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PATCH-CONTINUITY OF NORMALIZED RANKS OF MODULES OVER ONE-SIDED NOETHERIAN RINGS

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For J. T. Stafford's continuity theorem, concerning normalized ranks of finitely generated modules at prime ideals of a noetherian ring, a new proof is developed, which is much simpler than the original proof and also extends the theorem to one-sided noetherian rings.

Introduction. The normalized (reduced) rank at a prime ideal P for a finitely generated right module A over a right noetherian ring R is the rational number

$$r_P(A) = \text{length}(A \otimes_R Q_P)/\text{length}(Q_P),$$

where Q_P is the right Goldie quotient ring of R/P. In the main result of the paper, we prove that $r_P(A)$ is a continuous function of P provided the patch topology is used on the prime spectrum of R. A relatively straightforward method of proof is developed, avoiding the technical machinery used by Stafford, to whom the two-sided noetherian case of the theorem is due. If a "generic regularity" condition is assumed, meaning that any element of R regular modulo some prime ideal P remains regular modulo all prime ideals in some patch-neighborhood of P, then $r_P(A)$ is actually a locally constant function of P. This was proved in the two-sided noetherian case by R. B. Warfield, Jr. To put these results in context, we conclude with a brief discussion of some of the known applications.

All rings in this paper are associative with unit, and all modules are unital. This research was partially supported by a National Science Foundation grant.

1. The continuity theorem. A standard result in commutative ring theory is that the local rank of a finitely generated projective module A over a commutative noetherian ring R is a Zariski-continuous function from Spec(R) to $\mathbf{Z}$. The reference to localization can be removed by observing that

$$\text{rank}(A_P) = \dim(A_P/PA_P) = \dim(A \otimes_R Q_P)$$

for all $P \in \text{Spec}(R)$, where we have used Q_P to denote the quotient field of R/P. In the noncommutative case, we are concerned with a finitely generated right module A over a right noetherian ring R. Then Q_P, for $P \in \text{Spec}(R)$, becomes the right Goldie quotient ring of R/P, and $\dim(A \otimes_R Q_P)$ could be replaced by the composition series length of the right Q_P-module $A \otimes_R Q_P$. To obtain a continuity theorem in this setting, we must normalize the length of $A \otimes_R Q_P$, and we must replace the Zariski topology on Spec(R) by the patch topology.

DEFINITION. Given any finitely generated module A over a simple artinian ring Q, we write $\lambda(A)$ for the normalized length of A, that is,

$$\lambda(A) = \text{length}(A)/\text{length}(Q).$$

Note that λ is additive on direct sums of Q-modules.

DEFINITION. Let P be a prime ideal in a right noetherian ring R. For any finitely generated right R-module A, we set

$$r_P(A) = \lambda(A \otimes_R Q_P) = \text{length}(A \otimes_R Q_P)/\text{length}(Q_P).$$

Note that r_P is additive on direct sums of right R-modules. Although r_P need not be additive on short exact sequences, it is subadditive: namely, if $0 \to A \to B \to C \to 0$ is any exact sequence of finitely generated right R-modules, then $r_P(B) \leq r_P(A) + r_P(C)$, because of the exactness of the sequence

$$A \otimes_R Q_P \to B \otimes_R Q_P \to C \otimes_R Q_P \to 0.$$

Also, since Q_P is a flat left (R/P)-module, r_P is additive on short exact sequences of right (R/P)-modules.

DEFINITION. Let R be a ring. For any ideal I of R, define

$$V(I) = \{P \in \text{Spec}(R) \mid P \supseteq I\} \quad \text{and} \quad W(I) = \{P \in \text{Spec}(R) \mid P \not\supseteq I\}.$$

The sets $V(I)$ are exactly the Zariski-closed subsets of Spec(R), and the sets $W(I)$ are exactly the Zariski-open subsets. The *patch topology* on Spec(R) is the topology with a subbasis of closed sets consisting of all the Zariski-closed sets together with all the Zariski-compact Zariski-open sets [**4**, §§2, 8].

With the patch topology, Spec(R) is a compact, Hausdorff, totally disconnected space [**4**, Proposition 4]. In case R has the ACC on ideals, all the Zariski-open subsets of Spec(R) are Zariski-compact [**2**, Proposition 4.3]. Consequently, all the sets $V(I)$ and $W(I)$ are patch-closed and patch-open, and any point $P \in \text{Spec}(R)$ has a patch-neighborhood base consisting of the patch-open sets $V(P) \cap W(I)$ for ideals I properly containing P.

