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FITTING STRUCTURES

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Motivated by papers of H. Fitting, the problem arises whether there exists a ring which contains a given ring and a semigroup acting on each other. This problem is solved in the affirmative by the construction of a "universal envelopment". Furthermore, the situation investigated gives rise to a generalized wreath product which is used for a description of certain automorphism groups.

0. Introduction. The endomorphisms of an abelian group form a ring in a natural and well-known way, whereas in the case of a nonabelian group one has no general "addition" of endomorphisms. One easily proves that the "sum" of two endomorphisms α, β of a group G,

$$\alpha + \beta \colon g \mapsto g^{\alpha} g^{\beta} \quad \text{for all } g \in G,$$

is an endomorphism of G if and only if $[G^{\alpha}, G^{\beta}] = 1$. Hence the endomorphisms that can be added to any endomorphism are exactly the homomorphisms of G into its center $Z(G)$. In this sense, the ring $\mathrm{Hom}(G, Z(G))$ is a "pleasant" substructure of $\mathrm{End}(G)$. Long ago, Fitting [**2**] described the structure of $\mathrm{End}_G(G)$ which, though not a ring, still has numerous ring properties; if we put $H := \mathrm{End}_G(G)$, $S := \mathrm{Hom}(G, Z(G))$, then e.g.

$$(1) \qquad (s_1 h) s_2 = s_1 (h s_2) \quad \text{for all } h \in H,\ s_1, s_2 \in S,$$

$$(2) \qquad (h_1 + s) h_2 = h_1 h_2 + s h_2, \qquad h_1 (h_2 + s) = h_1 h_2 + h_1 s \quad \text{for all } h_1, h_2 \in H,\ s \in S,$$

$$(3) \qquad (h + s_1) s_2 = h s_2 + s_1 s_2, \qquad s_1 (h + s_2) = s_1 h + s_1 s_2 \quad \text{for all } h \in H,\ s_1, s_2 \in S.$$

Keeping these rules as axioms, we introduce so-called *Fitting structures* in the first chapter of this paper and show that firstly there does exist a ring R containing H and S such that (1), (2), (3) are special cases of its associative and distributive laws, and that secondly any ring with this property (if—which is a non-essential restriction—it is generated by H) is a homomorphic image of R. Emanating naturally from Fitting's notion of "Bereich" [**2**], the problem of the existence of enveloping rings for Fitting structures, which has been solved for $H = \mathrm{End}_G(G)$ by Fitting in a special way, thus finds a general positive answer. Chapter 2 shows that even if there is not given an addition of elements of H and S, such an addition can be defined after a certain enlargement of H, and that this process of making H and S into a Fitting structure is essentially uniquely determined by the actions of H on S. In Chapter 3, we introduce to each Fitting structure a *generalized wreath product* containing the usual wreath product of (semi-)groups as a special case, and use this concept to give a simple description of certain automorphism groups, applying a result of another paper by Fitting [**1**].

1. Fitting structures. For every ring[1] S, we put

$$\mathrm{End}_\Lambda(S) := \{\alpha \mid \alpha \in \mathrm{End}(S,+),\ (s_1s_2)^\alpha = s_1(s_2^\alpha) \text{ for all } s_1, s_2 \in S\},$$
$$\mathrm{End}_\mathrm{P}(S) := \{\alpha \mid \alpha \in \mathrm{End}(S,+),\ (s_1s_2)^\alpha = (s_1^\alpha)s_2 \text{ for all } s_1, s_2 \in S\}.$$

Obviously, $\mathrm{End}_\Lambda(S)$ and $\mathrm{End}_\mathrm{P}(S)$ are subrings of $\mathrm{End}(S,+)$.

1.1. DEFINITION. Let H be a semigroup, S a ring, φ a homomorphism of H into the multiplicative semigroup of $\mathrm{End}_\Lambda(S)$, ψ an antihomomorphism of H into the multiplicative semigroup of $\mathrm{End}_\mathrm{P}(S)$ such that H^φ and H^ψ commute elementwise. Let σ be a homomorphism of $(S,+)$ into the symmetric group $\mathfrak{S}_H$ on H such that $h^{s^\sigma} = h$ implies $s = 0$ for all $h \in H$, $s \in S$ (i.e., $(S,+)$ "acts freely" on H).

If $h \in H$, $s \in S$, we write sh for s^{h^φ}, hs for s^{h^ψ}, $h + s$ for h^{s^σ}. The 5-tuple $(H, S, \varphi, \psi, \sigma)$ is called a *Fitting structure* if (1), (2), (3) hold.[2]

By definition, we have

$$(4) \qquad (sh_1)h_2 = s(h_1h_2), \qquad h_1(h_2s) = (h_1h_2)s \qquad \text{for all } h_1, h_2 \in H,\ s \in S,$$

$$(5) \qquad (h + s_1) + s_2 = h + (s_1 + s_2) \quad \text{for all } h \in H,\ s_1, s_2 \in S,$$

$$(6) \qquad (s_1s_2)h = s_1(s_2h), \qquad h(s_1s_2) = (hs_1)s_2 \qquad \text{for all } h \in H,\ s_1, s_2 \in S,$$

$$(7) \qquad (h_1s)h_2 = h_1(sh_2) \quad \text{for all } h_1, h_2 \in H,\ s \in S$$

$$(8) \qquad h + s = h \Leftrightarrow s = 0 \quad \text{for all } h \in H,\ s \in S.$$

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[1]All rings in this paper are associative, but do not necessarily have an identity element.

[2]If H is a "Bereich" in the sense of Fitting [**2**], let S be the set of all elements of H which can be added to any element of H. Then S is a ring, and we get a Fitting structure with the additional property that S is contained in H, and H has an identity element. Any further possibilities to add elements of H (which might exist in Fitting's "Bereich") are treated as non-existent in our Fitting structures.

If H has an identity element 1, then (2) and (8) imply $1^{\varphi} = \mathrm{id} = 1^{\psi}$.

Fitting structures $\mathcal{F} = (H, S, \varphi, \psi, \sigma)$, $\mathcal{F}' = (H', S', \varphi', \psi', \sigma')$ are called *isomorphic* if there are isomorphisms α of H onto H', β of S onto S' with the properties $\beta\sigma' = \sigma\bar{\alpha}$, $\alpha\varphi' = \varphi\bar{\beta}$, $\alpha\psi' = \psi\bar{\beta}$, where $\bar{\alpha}$ is the isomorphism of $\mathfrak{S}_H$ onto $\mathfrak{S}_{H'}$ induced by α (such that $\pi^{\bar{\alpha}} = \alpha^{-1}\pi\alpha$ for all $\pi \in \mathfrak{S}_H$), and $\bar{\beta}$ is the isomorphism of End(S) onto End(S') induced by β (such that $\zeta^{\bar{\beta}} = \beta^{-1}\zeta\beta$ for all $\zeta \in \mathrm{End}(S)$).

$\mathcal{F}'$ is called a *Fitting substructure* of $\mathcal{F}$ if H' is a subsemigroup of H, S' is a subring of S, and $\varphi' = \varphi|_{H'}$, $\psi' = \psi|_{H'}$, $\sigma' = \sigma|_{S'}$. If H' is a subsemigroup of H, S' a subring of S, then $(H', S', \varphi|_{H'}, \psi|_{H'}, \sigma|_{S'})$ is a Fitting substructure of $\mathcal{F}$ if and only if $S'H' \subseteq S'$, $H'S' \subseteq S'$, and $H' + S' = H'$.

