$ (with (c) and (d) holding vacuously) and the conclusions of Theorem A follow from (b) and the properties of the generalized degree for A-proper mappings. It is useful to note that in this case the approximate equation (1.3) with $T = L - N$ reduces to the equation

$$Lx - Q_n Nx = Q_n y \qquad (x \in X_n, Q_n y \in Y_n), \tag{1.5}$$

which is particularly convenient in actual applications of the Galerkin type method to Eq. (1.1).

2. In this section we use Theorem A to establish the a-solvability and, in particular, the existence of solutions in $C^2([0, T])$ to the BV Problems (I) to (V) for some $y \in C([0, T])$ under suitable conditions on the nonlinearity $f(t, x, r, q)$. Some concrete examples of the above boundary value problems will be given in §3.

The following four lemmas whose proofs follow the approach of [**8**] will be used to establish the a priori bounds in the C^2-norm for the solutions to problems (I) to (V) (i.e., under suitable conditions on $f(t,x,r,q)$ we establish the existence of the set $G = B(0,r)$ in X_J used in Theorem A for which conditions (a) to (d) hold). For the convenience of the reader we give the detailed proofs of these lemmas here to expand on the approach used in [**8**] and also because our problems and conditions are more general than those studied in [**8**].

LEMMA 2.1. *Let* $f\colon [0,T]\times R^3 \to R$ *be continuous and such that*

(a) *there exist constant* $M > 0$ *and* a, $b \in R$ *such that condition* (ii) *holds* (*i.e.,* $b \le y_m \le y_M \le a$ *and* $x \ge M \Rightarrow f(t,x,0,q) > a$ *if* $t \in [0,T]$ *and* $q \in R$, *while* $x \le -M \Rightarrow f(t,x,0,q) < b$ *if* $t \in [0,T]$ *and* $q \in R$).

If $x(t)$ *is a solution in* $C^2([0,T])$ *of the differential equation* (A), *i.e, if*

$$x''(t) = f(t,x,x',x'') - y(t), \qquad 0 \le t \le T, \tag{2.1}$$

and $|x(t)|$ *does not achieve its maximum at* $t = 0$ *or* $t = T$, *then*

$$|x(t)| \le M \quad \text{for } t \in [0,T].$$

Proof. Under the conditions of Lemma 2.1, $|x|$ must achieve a positive maximum at some point $t_0 \in (0,T)$. We claim that $|x(t_0)| \le M$. If not, then $|x(t_0)| > M$. Assume first that $x(t_0) > M$. Then $x'(t_0) = 0$ and $x''(t_0) < 0$ by the second derivative test and, therefore, since x is a solution of (2.1), the condition (a) implies that

$$0 > x''(t_0) = f\big(t_0, x(t_0), 0, x''(t_0)\big) - y(t_0) > a - y_M.$$

Hence $y_M > a$, in contradiction to the last assumption in (a). If $x(t_0) < -M$, then $x'(t_0) = 0$ and $x''(t_0) > 0$ by the same test and so (a) implies that

$$0 < x''(t_0) = f\big(t_0, x(t_0), 0, x''(t_0)\big) - y(t_0) < b - y_m,$$

i.e., $y_m < b$, in contradiction to the last assumption in (a). Thus, $|x(t_0)| \le M$ and Lemma 2.1. is proved.

LEMMA 2.2. *Suppose that condition* (a) *of Lemma* 2.1 *holds. Then any solution* x *to problems* (I) *to* (V) *satisfies*

$$|x(t)| \le M \quad \text{for all } t \in [0,T].$$

Proof. For problem (I) and problems (IV) for the case when either (IVa) $(\alpha > 0, \beta > 0, a > 0, b > 0)$, (IVb) $(\beta = 0, a > 0, b > 0)$ or (IVc) $(b = 0, \alpha > 0, \beta > 0)$ the boundary conditions force nonzero extreme values to be assumed in $(0, T)$. Thus, in each of these problems, the a priori bounds on x asserted in Lemma 2.2 follow from Lemma 2.1.

Let x be a solution to either problem (II), (III), (V) or (IV) for the other cases when either (IVd) $(\beta = 0, a = 0)$, (IVe) $(b = 0, \alpha = 0)$, (IVf) $(\alpha = 0, a > 0, b \geq 0)$ or (IVg) $(a = 0, \alpha > 0, \beta \geq 0)$. We claim:

(i) If $|x|$ assumes its maximum at $t_0 = 0$ or $t_0 = T$, then $|x(t_0)| \leq M$.

Clearly, Lemma 2.1 and (i) imply that $|x(t)| \leq M$ for $t \in [0, T]$. Thus, it remains to prove (i) for problems (II), (III), (V) and the other cases of problem (IV) listed above.

Let first x be a solution of problem (II) for which $|x(0)|$ is the maximum value of $|x(t)|$ on $[0, T]$. We shall show that $|x(0)| > M$ leads to contradiction. Indeed, since $x'(0) = 0$ and assuming that the solution x to problem (II) is such that $x(0) > M$, we see that condition (a) implies the inequality

$$x''(0) = f\big(0, x(0), 0, x''(0)\big) - y(0) > a - y_M \geq 0.$$

Hence $x''(0) > 0$ and so $x'(t)$ is strictly increasing for $t > 0$ near 0. Thus $x'(t) > x'(0) = 0$ for $t > 0$ near 0, and so $x(t)$ is strictly increasing near 0, and $x(0) = |x(0)|$ is not the maximum on $[0, T]$, a contradiction. If, on the other hand $x(0) < -M$, then the fact that x is a solution of (II) and (a) imply that

$$x''(0) = f\big(0, x(0), 0, x''(0)\big) - y(0) < b - y_m \leq 0.$$

Hence $x''(0) < 0$ and so $x'(t)$ is strictly decreasing for $t > 0$ near 0. Thus $x'(t) < x'(0) = 0$ for $t > 0$ near 0, and so $x(t)$ is strictly decreasing for $t > 0$ near 0 and $x(0) < -M$. Hence $|x(0)|$ cannot be the maximum on x on $[0, T]$. The corresponding assertion for $x(T)$ is proved similarly. This proves (i) for problem (II).

Suppose next that x is a solution of problem (III). If $x'(0) = x'(T) \neq 0$, then it follows from $x(0) = x(T)$ that x cannot achieve its maximum at $t_0 = 0$ or $t_0 = T$. Since, by (i), $|x(t)|$ is assumed to achieve its maximum for $t_0 = 0$ or $t_0 = T$, we must have $x'(0) = 0$. Then x satisfies problem (II) and thus $|x(0)| = |x(T)| \leq M$ by what has been proved above. This proves (i) for problem (III). As was noted in [**8**], for problems (V), (IVd), (IVe), (*V*f) and (IVg), the assertion (i) holds by essentially the same argument. Thus (i) and Lemma 2.1 yield a priori bounds on solutions for these problems. This proves Lemma 2.2.

A priori estimates for derivatives. Using a variant of the argument in **[8]**, the a priori bounds are now established for the first and second derivatives of solutions x in $C^2([0,T])$ to (2.1) assuming the boundedness of $|x(t)|$.