Our objective in this section is to prove that for a finitely generated right module A over a right noetherian ring R, the rule $P \mapsto r_P(A)$ defines a patch-continuous function from Spec(R) to $\mathbf{Q}$. In case A is projective and R is right fully bounded or of right Krull dimension at most one, Warfield and the author proved that the maps $P \mapsto r_P(A)$ are actually locally constant with respect to the patch topology [**2**, Propositions 4.4 and 4.10]. This result has been extended by Warfield in the two-sided noetherian case to cover non-projective modules, and to allow somewhat more general rings, namely those satisfying the generic regularity condition [**13**, Lemma 3]. In the meantime, the continuity theorem was proved by Stafford for two-sided noetherian rings [**8**, Theorem 4.5].

That continuity rather than local constantness is to be expected in general can be seen in the case of the first Weyl algebra over $\mathbf{Z}$. If the ring $R = A_1(\mathbf{Z})$ is written as a differential operator ring $\mathbf{Z}[x][\theta; d/dx]$, and if $A = R/xR$, then $r_0(A) = 0$ yet R has prime ideals $P(q)$ for all prime integers q such that $\bigcap P(q) = 0$ and $r_{P(q)}(A) = 1/q$ [**8**, Proposition 7.3]. That the patch topology must be used instead of the Zariski topology can already be seen in the commutative case. For example, let $R = K[x, y]$ be a polynomial ring in two independent indeterminates over a field K, and let $A = xR + yR$. Then $r_M(A) = 1$ for all maximal ideals $M \neq A$, but $r_A(A) = 2$.

To prove the continuity theorem, we concentrate on the case in which R is a prime right noetherian ring and A is a finitely generated torsion right R-module, so that $r_0(A) = 0$. For any unfaithful subfactor B of A, we have $r_P(B) = 0$ for all $P \in W(\text{ann}(B))$. Hence, the main difficulty occurs with a fully faithful subfactor C. To see that $r_P(C)$ is arbitrarily small for P in suitable patch-neighborhoods of 0, we show that all finite direct sums of copies of C are essentially cyclic. The following two lemmas are used to implement this procedure. The first amounts to a simplified analog of the uniqueness of lengths of critical composition series [**3**, Corollary 2.8; **5**, Theorem 3.1]. It may be obtained as an easy consequence of the Schreier Refinement Theorem, or by a short direct argument, as follows.

LEMMA 1.1. *Let* $A = A_1 \oplus \cdots \oplus A_n$ *where* $A_1, \dots, A_n$ *are* α*-critical modules (for some ordinal* α*), and let*

$$B_0 = 0 \subset B_1 \subset B_2 \subset \cdots \subset B_n$$

be a chain of $n+1$ *submodules of* A. *If* $\text{K.dim}(B_i/B_{i-1}) = \alpha$ *for each* $i = 1, \dots, n$, *then* B_n *is essential in* A.

Proof. As A_1 is uniform, the case $n = 1$ is trivial. Now let $n > 1$, and assume that the lemma holds for direct sums of fewer α-critical modules.

If $B_n \cap A_n = 0$, the projection $p\colon A \to A_1 \oplus \cdots \oplus A_{n-1}$ maps B_n isomorphically into $A_1 \oplus \cdots \oplus A_{n-1}$. By the induction hypothesis, $p(B_{n-1})$ is essential in $A_1 \oplus \cdots \oplus A_{n-1}$, and so $p(B_{n-1}) \cap A_i \neq 0$ for $i = 1, \dots, n-1$. Then $A_i/(p(B_{n-1}) \cap A_i)$ has Krull dimension less than α for each $i = 1, \dots, n-1$, whence

$$\text{K.dim}((A_1 \oplus \cdots \oplus A_{n-1})/p(B_{n-1})) < \alpha.$$

But then $\text{K.dim}(p(B_n)/p(B_{n-1})) < \alpha$, contradicting our hypotheses.

Thus $B_n \cap A_n \neq 0$, and similarly $B_n \cap A_j \neq 0$ for each $j = 1, \dots, n$. Then each $B_n \cap A_j$ is essential in A_j, and therefore B_n is essential in A. □

DEFINITION. A *fully faithful* module is any nonzero module all of whose nonzero submodules are faithful. (In contrast, a *completely* faithful module is a nonzero module all of whose nonzero *subfactors* are faithful.)