1.2. Definition. Let $\mathcal{F}$ be a Fitting structure, R a ring, $\tilde{}$ a homomorphism of H into the multiplicative semigroup of R, and $\bar{}$ a homomorphism of S onto an ideal of R. The triple $(R, \tilde{}, \bar{})$ is called an *envelopment* of $\mathcal{F}$ if

$$(9) \qquad \tilde{h} + \bar{s} = \widetilde{h + s} \quad \text{for all } h \in H,\ s \in S$$

holds.

The envelopment $(R, \tilde{}, \bar{})$ is called *faithful* if $\tilde{}$ and $\bar{}$ are injective.

From (9), we conclude

$$(10) \qquad \tilde{h}\bar{s} = \overline{hs},\ \bar{s}\tilde{h} = \overline{sh} \quad \text{for all } h \in H,\ s \in S,$$

since

$$\widetilde{h^2} + \overline{sh} = \widetilde{h^2 + sh} = \widetilde{(h+s)h} = \widetilde{h+s}\tilde{h} = (\tilde{h} + \bar{s})\tilde{h} = \widetilde{h^2} + \bar{s}\tilde{h},$$

and the second part of (10) is proved similarly.

1.3. Definition. Let $\mathcal{F}$ be a Fitting structure and $(R, \tilde{}, \bar{})$, $(R', \approx, =)$ envelopments of $\mathcal{F}$. Then a mapping χ is called a *homomorphism* of $(R, \tilde{}, \bar{})$ into $(R', \approx, =)$ if χ is a homomorphism of R into R' such that $\tilde{h}^{\chi} = \tilde{\tilde{h}}$, $\bar{s}^{\chi} = \bar{\bar{s}}$ for all $h \in H, s \in S$. If χ is an isomorphism of R onto R', we call our envelopments *isomorphic*. An envelopment $\mathcal{U}$ of $\mathcal{F}$ is called *universal* if for any envelopment $\mathcal{V}$ of $\mathcal{F}$ there is a homomorphism of $\mathcal{U}$ into $\mathcal{V}$.

Universal envelopments of isomorphic Fitting structures are isomorphic. We now prove the following existence theorem:

1.4. Theorem. *Every Fitting structure has a faithful universal envelopment.*

Proof. Let $\mathscr{F} = (H, S, \varphi, \psi, \sigma)$ be the Fitting structure given. We put $T := \mathbb{Z}H$ and write $\hat{+}$, $\hat{\cdot}$ for the standard addition and multiplication in T. (Then $h_1 \hat{\cdot} h_2 = h_1 h_2$ for all $h_1, h_2 \in H$.) For $h \in H$, $s \in S$ define

$$\delta(h, s) := (h + s) \dot{-} h$$

and let K be the additive subgroup of T generated by $\{\delta(h_1, s) \dot{-} \delta(h_2, s) \mid h_1, h_2 \in H,\ s \in S\}$. Then

(11) $\quad h_1 \hat{\cdot} \delta(h_2, s) \dot{-} \delta(h_2, h_1 s),\ \delta(h_2, s) \hat{\cdot} h_1 \dot{-} \delta(h_2, s h_1) \in K$ for all $h_1, h_2 \in H$, $s \in S$,

as

$$\begin{aligned} h_1 \hat{\cdot} (h_2, s) \dot{-} \delta(h_2, h_1 s) &= h_1 \hat{\cdot} ((h_2 + s) \dot{-} h_2) \dot{-} \delta(h_2, h_1 s) \\ &= h_1(h_2 + s) \dot{-} h_1 h_2 \dot{-} \delta(h_2, h_1 s) \\ &= \delta(h_1 h_2, h_1 s) \dot{-} \delta(h_2, h_1 s) \in K, \end{aligned}$$

the second part of (11) being proved analogously. Obviously, (11) yields

(12) $\quad h_1 \hat{\cdot} (\delta(h_2, s) \dot{-} \delta(h_3, s)),\ (\delta(h_2, s) \dot{-} \delta(h_3, s)) \hat{\cdot} h_1 \in K$ for all $h_1, h_2, h_3 \in H$, $s \in S$.

Thus K is an ideal of T, and, by definition of K,

(13) $\quad \delta(h_1, s) \hat{+} K = \delta(h_2, s) \hat{+} K$ for all $h_1, h_2 \in H$, $s \in S$.

We now define

$$R := T/K,$$

$$\tilde{}: H \to R,\ h \mapsto h \hat{+} K,$$

and for an arbitrary $h \in H$

$$\bar{}: S \to R, \quad s \mapsto \delta(h, s) \hat{+} K.$$

(By (13), $\bar{}$ is independent of the choice of h.) Obviously, $\tilde{}$ is a homomorphism of H into the multiplicative semigroup of R, and

$$\begin{aligned} \widetilde{h + s} &= (h + s) \hat{+} K = h \hat{+} (h + s) \dot{-} h \hat{+} K \\ &= (h \hat{+} K) \hat{+} (\delta(h, s) \hat{+} K) = \tilde{h} \hat{+} \bar{s} \quad \text{for all } h \in H,\ s \in S, \end{aligned}$$

whence (9) holds. We want to show that $\bar{}$ is a ring homomorphism of S into an ideal of R, and start with

(14) $\delta(h, s_1 + s_2) \dot{-} \delta(h, s_1) \dot{-} \delta(h, s_2) \in K$ for all $h \in H$, $s_1, s_2 \in S$.

For

$$\begin{aligned} &\delta(h, s_1 + s_2) \dot{-} \delta(h, s_1) \dot{-} \delta(h, s_2) \\ &\quad = (h + s_1 + s_2) \dot{-} h \dot{-} (h + s_1) \hat{+} h \dot{-} \delta(h, s_2) \\ &\quad = \delta(h + s_1, s_2) \dot{-} \delta(h, s_2) \in K. \end{aligned}$$

Furthermore,

(15) $\delta(h, s_1 s_2) \mathbin{\hat{-}} \delta(h, s_1) \mathbin{\hat{\cdot}} \delta(h, s_2) \in K$ for all $h \in H$, $s_1, s_2 \in S$,

since

$$
\begin{aligned}
&\delta(h, s_1 s_2) \mathbin{\hat{-}} \delta(h, s_1) \mathbin{\hat{\cdot}} \delta(h, s_2) \\
&\quad = \delta(h, s_1 s_2) \mathbin{\hat{-}} ((h + s_1) \mathbin{\hat{-}} h) \mathbin{\hat{\cdot}} ((h + s_2) \mathbin{\hat{-}} h) \\
&\quad = \delta(h, s_1 s_2) \mathbin{\hat{-}} (h + s_1)(h + s_2) \mathbin{\hat{+}} (h + s_1)h \mathbin{\hat{+}} h(h + s_2) \mathbin{\hat{-}} h^2 \\
&\quad = \delta(h, s_1 s_2) \mathbin{\hat{-}} \delta((h + s_1)h, (h + s_1)s_2) \mathbin{\hat{+}} \delta(h^2, hs_2) \in K,
\end{aligned}
$$

by (13) and (14).