LEMMA 2.3. (i) *Suppose there exists a constant* $M > 0$ *such that* $|x(t)| \le M$ *for each* $t \in [0,T]$ *and for each* $x \in C^2([0,T])$ *which satisfies the differential equation* (2.1).

(ii) *Suppose there are constants* A, $C > 0$ *and* $B \in [0,1)$ *such that* $|f(t,x,r,q)| \le Ar^2 + B|q| + C$ *for* $(t,x) \in [0,T][-M,M]$ *and* r, $q \in R$.

Then there are constants M_1, $M_2 > 0$ *depending only on* M, A, B, C *and* y_M *such that*

$$|x'(t)| \le M_1, |x''(t)| \le M_2 \quad \forall t \in [0,T]$$

for each solution $x \in C^2([0,T])$ *of Eq.* (2.1) *whose derivative* $x'(t)$ *vanishes at least once in* $[0,T]$.

Proof. Since $x'(t)$ vanishes at least once in $[0,T]$, each point t in $[0,T]$ for which $x'(t) \neq 0$ belongs to an interval $[\mu,\gamma]$ such that $x'(t)$ maintains a fixed sign on $[\mu,\gamma]$ and $x(\mu)$ and/or $x(\gamma)$ is 0. To be definite, assume that $x'(\mu) = 0$ and $x'(t) \ge 0$ on $[\mu,\gamma]$. Then it follows that x satisfies the equation

$$x''(t) = f\big(t, x(t), x'(t), x''(t)\big) - (t), \qquad t \in [\mu,\gamma], \tag{2.2}$$

with $x'(t) \ge 0$ on $[\mu,\gamma]$. Hence, since $(t,x) \in [0,T] \times [-M,M]$, it follows from (2.2) and (ii) that

$$|x''(t)| \le Ax'^2 + B|x''| + C + d, \tag{2.3}$$

where $d = \max\{|y_m|, |y_M|\}$. Since $x'(t) \ge 0$ on $[\mu,\gamma]$, multiplying (2.3) by $x'(t)$ and rearranging the terms we get

$$(1 - B)|x''x'| \le (Ax'^2 + C + d)x'(t)$$

or

$$x''x' \le |x''x'| \le \big(\alpha x'^2 + \beta\big)x'(t) \qquad \left(\alpha \equiv \frac{A}{1-B}, \beta \equiv \frac{C+d}{1-B}\right).$$

Hence

$$\frac{2\alpha x''x'}{\alpha x'^2 + \beta} \le 2\alpha x',$$

and thus integrating the last inequality from μ to t and using the fact that $|x(t)| \le M$, we get

$$\ln\left(\frac{\alpha x'^2 + \beta}{\beta}\right) \le 4\alpha M.$$

The last inequality implies that

$$|x'(t)| \le \left[\frac{\beta}{\alpha}(e^{4\alpha M} - 1)\right]^{1/2} \equiv M_1. \tag{2.4}$$

If, on the other hand, $x'(\mu) = 0$ and $x'(t) \le 0$ on $[\mu, \gamma]$, then $-x'(t) = |x'(t)|$ on $[\mu, \gamma]$ and from (2.2) and (ii) we get

$$|x'x''| \le (Ax'^2 + C + d)|x'| + B|x''x''|$$

from which, as before, we obtain the bound M_1 on $x'(t)$.

It follows from the bounds on $|x|$ and $|x'|$ and from (2.1) and (ii) that

$$|x''(t)| \le \alpha M_1^2 + \beta \equiv M_2 \quad \text{for } t \in [0, T].$$

This completes the proof of Lemma 2.3.

Note that the boundary conditions for problems (I) to (V) imply that the derivative x' of each solution x to one of these problems must vanish at least once in $[0, T]$. Thus Lemmas 2.1 to 2.3 imply:

PROPOSITION 2.1. *Suppose there are constants $M > 0$ and a, $b \in R$ (depending on y) such that condition* (a) *of Lemma* 2.1 *holds. Suppose further that there are continuous functions $A(t, x)$, $C(t, x) > 0$, bounded on compact subsets of $[0, T] \times R$, and a constant $B \in [0, 1)$ such that for r, $q \in R$*

$$|f(t, x, r, q)| \le A(t, x)r^2 + B|q| + C(t, x).$$

Then there are constants M_1 and M_2 such that for any solution $x(t)$ to problems (I), (II), (III), (IV) *or* (V) *one has*

$$|x(t)| \le M, \quad |x'(t)| \le M_1, \quad |x''(t)| \le M_2 \qquad \textit{for } t \in [0, T].$$

Now the a-solvability and/or the existence proof below, which is based on Theorem A, requires the a priori bounds for the following family of problems:

$$\begin{cases} x''(t) = \lambda f(t, x, x', x'') - \lambda y(t), & 0 \le t \le T, \\ x \in B_J, \end{cases} \tag{2.5}$$

depending on the parameter λ, $0 < \lambda \le 1$. Here B_J denotes either the boundary condition $J = \mathrm{I}$, $J = \mathrm{II}$, $J = \mathrm{III}$, $J = \mathrm{IV}$, or $J = \mathrm{V}$ satisfied by a solution $x \in C^2([0, T])$ to the equation in (2.5).

LEMMA 2.4. (i) *Suppose there are constants* $M > 0$ *and* $a,\ b \in R$ *(depending on* y*) such that condition* (a) *of Lemma* 2.1 *holds.*

(ii) *Suppose further that there are constants* $A,\ C > 0$ *and* $B \in [0,1)$ *such that for all* $r,\ q$ *in* R:

$$|f(t,x,r,q)| \leq Ar^2 + B|q| + C \quad \text{for } (t,x) \in [0,T] \times [-M,M].$$

Then there are constants $M_0 = M,\ M_1$ *and* M_2 *independent of* λ *and* x *such that for* $\lambda \in (0,1]$ *and each solution* x_λ *to* (2.5) *we have*

$$|x_\lambda(t)| \leq M_0, \quad |x_\lambda'(t)| \leq M_1, \quad |x_\lambda''(t)| \leq M_2 \qquad \text{for } t \in [0,T].$$

Proof. The bounds for x_λ, x_λ' and x_λ'' satisfying (2.5) are established by using Lemmas 2.1 and 2.3 with λf replacing f and λy replacing y.

To state the main solvability result for problems (I) to (V), let $Y = C([0,T])$ be the Banach space of continuous functions x on $[0,T]$ with the norm $|x|_0 = \sup\{|x(t)|\colon 0 \leq t \leq T\}$ and let $C^k([0,T])$ be the Banach space of k-times continuously differentiable functions x on Y with the norm $|x|_k = \max\{|x^{(j)}|_0\colon 0 \leq j \leq k\}$. Let X_J be the closed subspace of $C^2([0,T])$ given by $X_J = \{x \in C^2([0,T])\colon x \in B_J\}$, and let $L\colon X_J \to Y$ be defined by $Lx = x''(t)$ for $t \in [0,T]$ and $x \in X_J$. It is known (see e.g. **[12]**) that L is Fredholm of $\operatorname{ind}(L) = 0$, $N(L) = \{x \in X_J\colon x(t) \equiv \text{constants}\}$, $R(L) = \{y \in Y\colon \int_0^T y\,dt = 0\}$, $X = N(L) \oplus X_1$ and $Y = N(L) \oplus R(L)$. It is not hard to show that for problems (I), (IV) and (V) one has $N(L) = \{0\}$, $R(L) = Y$ and L is a linear homeomorphism in these cases. In what follows we let $c \geq 0$ be a constant such that $K_c \equiv L - cI\colon X_J \to Y$ is a linear homeomorphism (in most cases, $c = 0$, $c = 1$ or $c > 0$ depending on the boundary conditions and on T), where I is the inclusion map of X_J into Y which is compact by the Arzela-Ascoli Theorem.