As (14) and (15) show, $\bar{\ }$ is a ring homomorphism, and by (12), $\bar{S}$ is an ideal of R. Therefore, $(R, \tilde{\ }, \bar{\ })$ is an envelopment of $\mathscr{F}$. We need some preliminaries to show that $\tilde{\ }$ and $\bar{\ }$ are injective:

Let D be the additive subgroup of T generated by $\{\delta(h, s) | h \in H, s \in S\}$. Then $K \leq D$, and $D/K = \bar{S}$. Since

(16) $\hat{-}\, \delta(h, s) = \delta(h + s, -s)$ for all $h \in H$, $s \in S$,

every element of D has the form $\Sigma_j \delta(h_j, s_j)$ for appropriate $h_j \in H$, $s_j \in S$. We claim:

(17) $$\sum_{j=1}^{k} \delta(h_j, s_j) = 0 \Rightarrow \sum_{j=1}^{k} s_j = 0$$

for all $h_1, \ldots, h_k \in H$, $s_1, \ldots, s_k \in S$.

Suppose $\Sigma_{j=1}^{k} \delta(h_j, s_j) = 0$. Then we have $\Sigma_{j=1}^{k} (h_j + s_j) = \Sigma_{j=1}^{k} h_j$. Since $(T, \hat{+})$ is free over H, there is a permutation π of $\{1, \ldots, k\}$ such that $h_j + s_j = h_{j\pi}$ for all $j \in \{1, \ldots, k\}$. For $i \in \{1, \ldots, k\}$, let f_i be the smallest positive integer such that $i\pi^{f_i} = i$. Then $\{i, i\pi, \ldots, i\pi^{f_i - 1}\}$ is the orbit of i under π, and $h_i + s_i + s_{i\pi} + \cdots + s_{i\pi^l} = h_{i\pi^{l+1}}$ for $0 \leq l < f_i$, hence in particular $h_i + s_i + s_{i\pi} + \cdots + s_{i\pi^{f_i - 1}} = h_i$, and $s_i + s_{i\pi} + \cdots + s_{i\pi^{f_i - 1}} = 0$ by (8). Now if X denotes a full set of representatives of the orbits of π in $\{1, \ldots, k\}$,

$$\sum_{j=1}^{k} s_j = \sum_{i \in X} (s_i + s_{i\pi} + \cdots + s_{i\pi^{f_i - 1}}) = 0,$$

proving (17).

As a consequence, we have

(18) $$\sum_{j=1}^{k} \delta(h_j, s_j) = \sum_{j=1}^{k'} \delta(h'_j, s'_j) \Rightarrow \sum_{j=1}^{k} s_j = \sum_{j=1}^{k'} s'_j$$

for all $h_1, \ldots, h_k, h'_1, \ldots, h'_{k'} \in H$, $s_1, \ldots, s_k, s'_1, \ldots, s'_{k'} \in S$.

Therefore, ρ: $\sum_{j=1}^{k}\delta(h_j, s_j) \mapsto \sum_{j=1}^{k} s_j$ defines a mapping of D into S which obviously is an additive homomorphism. We claim:

$$K = \ker\rho. \tag{19}$$

By (16) and the definition of K, we have $K \subseteq \ker\rho$. One generally has $\delta(h_j, s_j) \equiv \delta(h_1 + s_1 + \cdots + s_{j-1}, s_j) \bmod K$ for $h_1, \ldots, h_j \in H$, $s_1, \ldots, s_j \in S$. If now $\sum_{j=1}^{k}\delta(h_j, s_j) \in \ker\rho$, then $\sum_{j=1}^{k} s_j = 0$, and consequently

$$\begin{aligned}\sum_{j=1}^{k}\delta(h_j, s_j) &\equiv \sum_{j=1}^{k}\delta(h_1 + s_1 + \cdots + s_{j-1}, s_j) \quad \bmod K\\ &= ((h_1 + s_1) \mathbin{\hat{-}} h_1) \mathbin{\hat{+}} ((h_1 + s_1 + s_2) \mathbin{\hat{-}} (h_1 + s_1))\\ &\quad \mathbin{\hat{+}} \cdots \mathbin{\hat{+}} ((h_1 + s_1 + \cdots + s_k) \mathbin{\hat{-}} (h_1 + s_1 + \cdots + s_{k-1}))\\ &= \mathbin{\hat{-}} h_1 \mathbin{\hat{+}} (h_1 + s_1 + \cdots + s_k) = 0,\end{aligned}$$

i.e., $\sum_{j=1}^{k}\delta(h_j, s_j) \in K$.

If $h \in H$ and $s \in S \setminus \{0\}$, then $\delta(h, s) \notin K$ by (19), and this means

$$\bar{\ } \text{ is injective.} \tag{20}$$

We now want to show that $\tilde{\ }$ is injective which we shall conclude from

$$h \mathbin{\hat{-}} h' \in D \Rightarrow h = h' + s \quad \text{with } s \in S, \text{ for all } h, h' \in H. \tag{21}$$

We reformulate (21) in the following form to make it accessible to an induction argument:

(22) Suppose $h, h' \in H$ and $r \in \mathbb{N}$. If there are $h_1, \ldots, h_r \in H$, $s_1, \ldots, s_r \in S$ such that $h \mathbin{\hat{-}} h' = \sum_{j=1}^{r}\delta(h_j, s_j)$, then there is an element $s \in S$ such that $h = h' + s$.

If $r = 1$, then $h = h_1 + s_1 = h' + s_1$, as $(T, \hat{+}\ )$ is free over H. Now suppose $r > 1$ and (22) is true for $r - 1$ instead of r. Since $h \mathbin{\hat{-}} h' = \sum_{j=1}^{r}\delta(h_j, s_j)$, we may assume $h = h_1 + s_1$, $h' = h_r$. Furthermore, $h_1 = h_i + s_i$ with $i \in \{2, \ldots, r\}$. This yields

$$\begin{aligned} h \mathbin{\hat{-}} h' &= (h_i + s_i + s_1) \mathbin{\hat{-}} (h_i + s_i) \mathbin{\hat{+}} \sum_{j=2}^{r}\delta(h_j, s_j)\\ &= (h_i + s_i + s_1) \mathbin{\hat{-}} h_i \mathbin{\hat{+}} \sum_{\substack{j=2\\ j\neq i}}^{r}\delta(h_j, s_j)\\ &= \delta(h_i, s_i + s_1) \mathbin{\hat{+}} \sum_{\substack{j=2\\ j\neq i}}^{r}\delta(h_j, s_j),\end{aligned}$$

and an application of the induction hypothesis yields our claim. This proves (22) and the equivalent assertion (21) by means of which we conclude

(23) $\tilde{}$ is injective:

For if $h, h' \in H$ and $h \hat{-} h' \in K$, then a fortiori $h \hat{-} h' \in D$ and therefore $h = h' + s$ with $s \in S$. But then $h \hat{-} h' = \delta(h', s)$, and (20) implies $s = 0$. Therefore we have $h = h'$, and the proof of (23) is complete.

It remains to show that the envelopment $(R, \tilde{}, \bar{})$ is universal. To this end let $(R', \tilde{\tilde{}}, \bar{\bar{}})$ be an envelopment of $\mathscr{F}$. Since $(T, \hat{+})$ is free over H,

$$\chi_0 : T \to R', \qquad \sum_j z_j h_j \mapsto \sum_j z_j \tilde{\tilde{h}}_j \quad (z_j \in \mathbb{Z})$$

defines a ring homomorphism. We show

(24) $$K \subseteq \ker \chi_0.$$

Let $h_1, \dots, h_k \in H$, $s_1, \dots, s_k \in S$ and $\sum_{j=1}^{k} \delta(h_j, s_j) \in K$. Then

$$\begin{aligned}\left(\sum_{j=1}^{k} \delta(h_j, s_j)\right)^{\chi_0} &= \sum_{j=1}^{k} \left((h_j + s_j)^{\chi_0} - h_j^{\chi_0}\right) \\ &= \sum_{j=1}^{k} \left(\widetilde{\widetilde{h_j + s_j}} - \tilde{\tilde{h}}_j\right) = \sum_{j=1}^{k} \left(\tilde{\tilde{h}}_j + \bar{\bar{s}}_j - \tilde{\tilde{h}}_j\right) \\ &= \sum_{j=1}^{k} \bar{\bar{s}}_j = \overline{\overline{\sum_{j=1}^{k} s_j}} = 0,\end{aligned}$$

by (9) and (19).

By (24),

$$\chi : R \to R', \qquad \sum_j z_j h_j \hat{+} K \mapsto \sum_j z_j \tilde{\tilde{h}}_j \quad (z_j \in \mathbb{Z})$$

defines a homomorphism, and for all $h \in H$, $s \in S$ we have

$$\begin{aligned}&\tilde{h}^{\chi} = (h \hat{+} K)^{\chi} = \tilde{\tilde{h}}, \\ &\bar{s}^{\chi} = \left(\delta(h, s) \hat{+} K\right)^{\chi} = \widetilde{\widetilde{h + s}} - \tilde{\tilde{h}} = \tilde{\tilde{h}} + \bar{\bar{s}} - \tilde{\tilde{h}} = \bar{\bar{s}},\end{aligned}$$

by (9). This completes the proof of our theorem.

2. Fitting pre-structures.

2.1. Definition. Let H be a semigroup, S a ring, φ a homomorphism of H into the multiplicative semigroup of $\mathrm{End}_{\Lambda}(S)$, ψ an antihomomorphism of H into the multiplicative semigroup of $\mathrm{End}_{\mathrm{P}}(S)$ such that H^φ and H^ψ commute elementwise. (We use the notations introduced in 1.1.) The 4-tuple $\mathscr{F}_0 := (H, S, \varphi, \psi)$ is called a *Fitting pre-structure* if condition (1) holds.

Let $\mathscr{F} = (H^*, S^*, \varphi^*, \psi^*, \sigma^*)$ be a Fitting structure with $S = S^*$, and μ a monomorphism of H into H^*. The pair $(\mathscr{F}, \mu)$ is called a *continuation* of $\mathscr{F}_0$ if

$$(25) \qquad \varphi = \mu\varphi^*, \qquad \psi = \mu\psi^*$$

holds.

Isomorphisms of Fitting pre-structures and Fitting sub-pre-structures are defined in complete analogy to the corresponding notions for Fitting structures, the conditions on σ, σ' being omitted.

If $(\mathscr{F}_1, \mu_1)$ and $(\mathscr{F}_2, \mu_2)$ with $\mathscr{F}_1 = (H^*, S^*, \varphi^*, \psi^*, \sigma^*)$, $\mathscr{F}_2 = (H^{**}, S^{**}, \varphi^{**}, \psi^{**}, \sigma^{**})$ are continuations of $\mathscr{F}_0$, then a *homomorphism* of $(\mathscr{F}_1, \mu_1)$ into $(\mathscr{F}_2, \mu_2)$ is defined to be a homomorphism ω of H^* into H^{**} with the property

$$(26) \qquad \mu_2 = \mu_1\omega.$$

If ω is a bijection of H^* onto H^{**}, our continuations are called *isomorphic*. A continuation $\mathscr{F}$ of $\mathscr{F}_0$ is called *universal* if for any continuation $\mathscr{F}'$ of $\mathscr{F}_0$ there is a homomorphism of $\mathscr{F}$ into $\mathscr{F}'$.

Universal continuations of isomorphic Fitting pre-structures are isomorphic. We now prove the following existence theorem:

2.2. THEOREM. *Every Fitting pre-structure has a universal continuation.*

Proof. Let $\mathscr{F}_0 = (H, S, \varphi, \psi)$ be the Fitting pre-structure given. We put $H^{\mathscr{F}_0} := H \times S$ and define

$$(h_1, s_1)(h_2, s_2) := (h_1h_2,\ s_1h_2 + h_1s_2 + s_1s_2)$$

for all $h_1, h_2 \in H$, $s_1, s_2 \in S$. One readily verifies that $H^{\mathscr{F}_0}$ is a semigroup and the mapping

$$\mu\colon H \to H^{\mathscr{F}_0}, \quad h \mapsto (h, 0)$$

is a monomorphism. We call $H^{\mathscr{F}_0}$ the *continuation semigroup* of $\mathscr{F}_0$. We put $S^* := S$, and define for all $h \in H$, $s \in S$

$$(h, s)^{\varphi^*}\colon S \to S, \quad r \mapsto rh + rs$$

$$(h, s)^{\psi^*}\colon S \to S, \quad r \mapsto hr + sr.$$

Then $(h, s)^{\varphi^*} \in \mathrm{End}_\Lambda(S)$, $(h, s)^{\psi^*} \in \mathrm{End}_P(S)$, since

$$(r_1r)h + (r_1r)s = r_1(rh) + r_1(rs) = r_1(rh + rs),$$

$$h(rr_1) + s(rr_1) = (hr)r_1 + (sr)r_1 = (hr + sr)r_1$$

for all $h \in H$, r, r_1, $s \in S$. We show

(27) φ^* is a homomorphism of $H^{\mathscr{F}_0}$ into $\mathrm{End}_\Lambda(S)$, ψ^* is an antihomomorphism of $H^{\mathscr{F}_0}$ into $\mathrm{End}_\mathrm{P}(S)$, and (25) holds.

We confine ourselves to the assertions about φ, φ^* and leave the proof of the assertions about ψ, ψ^* to the reader. For all $h, h_1, h_2 \in H$, $r, s_1, s_2 \in S$ we have

$$r^{((h_1,s_1)(h_2,s_2))^{\varphi^*}} = r^{(h_1h_2, s_1h_2 + h_1s_2 + s_1s_2)^{\varphi^*}}$$
$$= (rh_1 + rs_1)h_2 + (rh_1 + rs_1)s_2 = r^{(h_1,s_1)^{\varphi^*}(h_2,s_2)^{\varphi^*}}$$

and

$$r^{h^\varphi} = rh + r \cdot 0 = r^{(h,0)^{\varphi^*}} = r^{h^{\mu\varphi^*}}.$$

In the following we write (as in 1.1) $r(h,s)$ for $r^{(h,s)^{\varphi^*}}$, $(h,s)r$ for $r^{(h,s)^{\psi^*}}$ and verify

(28) H^{φ^*} and H^{ψ^*} commute elementwise,

as for all $h_1, h_2 \in H$, $r, s_1, s_2 \in S$

$$((h_1,s_1)r)(h_2,s_2) = (h_1r + s_1r)(h_2,s_2)$$
$$= (h_1r + s_1r)h_2 + (h_1r + s_1r)s_2$$
$$= h_1(rh_2 + rs_2) + s_1(rh_2 + rs_2)$$
$$= (h_1,s_1)(rh_2 + rs_2) = (h_1,s_1)(r(h_2,s_2)).$$

A similar standard calculation yields

(29) $(r_1(h,s))r_2 = r_1((h,s)r_2)$ for all $h \in H$, r_1, r_2, $s \in S$,

i.e., condition (1) is satisfied.