Let $\{Y_n, Q_n\}$ be a projectionally complete scheme for Y, and let $\{\tilde{X}_n\} \subset X_J$ be such that $Y_n = K_c(\tilde{X}_n)$ for each $n \in Z^+$. Then $Q_n y \to y$ for each y in Y, $\operatorname{dist}(x, \tilde{X}_n) = \inf\{|x - v|_2\colon v \in \tilde{X}_n\} \to 0$ as $n \to \infty$ for each $x \in X_J$, the scheme $\Gamma_L \equiv \{\tilde{X}_n, Y_n, Q_n\}$ is admissible for maps from X_J to Y, and $L\colon X_J \to Y$ is A-proper w.r.t. Γ_L (see **[17]**). It is easy to prove that the map $N\colon X_J \to Y$, defined by $(Nx)(t) = f(t,x,x',x'')$ for $t \in [0,T]$ and $x \in X_J$, is continuous and maps bounded sets in X_J into bounded sets in Y.

We are now in the position to use Theorem A and Lemmas 2.1 to 2.4, to prove the following solvability results for problems (I) to (V) which, as we shall show below, extend to Eq. (2.1) subject to boundary conditions

B_J the basic existence results of [8] for the equation (C) subject to the same boundary conditions B_J. The relation of our present results for problems (I) to (V) to some of the earlier results obtained by other authors will also be indicated.

THEOREM 2.1. *Let y be a given element in Y, let $f(t, x, r, q)$ be a continuous function on $[0, T] \times R^3$, and let L, N: $X_J \to Y$ be maps and $\Gamma_L = \{\tilde{X}_n, Y_n, Q_n\}$ the schemes as defined above such that*:

(i) *$L - \lambda N$: $X_J \to Y$ is A-proper w.r.t. Γ_L for each $\lambda \in (0,1]$.*

(ii) *There are constants $M > 0$ and a, $b \in R$ such that condition* (a) *of Lemma* 2.1 *holds.*

(iii) *There are a constant $B \in [0,1)$ and continuous functions A, C: $[0, T] \times [-M, M] \to R^+$ such that*

$$|f(t, x, r, q)| \le A(t, x) r^2 + B|q| + C(t, x) \quad \text{for } r, q \in R.$$

Then each of the problems (I) *to* (V) *is feebly a-solvable w.r.t. Γ_L, i.e., there is $N_y \in Z^+$ such that for each $n \ge N_y$ the Galerkin type equation*

$$Q_n Lx = Q_n Nx - Q_n y \qquad (x \in \tilde{X}_n, Q_n y \in Y_n), \tag{2.6}$$

has a solution $x_n \in \tilde{X}_n$ such that $x_{n_j} \to x$ in X_J and (2.1) *holds. If, for some $y \in Y$, x is the unique solution, then $x_n \to x$ in X_J, i.e. the Galerkin method is applicable to problems* (I) *to* (V).

Proof. Since $L - \lambda N$: $X_J \to Y$ is A-proper for each $\lambda \in (0,1]$ by hypothesis (i), to deduce Theorem 2.1 from Theorem A it suffices to establish the existence of a bounded open set $G \equiv B(0, r)$ in X_J such that conditions (b), (c) and (d) of Theorem A hold.

Now, if $x \in X_J$ is a solution of $Lx = \lambda Nx - \lambda y$ for some $\lambda \in (0,1]$, then we see that, in view of conditions (ii) and (iii), Proposition 2.1 and Lemma 2.4 imply the existence of a constant $\overline{M} > 0$ such that $|x(t)| \le M$, $|x'(t)| \le \overline{M}$ and $|x''(t)| \le \overline{M}$ for all $t \in [0, T]$. Hence, if we take a fixed $r > \max\{M, \overline{M}\}$ and choose $G = B(0, r) = \{x \in X_J\colon |x|_2 < r\}$, then $Lx \ne \lambda Nx - \lambda y$ for $\lambda \in (0,1]$ and $x \in \partial G$, i.e., (b) holds. To verify (c) and (d), let $Qu = (1/T)\int_0^T u\,dt$ for u in Y and let the bilinear form on $Y \times X_J$ be defined by

$$[u, x] = \int_0^T u(t) x(t)\,dt.$$

Then Q is a projection of Y onto $N(L)$ and, as is not hard to show, $[\cdot, \cdot]$ is continuous and satisfies condition (1.4) for $J = I$ to V. Note that (c) and (d) of Theorem A hold since if $x \in N(L) \cap \partial G$, then $x(t)$ is a constant function, say $x(t) = c$, and $|x|_2 = |c| = r > M$. Therefore, it is easy to show that condition (ii) implies that

$$QN(c) - Qy = \frac{1}{T}\int_0^1 \{f(t,c,0,0) - y\}\, dt \neq 0$$

and

$$[QN(c) - Qy, c] = \int_0^T \{f(t,c,0,0) - y\}\, c\, dt > 0.$$

i.e., conditions (c) and (dl) hold.

Thus, all the hypotheses of Theorem A have been verified and hence the conclusions of Theorem 2.1 follow from Theorem A. □

REMARK 2.1. It should be noted that in case the boundary conditions B_J are such that L: $X_J \to Y$ is a homeomorphism, then Remark 1.2 applies to this boundary value problem and in this case (which happens when $J = \mathrm{I}$, IV or V) we can choose $c = 0$ and let $K = L$ in the construction of Γ_L so that the approximate solution $x_n \in \tilde{X}_n \subset X_J$ is determined by the equation

$$(2.6_0) \qquad Lx_n = Q_n N x_n - Q_n y \qquad (n \geq N_y),$$

which is of practical importance when one applies the method to concrete applications. From the numerical analysis point of view such schemes are very appropriate.

Uniqueness. The last assertion of Theorem 2.1 shows that it is important to find conditions on the nonlinearity f which would ensure the uniqueness of a solution x in X_J for a given boundary condition J and a given y in Y.