For all $r \in S$ we put

$$r^{\sigma^*}: H^{\mathscr{F}_0} \to H^{\mathscr{F}_0}, \quad (h,s) \mapsto (h, r+s).$$

Then r^{σ^*} is a permutation of $H^{\mathscr{F}_0}$. As before, we write $(h,s) + r$ for $(h,s)^{r^{\sigma^*}}$ and observe

(30) σ^* is a homomorphism of $(S, +)$ into $\mathfrak{S}_{H^{\mathscr{F}_0}}$

(31) $(h,s) + r = (h,s) \Leftrightarrow r = 0$, for all $h \in H$, $r, s \in S$.

We put $\mathscr{F} := (H^{\mathscr{F}_0}, S^*, \varphi^*, \psi^*, \sigma^*)$. In order to show that $\mathscr{F}$ is a Fitting structure it remains to check conditions (2) and (3) which here turn into

(32) $((h_1,s_1) + r)(h_2,s_2) = (h_1,s_1)(h_2,s_2) + r(h_2,s_2)$,
$(h_1,s_1)((h_2,s_2) + r) = (h_1,s_1)(h_2,s_2) + (h_1,s_1)r$

for all $h_1, h_2 \in H$, $r, s_1, s_2 \in S$,

(33) $$((h,s)+r_1)r_2 = (h,s)r_2 + r_1r_2,$$
$$r_1((h,s)+r_2) = r_1(h,s) + r_1r_2,$$

for all $h \in H,\ s, r_1, r_2 \in S$,

both being immediate consequences of our definitions. Now (27) and (30) show that $(\mathscr{F}, \mu)$ is a continuation of $\mathscr{F}_0$, and we claim:

(34) $\mathscr{F}$ is universal.

For if $(\mathscr{F}', \mu')$ with $\mathscr{F}' = (H^{**}, S^{**}, \varphi^{**}, \psi^{**}, \sigma^{**})$ is a continuation of $\mathscr{F}_0$, we put

$$\omega\colon H^{\mathscr{F}_0} \to H^{**}, \quad (h,s) \mapsto h^{\mu'} + s$$

and calculate for $h_1, h_2 \in H$, $s_1, s_2 \in S$ by means of (25):

$$\begin{aligned}((h_1,s_1)(h_2,s_2))^{\omega} &= \left(h_1h_2, s_2^{h_1\psi^*} + s_1^{h_2\varphi^*} + s_1s_2\right)^{\omega}\\ &= (h_1h_2)^{\mu'} + s_2^{h_1\psi^*} + s_1^{h_2\varphi^*} + s_1s_2\\ &= h_1^{\mu'}h_2^{\mu'} + s_2^{h_1^{\mu'}\psi^{**}} + s_1^{h_2^{\mu'}\varphi^{**}} + s_1s_2\\ &= \left(h_1^{\mu'} + s_1\right)\left(h_2^{\mu'} + s_2\right) = (h_1,s_1)^{\omega}(h_2,s_2)^{\omega},\end{aligned}$$

whence ω is a homomorphism. Since for all $h \in H$ we have $h^{\mu\omega} = (h,0)^{\omega} = h^{\mu'}$, we put $\mu' = \mu\omega$ so that (26) holds. Thus the proof of our theorem is complete.

We add some remarks on the continuation semigroup $H^{\mathscr{F}_0}$ of a Fitting pre-structure $\mathscr{F}_0$. For any ring S,

$$s_1 \circ s_2 := s_1 + s_2 + s_1s_2 \qquad (s_1, s_2 \in S)$$

defines an associative composition with identity element 0. As is well known, S is a radical ring if and only if $(S, \circ)$ is a group. We have:

(35) If H has an identity element 1, then

$$\nu\colon S \to H^{\mathscr{F}_0},\ s \mapsto (1,s)$$

is a monomorphism of $(S, \circ)$ into $H^{\mathscr{F}_0}$.

(36) If H has a zero element 0, then

$$\lambda\colon S \to H^{\mathscr{F}_0},\ s \mapsto (0,s)$$

is a monomorphism of $(S, \cdot)$ into $H^{\mathscr{F}_0}$.

(37) An element $(h_0, s_0) \in H^{\mathscr{F}_0}$ is an identity element of $H^{\mathscr{F}_0}$ if and only if h_0 is an identity element of H, $s_0H = 0 = Hs_0$ and $(h_0 + s_0)^{\varphi} = \mathrm{id}_S = (h_0 + s_0)^{\psi}$,

since (h_0, s_0) is an identity element of $H^{\mathscr{F}_0}$ if and only if $h_0h = h = hh_0$ and $h_0s + s_0h + s_0s = s = hs_0 + sh_0 + s_0s$ for all $h \in H$, $s \in S$.

We obviously have

(38) If $H^{\mathscr{F}_0}$ is a group, then so is H.

(39) If H has an identity element 1 such that $1^\varphi = \mathrm{id}_S = 1^\psi$, then $(1,0)$ is an identity element of $H^{\mathscr{F}_0}$, and $H^{\mathscr{F}_0}$ is a group if and only if H is a group and S is a radical ring. In this case $H^{\mathscr{F}_0}$ is a semdirect product of H and $(S, \circ)$.

Herein the statement about $(1,0)$ follows from (37). Now let H be a group and S a radical ring. If for $h \in H$, $s \in S$ the $\circ$-inverse element of sh^{-1} is denoted by $(sh^{-1})^-$, we have

$$(h,s)\big(h^{-1}, h^{-1}(sh^{-1})^-\big) = \big(1, 1(sh^{-1})^- + sh^{-1} + sh^{-1}(sh^{-1})^-\big) = (1,0).$$

Therefore $H^{\mathscr{F}_0}$ is a group. As to the converse, observing (38), it suffices to show that S is a radical ring. But if $s \in S$ and $(h_1, s_1) \in H^{\mathscr{F}_0}$ is the inverse of $(1, s)$, then

$$(1,0) = (1,s)(h_1, s_1) = (h_1, 1 \cdot s_1 + sh_1 + ss_1),$$

$$(1,0) = (h_1, s_1)(1,s) = (h_1, h_1 s + s_1 \cdot 1 + s_1 s),$$

hence $h_1 = 1$ and $s_1 \circ s = 0 = s \circ s_1$. Thus s is $\circ$-invertible. Let finally μ be the embedding of H into $H^{\mathscr{F}_0}$ as in the proof of 2.2 and ν as in (35). Then S^ν is a normal subgroup of $H^{\mathscr{F}_0}$ and isomorphic to $(S, \circ)$, H^μ is a subgroup of $H^{\mathscr{F}_0}$ which is isomorphic to H such that $S^\nu \cap H^\nu = \{(1,0)\}$, and for all $h \in H$, $s \in S$ we have

$$(h,s) = (1, sh^{-1})(h,0) = (sh^{-1})^\nu h^\mu,$$

whence $H^{\mathscr{F}_0} = S^\nu H^\mu$.