We will discuss the uniqueness for problems (I) to (V) under the following differentiability assumption:

(A1) *Suppose that the continuous function $f(t,x,r,q)$ has first partial derivatives with respect to x, r, and q which are bounded on compact subsets of $[0,T] \times R^3$ and such that $f_x \geq 0$ and $f_q < 1$ for $(t,x,r,q) \in [0,T] \times R^3$.*

Now, if x_1 and x_2 are both solutions of problems (I), (II), (III), (IV) or (V), then their difference $x = x_1 - x_2$ satisfies the same boundary conditions as x_1 and x_2 as well as the linear differential equation

$$(2.7) \qquad x'' = \bar{f}_x x + \bar{f}_r x' + \bar{f}_q x'' \quad \text{or} \quad \left(1 - \bar{f}_q\right)x'' = \bar{f}_x x + \bar{f}_r x',$$

where the bar indicates that f_x, f_r and f_q are evaluated at intermediate points (depending on x_1 and x_2). Since, by assumption (A1), $(1 - \bar{f}_q) > 0$ and $\bar{f}_x \geq 0$ for $(t,x,r,q) \in [0,T] \times R^3$ it follows from (2.7) and the maximum principle (see **[22]**) that the following lemma is true (cf. **[8]**).

LEMMA 2.5. *The difference* $x = x_1 - x_2$ *cannot achieve a positive (local) maximum or a negative (local) minimum on* $(0, T)$ *unless it is identically constant.*

THEOREM 2.2. *Suppose that* $f(t, x, r, q)$ *satisfies condition* (A1). *Then each of the problems* (I), (IVa), (IVb), *and* (IVc) *has at most one solution.*

Proof. For problems (I), (IVa), (IVb) and (IVc) the boundary conditions (defined above) imply that x assumes its extreme values in the interior of $(0, T)$. If $x(t) \not\equiv 0$, then Lemma 2.1 implies that x is a constant. This constant must be zero, a contradiction. Thus, $x(t) \equiv 0$ and uniqueness is proven.

Combining Theorem 2.1 with Theorem 2.2 we have the following constructive result.

THEOREM 2.3. *Suppose that* $y \in Y$ *and* $f(t, x, r, q)$ *satisfy conditions* (i), (ii) *and* (iii) *of Theorem* 2.1. *Suppose further that* f *satisfies condition* (A1). *Then each of the problems* (I), (IVa), (IVb) *and* (IVc) *is strongly a-solvable w.r.t.* Γ_L, *i.e., for each of the above problems there exists an integer* $N_y \in Z^+$ *such that the equation*

$$Lx = Q_n Nx - Q_n y \qquad (x \in \tilde{X}_n, Q_n y \in Y_n) \tag{2.8}$$

has a solution $x_n \in \tilde{X}_n \cap B(0, r)$ *for each* $n \geq N_y$, $x_n \to x$ *as* $n \to \infty$ *in the* C^2*-norm, and* x *is the unique solution of the problem* (I), (IVa), (IVb), *or* (IVc).

It is known that uniqueness need not hold for Neumann, periodic, and some of the other boundary value problems. However, as in [**8**] we note that under some additional conditions on f we can prove

THEOREM 2.4. *Suppose* f *satisfies condition* (A1) *and assume further that* $(1 - \bar{f}_q) \geq a$ *for some constant* $a > 0$ *and all* $(t, x, r, q) \in [0, T] \times R^3$. *Then each of the problems* (IVd) *and* (IVe) *has at most one solution in* $C^2([0, T])$.

The proof of Theorem 2.4 is based on the following slight extension of Lemma 5.4 in [**8**] whose proof follows the argument of [**8**]. This lemma will also be used to obtain a uniqueness theorem for remaining boundary value problems.

LEMMA 2.6. *If the difference $x = x_1 - x_2$ satisfies either the boundary conditions $x(0) = 0$, $x'(T) = 0$ or the conditions $x'(0) = 0$, $x(T) = 0$, then $x(t) \equiv 0$.*

Proof. Clearly it is sufficient to consider the boundary conditions $x(0) = 0$, $x'(T) = 0$. The other case may be treated similarly. If $x(t) = 0$, then $x(t) \equiv 0$ by the proof of Theorem 2.2. So assume that $x(T) \neq 0$. Without loss of generality let $x(T) > 0$ (if not replace x by $-x$). By Lemma 2.5, either $x(T)$ is the (positive) maximum value of x or x is a constant. In the latter case $x(t) \equiv x(0) = 0$ and the proof is complete. Thus, we may assume the former case to hold. Furthermore, x cannot assume the negative value by Lemma 2.5 because $x(0) = 0 < x(T)$. Thus $x(t) \geq 0$ on $[0, T]$. Since $x(0) < x(T)$, there is $t_0 \in (0, T)$ with $x'(t_0) > 0$ by the mean value theorem. By applying the maximum principle to the subinterval of $[t_0, T]$ one can easily show that $x'(t) \geq 0$ for $t \in [t_0, T]$. By assumption (A1), there exist a bound $b > 0$ and a number $a > 0$ such that $|\bar{f}_r| \leq b$ on $[0, T]$ and $(1 - \bar{f}_q) \geq a$ for $(t, x, r, q) \in [0, T] \times R^3$. Since $x(t) \geq 0$ and $x'(t) \geq 0$ on $[t_0, T]$, we have on $[t_0, T]$,

$$x'' = \frac{\bar{f}_x}{1 - \bar{f}_q} x + \frac{\bar{f}_r}{1 - \bar{f}_q} x' \geq \frac{-b}{a} x',$$

and since $x'(T) = 0$, it follows from the above that

$$x'(t) \leq \frac{b}{a} \int_t^T x'(s)\, ds.$$

This implies that

$$\left(-e^{(b/a)t} \int_t^T x'(s)\, ds\right)' \leq 0.$$

Integrating the above inequality from t to T yields

$$e^{(b/a)t} \int_t^T x'(s)\, ds \leq 0.$$

Since $x'(t) \geq 0$ on $[t_0, T]$, this implies that $x'(t) \equiv 0$ on $[t_0, T]$. Thus, $x(t_0) = x(T)$ the positive maximum of x on $[0, T]$. Then Lemma 2.5 implies that x is constant. Thus, $x(t) \equiv x(0) = 0$, and Lemma 2.6 is proven.

Proof of Theorem 2.4. Let $x = x_1 - x_2$ be the difference of two solutions to problem (IVd) or (IVe). Then $x(0) = 0$ and $x'(T) = 0$ or $x(T) = 0$ and $x'(0) = 0$. Thus, by Lemma 2.6, $x(t) \equiv 0$ and the proof of Theorem 2.4 is complete.

THEOREM 2.5. *Suppose f satisfies the conditions of Theorem* 2.4. *Then any two solutions to problem* (II) *or problem* (III) *differ by a constant, while the problem* (V) *has at most one solution in* $C^2([0,T])$. *If, in addition,* $f_x(t_0, x, r, q) > 0$ *for some fixed* $t_0 \in (0,T)$, *then the problems* (II) *and* (III) *have also at most one solution in* $C^2([0,T])$.