In order to give examples of Fitting structures, it is sufficient, by 2.2, to construct Fitting pre-structures:

2.3. Example. Let M be a set and A a subset of M which is an abelian group with respect to some composition $+$. Then the set

$$S(M,A) := \{s \mid s\colon M \to A,\ s|_A \in \mathrm{End}(A)\}$$

is a ring with respect to the compositions

$$\begin{aligned} s_1 + s_2\colon\quad & M \to A \\ & m \mapsto m^{s_1} + m^{s_2} \\ s_1 s_2\colon\quad & M \to A \\ & m \mapsto r(m^{s_1})^{s_2}, \end{aligned}$$

and the set

$$H(M,A) := \{h \mid h\colon M \to M,\ h|_A \in \operatorname{End}(A)\}$$

is a semigroup with respect to the composition

$$\begin{aligned} h_1 h_2\colon\ & M \to M \\ & m \mapsto (m^{h_1})^{h_2}. \end{aligned}$$

For all $h \in H(M,A)$, $s \in S(M,A)$ let $s^{h^{\Phi}}$ (resp. $s^{h^{\Psi}}$) be the usual composition (of mappings) sh (resp. hs).

Then $(H(M,A), S(M,A), \Phi, \Psi)$ is a Fitting pre-structure.

We show that all "well-behaved" Fitting pre-structures can be subsumed under this type of example:

2.4. THEOREM. *Let H be a semigroup with identity element* 1, $\mathscr{F} = (H, S, \varphi, \psi)$ *a Fitting pre-structure such that* $1^{\psi} = \mathrm{id}_S$. *Then there are* M, A *as in* 2.3 *such that* $\mathscr{F}$ *is isomorphic to a Fitting sub-pre-structure of* $(H(M,A), S(M,A), \Phi, \Psi)$.

Proof. W.l.o.g. we may assume $H \cap S = \varnothing$. Then we put $M := H \cup S$, $(A,+) := (S,+)$. For all $h \in H$, $s \in S$ we define

$$h^{\alpha}\colon M \to M, \quad m \mapsto mh = \begin{cases} mh & \text{for } m \in H \\ m^{h^{\varphi}} & \text{for } m \in S \end{cases}$$

$$s^{\beta}\colon M \to A, \quad m \mapsto ms = \begin{cases} s^{m^{\psi}} & \text{for } m \in H \\ ms & \text{for } m \in S. \end{cases}$$

Obviously, α is a homomorphism of H into $H(M,A)$, and β is a homomorphism of S into $S(M,A)$. For $h \in \ker\alpha$ we have $1 = 1^{h^{\alpha}} = 1 \cdot h = h$; thus α is injective. Similarly, $s \in \ker\beta$ implies $0 = 1^{s^{\beta}} = 1 \cdot s = s$, hence β is injective, too. For all $m \in M$, $h \in H$, $s \in S$ we have

$$m^{h^{\alpha}s^{\beta}} = (mh)s = m(hs) = m^{(hs)^{\beta}},$$

$$m^{s^{\beta}h^{\alpha}} = (ms)h = m(sh) = m^{(sh)^{\beta}},$$

$$\begin{aligned} m^{(s^{\beta})^{(h^{\alpha})^{\Phi}}} &= m^{s^{\beta}h^{\alpha}} = (ms)h = m(sh) \\ &= m^{(sh)^{\beta}} = m^{s^{(h^{\varphi}\beta)}} = m^{(s^{\beta})^{\beta^{-1}h^{\varphi}\beta}} = m^{(s^{\beta})^{(h^{\varphi})^{\bar\beta}}}, \end{aligned}$$

$$\begin{aligned} m^{(s^{\beta})^{(h^{\alpha})^{\Psi}}} &= m^{h^{\alpha}s^{\beta}} = (mh)s = m(hs) \\ &= m^{(hs)^{\beta}} = m^{s^{(h^{\psi}\beta)}} = m^{(s^{\beta})^{\beta^{-1}h^{\psi}\beta}} = m^{(s^{\beta})^{(h^{\psi})^{\bar\beta}}}. \end{aligned}$$

Hence $H^\alpha S^\beta \subseteq S^\beta$, $S^\beta H^\alpha \subseteq S^\beta$, $\alpha\Phi = \varphi\bar{\beta}$, $\alpha\Psi = \psi\bar{\beta}$. Therefore $(H^\alpha, S^\beta, \Phi|_{H^\alpha}, \Psi|_{H^\alpha})$ is a Fitting sub-pre-structure of $(H(M, A), S(M, A), \Phi, \Psi)$ and isomorphic to (H, S, φ, ψ). The hypothesis that H has an identity element 1 such that $1^\psi = \mathrm{id}_S$ has only been used to prove that α and β are injective. As is easily seen, for that purpose even weaker hypotheses on H are sufficient.

Let G be a group and A a characteristic abelian normal subgroup of G. Then $\mathscr{F} := (H(G, A), S(G, A), \Phi, \Psi)$ is a Fitting pre-structure. If we put $H := \mathrm{Aut}(G)$, $S := \mathrm{Hom}(G, A)$, then we obviously have $HS \subseteq S$, $SH \subseteq S$, whence (H, S, φ, ψ) with $\varphi = \Phi|_H$, $\psi = \Psi|_H$ is a Fitting sub-pre-structure of $\mathscr{F}$. If we put for $h \in H(G, A)$, $s \in S(G, A)$

$$h^{s^\Sigma}: G \to G, \quad g \mapsto g^h g^s,$$

then $s^\Sigma \in \mathfrak{S}_{H(G,A)}$, and Σ is a homomorphism of $(S(G, A), +)$ into $\mathfrak{S}_{H(G,A)}$ such that $h^{s^\Sigma} = h \Leftrightarrow s = 0$ for all $h \in H(G, A)$, $s \in S(G, A)$. It is easy to see that (2) and (3) hold; thus $(H(G, A), S(G, A), \Phi, \Psi, \Sigma)$ is a Fitting structure. We put $\sigma := \Sigma|_H$. In general, $(H, S, \varphi, \psi, \sigma)$ need not be a Fitting structure. But we have:

2.5. THEOREM. *Let G be a group which has no nontrivial direct abelian factor. Assume $Z(G)$ is finite. Then* $(\mathrm{Aut}(G), \mathrm{Hom}(G, Z(G)), \varphi, \psi, \sigma)$ *is a Fitting structure.*

(*Here φ, ψ, σ have the meaning introduced above.*)

Proof. By our preparatory considerations it suffices to show:

(40) $\quad \alpha^{\zeta^\sigma} \in \mathrm{Aut}(G) \quad$ for all $\alpha \in \mathrm{Aut}(G)$, $\zeta \in \mathrm{Hom}(G, Z(G))$.