Proof. Suppose first that $x = x_1 - x_2$ is the difference of two solutions to the Neumann problem (II). If $x(0) = 0$ or $x(T) = 0$, then by Lemma 2.6 we see that $x(t) \equiv 0$. Thus, we may assume that $x(0)$, $x(T) \neq 0$ and without loss of generality that $x(0) > 0$. If $x(t_0) = 0$ for some $t_0 \in (0,T)$, then $x(t) \equiv 0$ by Lemma 2.6 applied to $[0, t_0]$. This contradicts the fact that $x(0) \neq 0$. Hence no such t_0 exists and $x(t) > 0$ on $[0,T]$. Assume that $x(0) > x(T)$. Then there is $t_0 \in (0,T)$ with $x'(t_0) < 0$ by the mean value theorem. Hence $x'(t) \leq 0$ on $[0, t_0]$ since otherwise the maximum principle is contradicted on the subinterval $[0, t_0]$. Since $f_x x/(1 - f_q) \geq 0$ and $bx'/(1 - f_q) \geq (b/a)x'$ on $[0, t_0]$, we see from (2.7) that $x'' \geq (b/a)x'$ on $[0, t_0]$. Hence the argument as that used to prove Lemma 2.6 shows that

$$e^{-b/a} \int_0^t x'(s)\, ds \geq 0 \quad \text{for all } t \in [0, t_0].$$

Since $x'(t) \leq 0$ on $[0, t_0]$, it follows from the last inequality that $x'(t) \equiv 0$ on $[0, t_0]$. Then $x(t_0) = x(0)$ and $x(0)$ must be the positive maximum of x because $x(0) > x(T)$. This contradicts Lemma 2.5. Thus, $x(0) > x(T)$ is impossible and, likewise, $x(0) < x(T)$ cannot hold. Thus, $x(0) = x(T)$. The above discussion implies that either $x(t) \equiv x(0)$ or x assumes a positive minimum value strictly less than $x(0)$ at some point $t_0 \in (0,T)$. In the latter case, $x(t_0) < x(0)$ and $x'(t_0) = 0 = x'(0)$. This is impossible by the argument just used when applied to the interval $[0, t_0]$. Hence $x(t) \equiv x(0)$, a constant, and the first part of Theorem 2.5 is proved for problem (II).

Next assume that $x = x_1 - x_2$ is the difference of two solutions to the periodic problem (III). If $x'(0) \neq 0$, then x must assume either a positive maximum or a negative minimum in $(0,T)$. Then, by Lemma 2.5, $x(t)$ is a constant. On the other hand, if $x'(0) = 0$, x satisfies problem (II), and $x(t)$ is a constant by what has just been proved. Thus, if $x = x_1 - x_2$ is the difference of two solutions to problem (II) or problem (III), then $x(t)$ is a constant on $[0,T]$. Similar argument shows that if $x = x_1 - x_2$ is the difference of two solutions to the antiperiodic problem (V), then x must be a constant just as in the periodic case; however, this constant must be zero because $x(0) = -x(T)$. This proves the first part of Theorem 2.5.

Now, if $x = x_1 - x_2$ is the difference of two solutions to problem (II) or problem (III), then by the first part of Theorem 2.5 the function x is a constant and the differential equation (2.7) satisfied by x reduces in this case to $0 = \bar{f}_x x$. Since, by additional condition, $f_x(t_0, x, r, q) > 0$ for some $t_0 \in (0, T)$, it follows that $x(t) \equiv 0$. This completes the proof of Theorem 2.5. $\square$

When we now combine the assertions of Theorem 2.1 with those of Theorems 2.4 and 2.5, we obtain the following constructive result (i.e. the applicability of the Galerkin method) for the other BV Problems not included in Theorem 2.3.

THEOREM 2.6. *Suppose that $y \in Y$ and $f(t, x, r, q)$ satisfy conditions* (i), (ii) *and* (iii) *of Theorem* 2.1. *Suppose further that f satisfies condition* (A1) *and $(1 - f_q) \geq a > 0$ for all $(t, x, r, q) \in [0, T] \times R^3$.*

Then each of the problems (IVd), (IVe) *and* (V) *is strongly a-solvable w.r.t. Γ_L. If additionally we assume that $f_x(t_0, x, r, q) > 0$ for some $t_0 \in (0, T)$, then problems* (II) *and* (III) *are also strongly a-solvable w.r.t. Γ_L.*

3. *Special cases.* Before we discuss some special cases of Eq. (A), we first make the following useful observation.

A simple argument shows that our Theorem 2.1 and the proofs of Lemmas 2.1 to 2.4 imply the validity of the following *new* results for the generalized *Lienard* boundary value problems:

$$x'' = g(x)x' + h(t, x, x', x'') - y(t), \qquad 0 \leq t \leq T, \tag{3.1}$$

$$x \in B_J \quad (\text{for } J = \text{I, II, III, IV, } or \text{ V}), \tag{3.2}$$

which has been studied in [15] when $J = \text{III}$ and in [16] when $J = \text{II}$.

THEOREM 3.1. *Let g: $R \to R$ be continuous and suppose that $y \in Y$ and the continuous function h: $[0, T] \times R^3 \to R$ is such that*

(3i) *$L - \lambda N_2$: $X_J \to Y$ is A-proper w.r.t. Γ_L for each $\lambda \in (0, 1]$ where $N_2(X) = h(t, x, x', x'')$ for $t \in [0, T]$ and $x \in X_J$.*

(3ii) *There are $M > 0$ and a, $b \in R$ with $b \leq y_m \leq y_M \leq a$ and $x \geq M \Rightarrow h(t, x, 0, q) > a$ if $t \in [0, T]$ and $q \in R$, while $x \leq -M \Rightarrow h(t, x, 0, q) < b$ if $t \in [0, T]$ and $q \in R$.*

(3iii) *There are $B \in [0, 1)$ and continuous A, C: $[0, T] \times [-M, M] \to R^+$ such that $|h(t, x, r, q) \leq \leq A(t, x)r^2 + B|q| + C(t, x)$ for $r, q \in R$.*

Then the conclusions of Theorem 2.1 *hold for the BV Problems* (3.1)–(3.2).

Proof. Since X_J is compactly imbedded into $C^1([0, T])$, it follows that N_1: $X_J \to Y$, defined by $N_1(x) = g(x)x'$ for $x \in X_J$, is completely continuous (see [13]). Hence, letting N_2: $X_J \to Y$ be defined by $N_2(x) = h(t, x, x', x'')$ for $t \in [0, T]$ and $x \in X_J$ and using our condition on h in (3i), we see that $L - \lambda N$: $X_J \to Y$ is A-proper w.r.t. Γ_L for each $\lambda \in (0, 1]$ with $N = N_1 + N_2$, i.e., (i) of Theorem 2.1 holds for $N(x) = f(t, x, x', x'') \equiv g(x)x' + h(t, x, x', x'')$. Furthermore, since $f(t, x, 0, x'') = h(t, x, 0, x'')$ for $t \in [0, T]$ and $x \in X_J$, (3ii) implies that the function f defined above satisfies condition (ii) of Theorem 2.1 since h does. Finally, it is not hard to show that because of (3iii) and the continuity of g the condition (iii) of Theorem 2.1 also holds when $f(t, x, x', x'') = g(x)x' + h(t, x, x', x'')$ with a different function $A(t, x)$. Consequently, Theorem 3.1 follows form Theorem 2.1. □

REMARK 3.1. Theorem 3.1, when $J = \text{III}$, is related to Theorem 2.1, Corollary 2.2 and Theorem 2.2 in Petryshyn-Yu [15], studied there under the assumption that the growth of $h(t, x, r, q)$ is at most linear in r and uniformly bounded in q. However, the other conditions imposed on h in [15] are different from condition (3ii).

We now make some relevant comments on the hypothesis (i) used in Theorem 2.1 and other theorems in §2. This will allow us to indicate some special cases of Theorem 2.1 and Theorems 2.3 and 2.6 and relate those to earlier results obtained by other authors.

When the function f in Eq. (A) (or Eq. (2.1)) is independent of x'', i.e., f is of the form $f(t, x, x')$, then the map N: $X_J \to Y$ given by $Nx = f(t, x, x')$ for $t \in [0, T]$ and $x \in X_J$, is completely continuous since X_J is compactly imbedded into $C^1([0, T])$ and so in this case the condition (i) of Theorem 2.1 always holds because $L - \lambda N$: $X_J \to Y$ is A-proper w.r.t. Γ_L for each $\lambda \in R$. Thus in this case Theorem 2.1 yields the following

COROLLARY 3.1. *Let $y \in Y$ and assume that f: $[0, T] \times R^2 \to R$ is continuous and*

(α) *There are $M > 0$ and $a, b \in R$ such that $b \leq y_m \leq y_M \leq a$ and $x \geq M \Rightarrow f(t, x, 0) > a$ if $t \in [0, T]$, while $x \leq -M \Rightarrow f(t, x, 0) < b$ if $t \in [0, T]$.*

($\alpha\alpha$) *There are continuous functions $A(t, x)$, $C(t, x) > 0$ bounded for $(t, x) \in [0, T] \times [-M, M]$ such that $|f(t, x, r)| \leq A(t, x)r^2 + C(t, x)$ for $r \in R$.*

Then the boundary value problem

$$(3.3) \qquad x'' = f(t, x, x') - y(t), \qquad 0 \le t \le T,$$

$$(3.4) \qquad x \in B_j \qquad (\textit{for } J = \mathrm{I, II, III, IV}, \textit{ or } \mathrm{V})$$

is feebly a-solvable w.r.t. Γ_L *and, in particular,* (3.3)–(3.4) *has a solution* $x \in X_J$. *If* x *is the unique solution for a given* y, *then* (3.3)–(3.4) *is strongly a-solvable w.r.t.* Γ_L (*i.e., the Galerkin method applies to* (3.3)).

REMARK 3.2. The existence part of Corollary 3.1 (when $a = b = 0$) has been proved by Granas, Guenther and Lee [**8**, Theorems 4.1 and 6.1] since in this case conditions (α) and (F) are equivalent. The constructive aspect of the second part of Corollary 3.1 is new even for (3.3)–(3.4) if we somehow know that (3.3)–(3.4) has a unique solution x for a given y. In particular, in view of Theorems 2.3 and 2.6, we have the following constructive versions of the corresponding theorems in [**8**] for (3.3)–(3.4).

COROLLARY 3.2. *Suppose* $y \in Y$ *and* $f(t, x, r)$ *satisfies conditions* (α) *and* ($\alpha\alpha$). *Suppose further that* f_x *and* f_r *exist, are bounded on compact subsets of* $[0, T] \times R^2$, *and are such that* $f_x \ge 0$ *for* $(t, x, r) \in [0, T] \times R^2$, *then the problems* (I), (IVa), (IVb), (IVd), (IVe) *and* (V) *for the Eq.* (3.3) *is strongly a-solvable w.r.t.* Γ_L. *If we additionally assume that* $f_x(t_0, x, r) > 0$ *for some* $t_0 \in (0, T)$, *then the problem* (II) *and* (III) *for Eq.* (3.3) *are also strongly a-solvable w.r.t.* Γ_L.

REMARK 3.3. It was also shown in [**8**] that if the continuous function $f(t, x, r)$ is strictly increasing in x for each fixed (t, r), then there is at most one solution in $C^2([0, T])$ of Eq. (3.3) subject to each of the boundary conditions (I) to (V). Hence, in this case, if we additionally assume that conditions (α) (i.e. (F)) and ($\alpha\alpha$) (i.e. (E)) hold, then (3.3)–(3.4) is strongly a-solvable w.r.t. Γ_L. Thus the constructive versions of the corresponding results in [**8**] follow from the second part of Theorem 2.1.

In view of the discussion preceding the statement of Corollary 3.1, an immediate consequence of Theorem 3.1 or Corollary 3.1 is the following

COROLLARY 3.3. *Suppose* $f(t, x, r)$ *satisfies the conditions* (α) *and* ($\alpha\alpha$) *of Corollary* 3.1 *and suppose that* g: $R \to R$ *is continuous. Then the conclusions of Corollary* 3.1 *hold for the boundary value problems of Lienard* *type*:

$$x'' = g(x)x' + h(t, x, x') - y(t), \qquad 0 \le t \le T, \tag{3.5}$$

$$x \in X_J \qquad (\textit{for } J = \mathrm{I}, \dots, \textit{or } \mathrm{V}). \tag{3.6}$$

For an extensive literature dealing with the solvability of Eq. (3.5) when $J = \mathrm{III}$ (i.e. periodic BV Problem for (3.5)) under various conditions on g and h see the books [**25**], [**7**], [**23**], the recent articles in [**4**, **10**, **11**] and the literature listed there.

Now, when the function $f(t, x, x', x'')$ in Eq. (A) or (2.1) is such that x'' cannot be eliminated from f, then the following simple analytic condition on f leads to the conclusion that $L - \lambda N$: $X_J \to Y$ is A-proper w.r.t. Γ_L for each $\lambda \in (0, 1]$ (a condition (i) required in Theorems 2.1 and others with $Lx = x''$ and $Nx = f(t, x, x', x'')$ for $x \in X_J$) whose proof is given here in detail (for short outline see [**15**]).

Before we prove our next result, we first recall some notions. Given any bounded set $Q \subset X$, the ball-measure of noncompactness $\beta_2(Q)$ of Q is given by $\beta_2(Q) = \inf\{r > 0 | Q \subset \bigcup_{j=1}^{p} B(x_j, r)$ with center $x_j \in X$ for $1 \le j \le p$ and some $p \in Z^+\}$. A continuous map N: $X \to Y$ is said to be k-ball-contractive if $\beta_0(N(Q)) \le \beta_2(Q)$ for any bounded set $Q \subset X$ and some constant $k \ge 0$, where β_2 and β_0 denote the ball-measures in X and Y, respectively, and we set $X = X_J$ for simplicity of notation.

LEMMA 3.1. *Suppose f: $[0, T] \times R^3 \to R$ is a continuous function such that for some constant $k > 0$*

$$|f(t, x, r, q) - f(t, x, r, \bar{q})| \le k|q - \bar{q}| \tag{3.7}$$
$$\forall t \in [0, T] \quad \forall x, r, q, \bar{q} \in R.$$

Then the mapping $L - \lambda N$: $X_J \to Y$ is A-proper w.r.t. Γ_L for each $\lambda \in (0, 1]$ provided k is sufficiently small.

Proof. Since L: $X \to Y$ is Fredholm of index 0, in view of Lemma 2 in [**21**], to prove Lemma 3.1 it suffices to show that $Nx = f(t, x, x', x'')$ is k-ball-contractive with k appearing in (3.7).