Since $\alpha^{\zeta^\sigma} \in \mathrm{Aut}(G)$ if and only if $\mathrm{id}_G^{(\alpha^{-1}\zeta)^\sigma} \in \mathrm{Aut}(G)$, for our proof of (40) we may assume $\alpha = \mathrm{id}_G$. Surely, $\mathrm{id}_G^{\zeta^\sigma}$ is a homomorphism. By our hypotheses on G and $Z(G)$, we have (see [1]) $\zeta^n = 0$ for an appropriate $n \in \mathbb{N}$. Since

$$\mathrm{id}_G^{\zeta^\sigma} \cdot \mathrm{id}_G^{(-\zeta+\zeta^2- \cdots \pm\zeta^{n-1})^\sigma} = \mathrm{id}_G = \mathrm{id}_G^{(-\zeta+\zeta^2- \cdots \pm\zeta^{n-1})^\sigma} \cdot \mathrm{id}_G^{\zeta^\sigma},$$

$\mathrm{id}_G^{\zeta^\sigma}$ is bijective, proving (40).

3. Wreath products over Fitting structures.

3.1. DEFINITION. Let $\mathscr{F} = (H, S, \varphi, \psi, \sigma)$ be a Fitting structure and $n \in \mathbb{N}$. For $\pi \in \mathfrak{S}_n$, a $(n \times n)$-matrix $A = (a_{ij})$ is called a *π-matrix* over $H \cup S$ if

$$a_{ij} \in \begin{cases} H & \text{for } j = i\pi \\ S & \text{for } j \neq i\pi. \end{cases}$$

If $\pi, \pi' \in \mathfrak{S}_n$ and (a_{ij}) is a π-matrix, (a'_{ij}) is a π'-matrix over $H \cup S$, we define, using the product and sum notations introduced in 1.1,

$$(a_{ij})(a'_{ij}) := (b_{ij}) \quad \text{with } b_{ij} = \sum_{k=1}^{n} a_{ik} a'_{kj} \qquad \text{for } 1 \leq i, j \leq n.$$

We observe:

(41) If $\pi, \pi' \in \mathfrak{S}_n$ and A is a π-matrix, A' is a π'-matrix over $H \cup S$, then AA' is a $(\pi\pi')$-matrix over $H \cup S$.

3.2. DEFINITION. Let $\mathscr{F} = (H, S, \varphi, \psi, \sigma)$ be a Fitting structure and $n \in \mathbb{N}$. Let X be a subgroup of $\mathfrak{S}_n$. We put

$$H \underset{S}{\wr} X := \{(A, \pi) | \pi \in X, A \text{ is a } \pi\text{-matrix over } H \cup S\},$$

and define for $(A, \pi), (A', \pi') \in H \underset{S}{\wr} X$

$$(A, \pi)(A', \pi') := (AA', \pi\pi').$$

By (41), this is a composition in $H \underset{S}{\wr} X$. We observe:

(42) $$H \underset{S}{\wr} X \text{ is a semigroup.}$$

We call $H \underset{S}{\wr} X$ the *wreath product of H and X over S*. If $H \cap S = \varnothing$ and $(A, \pi) \in H \underset{S}{\wr} \mathfrak{S}_n$, then π is uniquely determined by A. In this case the elements (A, π) of the wreath product can be identified with their first components, the matrices A.

We add a few simple remarks:

(43) If $H = S$ and X is the trivial subgroup of $\mathfrak{S}_n$, then $H \underset{S}{\wr} X$ is isomorphic to the multiplicative semigroup of the ring $(S)_n$ of all $(n \times n)$-matrices over S.

(44) If $\mathscr{F}' = (H', S', \varphi', \psi', \sigma')$ is a Fitting substructure of $\mathscr{F}$ and X' is a subgroup of X, then $H' \underset{S'}{\wr} X'$ is a subsemigroup of $H \underset{S}{\wr} X$.

(45) The standard wreath product $H \wr X$ is isomorphic to $H \underset{S_0}{\wr} X$ (writing S_0 for the trivial ring); thus it is contained in every wreath product $H \underset{S}{\wr} X$ as a subsemigroup.

For $(H, \{0\}, \varphi_0, \psi_0, \sigma|_{\{0\}})$ is a Fitting substructure of $(H, S, \varphi, \psi, \sigma)$ where we write φ_0, ψ_0 for the (unique) actions of H on $\{0\}$, whence the second part of (45) is a consequence of (44).

Matrix multiplications yield actions of $H \underset{\{0\}}{\wr} X$ on $(S)_n$: We put

$$B^{A^{\hat\varphi}} := BA, \qquad B^{A^{\hat\psi}} = AB \quad \text{for } (A,\pi) \in H \underset{\{0\}}{\wr} X,\ B \in (S)_n,$$

where these products are defined analogously to the matrix product introduced above. Then one readily verifies that $\mathscr{F}_0 := (H \underset{\{0\}}{\wr} X, (S)_n, \hat\varphi, \hat\psi)$ is a Fitting pre-structure. The mapping κ of the continuation semigroup $(H \underset{\{0\}}{\wr} X)^{\mathscr{F}_0}$ into $H \underset{S}{\wr} X$ such that $(A,B)^\kappa = A + B$ for all $(A,\pi) \in (H \underset{\{0\}}{\wr} X)^{\mathscr{F}_0}$, $B \in (S)_n$ is an epimorphism. (The addition of A and B means, as usual, addition of corresponding components, using the notations of 1.1 with regard to σ.) If H has an identity element 1, then

$$\ker\kappa = \left\{ \begin{pmatrix} 1+s_1 & & 0 \\ & \ddots & \\ 0 & & 1+s_n \end{pmatrix} \middle| s_j \in S \right\} \cong \Big(\underbrace{S \oplus \cdots \oplus S}_{n}, \circ\Big).$$

We claim

(46) Let $\mathscr{F} = (H, S, \varphi, \psi, \sigma)$ be a Fitting structure, $n \in \mathbb{N}$ and X a subgroup of $\mathfrak{S}_n$. Suppose H has an identity element 1. Then $H \underset{S}{\wr} X$ is a group if and only if H is a group and S is a radical ring.

For, if H is a group and S is a radical ring, then, by (45), $H \underset{\{0\}}{\wr} X$ is a group and, by [3, I, 7. Th. 3], $(S)_n$ is a radical ring. Therefore, (39) yields that $(H \underset{\{0\}}{\wr} X)^{\mathscr{F}_0}$ is a group, and so is a fortiori the semigroup $H \underset{S}{\wr} X$, being isomorphic to $(H \underset{\{0\}}{\wr} X)^{\mathscr{F}_0}/\ker\kappa$. Conversely, suppose $H \underset{S}{\wr} X$ is a group. Its identity element being denoted by (I, id), where I is the identity matrix, we have $(I + B, \mathrm{id}) \in H \underset{S}{\wr} X$ for every $B \in (S)_n$. If we put $C := (I + B)^{-1}$, then $BC \in (S)_n$, and

$$B \circ (-BC) = B - BC - B^2C = B - B(I+B)C = B - BI = 0,$$

$$(-BC) \circ B = -BC + B - BCB = B - BC(I+B) = B - BI = 0.$$

Therefore $(S)_n$, hence S, is a radical ring. Now let $h \in H$ and set

$$A := \begin{pmatrix} h & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

As before, we have $(A, \mathrm{id}) \in H \underset{S}{\wr} X$, and the entry c in the upper left corner of A^{-1} satisfies $hc = 1 = ch$. The assumption $c \in S$ would imply $1 \in S$ and, regarding the equations $1^\varphi = \mathrm{id}_S = 1^\psi$, 1 were an identity element of S. As S is a radical ring, this would yield $S = \{1\} \subseteq \{1\} \cdot H \subseteq S$, whence $S = \{1\} = H$, and everything would be trivial. But if $c \notin S$, then $c \in H$, and h is an invertible element of H, as desired.