To prove the latter, let V: $X \times X \to Y$ be defined by $V(x, u) = f(t, x, x', u'')$ and note that $N(x) = V(x, x)$ for $x \in X$ and N: $X \to Y$ is continuous. We now claim that N: $X \to Y$ is k-ball-contractive, i.e., $\beta_0(N(Q)) \le k\beta_2(Q)$ for any bounded set $Q \subset X$. So let $Q \subset X$ be any bounded set, let $r \equiv \beta_2(Q)$, and for any given $\varepsilon > 0$ cover Q by a finite number of balls in X with radius $r + \varepsilon/k$ and centers u_j in X; say

$B(u_j, r + \varepsilon/k) \subset X$ for $1 \le j \le p$, i.e. $Q \subset \bigcup_{j=1}^{p} B(u_j, r + \varepsilon/k)$. Since x is compactly imbedded into $C^1([0, T])$, Q is bounded in X, it follows that Q is precompact in $C^1([0, T])$ and the map $x \to V(x, u)$ is continuous as a map from $C^1([0, T])$ to Y for each fixed u in X. Hence the set $V(Q, u)$ is precompact in Y for any fixed u in X and so $\bigcup_{j=1}^{p} V(Q, u_j)$ is also precompact in Y. Therefore, for the given $\varepsilon > 0$ there exist points $z_1, \ldots, z_q$ in X such that

$$\bigcup_{n=1}^{p} V(Q, u_n) \subset \bigcup_{j=1}^{q} B(z_j, \varepsilon).$$

Now, given any x in Q we choose n such that $|x - u_n|_2 \le r + \varepsilon/k$ and observe that

$$\begin{aligned} |V(x,x) - V(x,u_n)| &= |f(t,x,x',x'') - f(t,x,x',u_n'')| \le k|x'' - u_n''| \\ &\le k|x'' - u_n''|_0 \le k|x - u_n|_2 \le k(r + \varepsilon/k). \end{aligned}$$

Moreover, we may choose $j \in [1, 2, \ldots, q]$ such that $|V(x, u_n) - z_j|_0 < \varepsilon$. Thus,

$$\begin{aligned} |N(s) - z_j|_0 &= |V(x,x) - z_j|_0 \\ &\le |V(x,x) - V(x,u_n)|_0 + |V(x,u_n) - z_j|_0 \\ &\le k(r + \varepsilon/k) + \varepsilon = kr + 2\varepsilon. \end{aligned}$$

It follows from this that $N(Q) \subset \bigcup_{j=1}^{q} B(z_j, kr + 2\varepsilon)$. In other words, $\beta_0(N(Q)) \le kr + 2\varepsilon$ for any given $\varepsilon > 0$. Hence $\beta_0(N(Q)) \le k\beta_2(Q)$, i.e., N is k-ball-contractive. □

Consequently, all theorems in §2 are valid when condition (i) is replaced by the assumption (3.7) for a given $T > 0$ and $\eta = \sup_n \|Q_n\|$ with k small depending on T, η and the boundary conditions J.

As an illustration of Theorem 2.3, we consider the following BV Problem mentioned in the Introduction for any fixed $\alpha \in (0, 1)$ to which the results in [**8**] do not apply but Theorem 2.3 does because of Lemma 3.1. Thus, we consider the BVP

$$x'' = x^3 + x'^2 + 1 + \alpha \sin x'' - y(t), \qquad 0 \le t \le 1, \tag{3.8}$$

$$x(0) = x(1) = 0. \tag{3.9}$$

Letting $f(t, x, r, q) = x^3 + r^2 + 1 + \alpha \sin q$, we see that (3.7) holds with $k = \alpha < 1$ and thus, by Lemma 3.1 and the fact that $\|L^{-1}\|_{Y \to X_I} = 1$. The map $L - \lambda N\colon X_I \to Y$ is A-proper w.r.t. $\Gamma_L = \{\tilde{X}_n, Y_n, Q_n\}$ for each

$\lambda \in (0,1]$ if we assume that $\eta = 1$. Further, since $f(t, x, 0, q) = x^3 + 1 + \alpha \sin q$ and $|\sin q| \leq 1$ for all $q \in R$, we see that if we take $M \geq \sqrt[3]{\alpha}$, $a = M^3 + 1 - \alpha$ and $b = -M^3 + 1 + \alpha$, then for any $y \in C([0,1])$ we can choose $M \geq \sqrt[3]{\alpha}$ such that $b \leq y(t) \leq a$, i.e., condition (ii) of Theorem 2.3 holds. Now, condition (iii) also holds since $|f(t, x, r, q)| \leq r^2 + |x|^3 + 1 + \alpha|q|$ with $\beta = \alpha < 1$. Finally, since $f_x = 3x^2$, $f_r = 2r$ and $f_q = \alpha \cos q$ we see that $f_x \geq 0$ and $1 - f_q \geq 1 - \alpha \equiv a > 0$ for all $(t, x, r, q) \in [0,1] \times R^3$. Hence Theorem 2.3 implies the following result for (3.8)–(3.9).

PROPOSITION 3.1. *Let $\alpha \in (0,1)$. Then the BV Problem* (3.8)–(3.9) *is strongly a-solvable w.r.t. Γ_L for each $y \in C([0,T])$.*

As our next and applicatively important example, we consider the following periodic BV Problem

$$x''(t) = a(t)x^3 - b(t)x - c(t)x' - |x'|x' - y(t), \quad 0 \leq t \leq T, \tag{3.10}$$

$$x(0) = x(T), \qquad x'(0) = x'(T), \tag{3.11}$$

which appears in the study of the dynmaics of wires, where $y \in Y = C([0,T])$ and $a(t)$, $b(t)$, and $c(t)$ are continuous functions defined on $[0,T]$. When $c(t) \equiv 1$ and $q(t)$ satisfies

(3.12) *There exists $\alpha > 0$ such that $a(t) \geq \alpha$ for all $t \in [0,T]$,*

the existence of a classical solution to (3.10)–(3.11) (which is T-periodic if $a(t)$, $b(t)$ and $y(t)$ are T-periodic) was established by Stoppelli [**26**]. The same problem for sufficiently small constants a and c has been solved in [**3**]. Recently the problem (3.10)–(3.11) (formulated as a BV Problem in the $L^2(0,T)$ space) has been treated by Sanchez [**24**] and by the author [**18**] under the additional condition that $a(t)$ is Lipschitz continuous in $[0,T]$ and $a(0) = a(T)$, but with $y \in L^2(0,T)$ and $b, c \in L^\infty(0,T)$.

Using Corollaries 3.1 and 3.2 we get the following extension of the result in [**26**] (and in [**3**]) which, in addition, is constructive if $b(t) \leq 0$ for $t \in [0,T]$.