We finally show that our notion of a generalized wreath product is useful for the description of automorphism groups of groups:

To this end let G be a direct indecomposable nonabelian group satisfying the minimum condition on central subgroups, and let $n \in \mathbb{N}$. We put $S := \mathrm{Hom}(G, Z(G))$. Then S is a nil ring, hence a radical ring. By 2.5, $(\mathrm{Aut}(G), S, \varphi, \psi, \sigma)$ is a Fitting structure, and the associated wreath product $(\mathrm{Aut}(G)) \underset{S}{\wr} \mathfrak{S}_n$ is a group, by (46). Since $\mathrm{Aut}(G) \cap S = \varnothing$, we may identify its elements (A, π) with their first components, the matrices A. For $A = (\alpha_{ij}) \in (\mathrm{Aut}(G)) \underset{S}{\wr} \mathfrak{S}_n$ we define

(47) $$(g_1, \dots, g_n)A := \left(\prod_{i=1}^{n} g_i^{\alpha_{i1}}, \dots, \prod_{i=1}^{n} g_i^{\alpha_{in}} \right) \quad \text{for } g_1, \dots, g_n \in G.$$

This gives a mapping α_A of $G \times \cdots \times G$ (n factors) into itself which can formally be regarded as the multiplication of the row $(g_1, \dots, g_n)$ and the matrix A. The properties of A imply:

(48) For $g_1, \dots, g_n \in G$, $j \in \{1, \dots, n\}$, there is at most one $i \in \{1, \dots, n\}$ such that $g_i^{\alpha_{ij}} \notin Z(G)$, viz. $i = j\pi^{-1}$, where π is the permutation determined by A.

Therefore for $g_1, \dots, g_n, h_1, \dots, h_n \in G$, $j \in \{1, \dots, n\}$ we have

$$\prod_{i=1}^{n} (g_i h_i)^{\alpha_{ij}} = \prod_{i=1}^{n} g_i^{\alpha_{ij}} h_i^{\alpha_{ij}} = \prod_{i=1}^{n} g_i^{\alpha_{ij}} \prod_{i=1}^{n} h_i^{\alpha_{ij}},$$

yielding

(49) For all $A \in (\mathrm{Aut}(G)) \underset{S}{\wr} \mathfrak{S}_n$, α_A is an endomorphism of $\underbrace{G \times \cdots \times G}_{n}$.

For $A = (\alpha_{ij})$, $B = (\beta_{ij}) \in (\mathrm{Aut}(G)) \underset{S}{\wr} \mathfrak{S}_n$, $g_1, \dots, g_n \in G$ and $j \in \{1, \dots, n\}$, we have, by (48)

$$\prod_{i=1}^{n} g_i^{\sum_{k=1}^{n} \alpha_{ik}\beta_{kj}} = \prod_{i=1}^{n} \left(\prod_{k=1}^{n} g_i^{\alpha_{ik}} \right)^{\beta_{kj}} = \prod_{k=1}^{n} \left(\prod_{i=1}^{n} g_i^{\alpha_{ik}} \right)^{\beta_{kj}}$$

which implies

(50) $$\alpha_A \alpha_B = \alpha_{AB} \quad \text{for all } A, B \in (\mathrm{Aut}(G)) \underset{S}{\wr} \mathfrak{S}_n.$$

This and the obvious statement

(51) $\alpha_I = \mathrm{id}$, where I is the identity element of $(\mathrm{Aut}(G)) \underset{S}{\wr} \mathfrak{S}_n$

imply:

(52) Associating to each $A \in (\mathrm{Aut}(G)) \wr_S \mathfrak{S}_n$ the automorphism α_A yields a homomorphism ι of $(\mathrm{Aut}(G)) \wr_S \mathfrak{S}_n$ into $\mathrm{Aut}(G \times \cdots \times G)$ (n factors).

If $A \in (\mathrm{Aut}(G)) \wr_S \mathfrak{S}_n$ such that $\alpha_A = \mathrm{id}_{G\times \cdots \times G}$ (n factors), then for all $i \in \{1, \ldots, n\}$, $g \in G$

$$(1, \ldots, 1, g, 1, \ldots, 1) = (1, \ldots, 1, g, 1, \ldots, 1)A = (g^{\alpha_{i1}}, \ldots, g^{\alpha_{in}}),$$

where g is in the ith place. Hence $A = I$. Thus we have

(53) ι is injective.

Finally we claim

(54) ι is surjective.

To this end we define for all $j \in \{1, \ldots, n\}$

$$\varepsilon_j: \quad G \to G \times \underset{n}{\cdots} \times G$$
$$g \mapsto (1, \ldots, 1, g, 1, \ldots, 1),$$
(where g is in the jth place)
$$\delta_j: \quad G \times \underset{n}{\cdots} \times G \to G$$
$$(g_1, \ldots, g_n) \mapsto g_j,$$

and put for all $\alpha \in \mathrm{Aut}(G \times \underset{n}{\cdots} \times G)$

$$A_\alpha := (\alpha_{ij}) \quad \text{with } \alpha_{ij} = \varepsilon_i \alpha \delta_j \quad \text{for } i, j \in \{1, \ldots, n\}.$$

Then α_{ij} is an endomorphism of G, and by [**1**, Satz 2] there is exactly one $\pi \in \mathfrak{S}_n$ such that, for $i \in \{1, \ldots, n\}$, $\alpha_{i,i\pi} \in \mathrm{Aut}(G)$ and $\alpha_{ij} \in S$ for $j \neq i\pi$. Hence $A_\alpha \in (\mathrm{Aut}(G)) \wr_S \mathfrak{S}_n$. By (47), $A_\alpha^\iota = \alpha$, proving (54).

Summarizing, we have proved:

3.3. Theorem. *Let G be a direct indecomposable nonabelian group and suppose $Z(G)$ is finite. Let $n \in \mathbb{N}$ and put $S :=$ Hom$(G, Z(G))$. Then*

$$\mathrm{Aut}\Big(G \times \underset{n}{\cdots} \times G\Big) \cong (\mathrm{Aut}(G)) \wr_S \mathfrak{S}_n.$$

If G satisfies the additional condition $\mathrm{Hom}(G, Z(G)) = 0$ (which is in the finite case equivalent to $(|G/G'|, |Z(G)|) = 1$), our Theorem yields via (45) the well-known statement:

$$\mathrm{Aut}\Big(G \times \underset{n}{\cdots} \times G\Big) \cong (\mathrm{Aut}(G)) \wr \mathfrak{S}_n.$$

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