PROPOSITION 3.2. *Let $a(t)$, $b(t)$, $c(t)$ be continuous functions on $[0,T]$ and such that $a(t)$ satisfies the condition* (3.12). *Then the BV Problem* (3.10)–(3.11) *is feebly a-solvable w.r.t. Γ_L for each y in Y. If we additionally assume that $b(t) \leq 0$ (and $b(t) \not\equiv 0$) for $t \in [0,T]$, then* (3.10)–(3.11) *is strongly a-solvable w.r.t. Γ_L for each y in Y (i.e. in this case the Galerkin method is applicable to* (3.10)–(3.11)).

Proof. Let $X_J \equiv X_0$ when $J \equiv$ III, let $f(t,x,x') = a(t)x^3 - b(t)x - c(t)x' - |x'|x'$ for $x \in X_0$, and let L, N: $X_0 \to Y$ be defined by $Lx = x''$ and $Nx = f(t,x,x')$ for $x \in X_0$. Then $L - \lambda N$: $X_0 \to Y$ is A-proper w.r.t. Γ_L for each $\lambda \in R$. Now, it follows from (3.12) that $f(t,x,0) = x^2[a(t)x - b(t)] \to +\infty$ $(-\infty)$ as $x \to +\infty$ $(-\infty)$. Hence, for any given y in Y, we can find $M > 0$ and a, $b \in R$ with $a \geq b$ such that condition $(\alpha\alpha)$ of Corollary 3.1 holds. Now, it is easy to see that $f(t,x,x')$ satisfies condition $(\alpha\alpha)$ of Corollary 3.1. Consequently, the first assertion of Proposition 3.2 follows from Corollary 3.1.

To prove the second part of Proposition 3.2, we apply Corollary 3.2. Indeed, since $f_r = -c(t) - 2|r|$ and $f_x = 3a(t)x^2 - b(t)$ we see that $f_x \geq 0$ for all $(t,x,r) \in [0,T] \times R^2$ and $f_x(t_0,x,r) = 3a(t_0)x^2 - b(t_0) > 0$ for some $t_0 \in (0,T)$ by our additional hypothesis on $b(t)$. Hence (3.10)–(3.11) is strongly a-solvable by Corollary 3.2. □

We note in passing that $f(t,x,r) = a(t)x^3 - b(t)x - c(t)r - |r|r$ is strictly increasing in x for each fixed (t,r) since when $x > \bar{x}$, then $f(t,x,r) - f(t,\bar{x},r) = -b(t)(x - \bar{x}) + a(t)(x^3 - \bar{x}^3) > 0$ because $-b(t) \geq 0$ and $a(t) \geq \alpha > 0$. Hence the conclusion of the second part of Proposition 3.2 follows also from Remark 3.3.

It is interesting to note that if instead of (3.10)–(3.11) we consider the BV Problem

$$x'' = a(t)x^3 - b(t)x - c(t)x' - |x'|^2 - y(t), \qquad 0 \leq t \leq T, \tag{3.13}$$

$$x(0) = x(T), \qquad x'(0) = x'(T), \tag{3.14}$$

then the conclusions of Proposition 3.2 are still valid for (3.13)–(3.14), but the method used in **[24]** (and in **[18]**) to obtain the a priori bounds is no longer applicable when the term $|x'|x'$ is replaced by $|x'|^2$.

We complete this section with the study of the problem (L), i.e., with

$$x'' = f_1(t,x,x') + g(t,x',x'') - y(t), \qquad 0 \leq t \leq 1, \tag{3.15}$$

$$x(0) = x(1), \qquad x'(0) = x'(1), \tag{3.16}$$

where the continuous functions f_1, g: $[0,1] \times R^2 \to R$ satisfy:

(a1) *There are* $M > 0$ *and* $\delta > 0$ *such that* $x \geq M \Rightarrow f_1(t,x,0) \geq \delta$ *if* $t \in [0,1]$, *while* $x \leq -M \Rightarrow f_1(t,x,0) \leq -\delta$ *if* $t \in [0,1]$.

(a2) $|f(t,x,r)| \leq A(t,x)r^2 + C(t,x)$ *for* $r \in R$, *where* $A(t,x)$, $C(t,x) > 0$ *are functions bounded on compact subsets of* $[0,1] \times R$.

(a3) *There is* $k \in (0,1)$ *and* $p \in (0,\delta)$ *such that* $|g(t,r,q)| \leq p$ *and* $|g(t,r,q) - g(t,r,\bar{q})| \leq k|q - \bar{q}|$ *or all* $(t,r,q) \in [0,1] \times R^2$.

Using Theorem 2.1 and Lemma 3.1 we have

PROPOSITION 3.3. *Suppose that* (a1), (a2) *and* (a3) *hold. Then for each* $y \in Y \equiv C([0, 1])$ *such that* $p - q \leq y(t) \leq \delta - p$ *the problem* (3.15)–(3.16) *is feebly a-solvable w.r.t.* Γ_L *and, in particular, it has a solution* x *in* X_J *for* $J =$ III. *If* x *is unique, then* (3.15)–(3.16) *is strongly a-solvable.*

Proof. Proposition 3.3. follows from Theorem 2.1 and Lemma 3.1. Indeed, letting $N_1x = f_1(t, x, x')$, $N_2x = g(t, x', x'')$ for $x \in X_J$, and $N = N_1 + N_2$ it follows from Lemma 3.1 because of (a3) and the compactness of N_1, that $L - \lambda N$: $X_J \to Y$ is A-proper w.r.t. Γ_L for each $\lambda \in (0, 1]$, i.e., (i) holds. Now, if we let $f(t, x, x', x'') \equiv f_1(t, x, x') + g(t, x', x'')$, then it follows from (a1) and the first part of (a3) that

$$x \geq M \Rightarrow f(t, x, 0, q) = f_1(t, x, 0) - q(t, 0, q) \geq \delta + p \equiv a,$$

while

$$x \leq -M \Rightarrow f(t, x, 0, q) \leq -\delta + p \equiv b,$$

i.e. (ii) holds for $b \leq y \leq a$. Since (iii) obviously follows from (a2) and (a3), we see that the conclusions of Proposition 3.3 follow from Theorem 2.1. □

REMARK 3.4. In [**6**] the solvability of (3.15)–(3.16) (with $y(t) = 0$) in $W_2^2(0, 1)$ was studied by means of the degree theory of condensing vector fields but under a much stronger condition on $f_1(t, x, x')$, where f_1 was required to have a sublinear growth (i.e. $|f(t, x, r)| \leq b(t) + \gamma(|x|^\beta + |r|^\beta)$ for some $\beta \in (0, 1)$ and all $(t, x, r) \in [0, 1] \times R^2$) and be such that $x \geq M \Rightarrow f_1(t, x, r) \geq \delta$ if $t \in [0, 1]$ and $r \in R$ and $x \leq -M \Rightarrow f_1(t, x, r) \leq -\delta$ if $t \in [0, 1]$ and $r \in R$.

It should be added that Proposition 3.3 does not include the author's result [**17**, Th. 3.1] for (3.15)–(3.16) since the function $g(t, r, q)$ is not required to be Lipschitzian in q. But the method used in [**17**] also requires f_1 to have a sublinear growth.

REMARK 3.5. The author is grateful to the referee for his useful suggestions.

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Received August 20, 1984 and in revised form February 25, 1985. Supported in part by the NSF grant MSC-83015167-01.

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