DEFINITION. An α*-homogeneous* module (for some ordinal α) is any nonzero module all of whose nonzero submodules have Krull dimension α.

The following lemma was inspired by the result of Eisenbud and Robson that all completely faithful modules of finite length are cyclic [**1**, Lemma 3.1], and by the result of Stafford that over a ring of Krull dimension greater than n, all completely faithful noetherian modules of Krull dimension n can be generated by $n+1$ elements [**7**, Theorem 1.3].

LEMMA 1.2. *Let* R *be a right noetherian ring, and let* A *be a finitely generated, fully faithful,* α*-homogeneous right* R*-module, for some ordinal* α. *If* $\text{r.K.dim}(R) > \alpha$, *then* A *has an essential cyclic submodule.*

Proof. Since A may be replaced by an essential submodule, there is no loss of generality in assuming that $A = A_1 \oplus \cdots \oplus A_n$ for some α-critical submodules A_i.

We claim that there are elements $x_i \in A_i$ for each $i = 1, \dots, n$ such that if $I_0 = R$ and $I_j = \operatorname{ann}_R(\{x_1, \dots, x_j\})$ then K.dim$(I_{j-1}/I_j) = \alpha$ for each $j = 1, \dots, n$. First choose any nonzero element $x_1 \in A_1$, and observe that I_0/I_1 has Krull dimension α because A_1 is α-critical.

Now suppose that $x_1, \dots, x_j$ have been chosen, for some $j < n$. Since

$$\text{K.dim}(R/I_j) = \alpha < \text{r.K.dim}(R),$$

we must have $I_j \neq 0$. As A is fully faithful, there exists $x_{j+1} \in A_{j+1}$ such that $x_{j+1} I_j \neq 0$, and

$$\text{K.dim}(I_j/I_{j+1}) = \text{K.dim}(x_{j+1} I_j) = \alpha.$$

This completes the proof of the claim.

Set $x = x_1 + \cdots + x_n$, and observe that $\operatorname{ann}_R(x) = I_n$, whence

$$xI_{j-1}/xI_j \simeq I_{j-1}/I_j$$

for each $j = 1, \dots, n$. Thus xR has a chain

$$xI_0 = xR \supset xI_1 \supset \cdots \supset xI_n = 0$$

of $n + 1$ submodules in which the successive factors xI_{j-1}/xI_j all have Krull dimension α. By Lemma 1.1, xR is essential in A. □

PROPOSITION 1.3. *Let R be a prime right noetherian ring, let A be a finitely generated torsion right R-module, and let ε be a positive real number. Then R contains a nonzero ideal I such that $r_P(A) < \varepsilon$ for all prime ideals P not containing I.*

Proof. Set $\alpha =$ K.dim(A), and note that the case $\alpha = -1$ is trivial. Now let $\alpha \geq 0$, and assume that the proposition holds for finitely generated torsion modules of Krull dimension less than α. There exists a chain

$$A_0 = 0 \subset A_1 \subset \cdots \subset A_m = A$$

of submodules of A in which the successive subfactors are all critical. Since the functions r_P are subadditive, it suffices to find nonzero ideals $I_1, \dots, I_m$ in R such that $r_P(A_j/A_{j-1}) < \varepsilon/m$ for all prime ideals $P \not\supseteq I_j$. (Take $I = I_1 I_2 \cdots I_m$.) Thus we may assume that A is α-critical.

First suppose that A contains a nonzero unfaithful submodule B, and set $I_1 = \operatorname{ann}_R(B)$. If P is any prime ideal not containing I_1, then B/BP is an unfaithful (R/P)-module and so is torsion over R/P, whence $r_P(B) = 0$. As K.dim$(A/B) < \alpha$, there exists a nonzero ideal I_2 in R such that $r_P(A/B) < \varepsilon$ for all prime ideals $P \not\subseteq I_2$. Thus $r_P(A) < \varepsilon$ for all prime ideals $P \not\supseteq I_1 I_2$.

Finally, suppose that A is fully faithful, and note that because A is a torsion module, r.K.dim$(R) > \alpha$. Choose a positive integer $n > 2/\varepsilon$. Then A^n is a fully faithful α-homogeneous right R-module, and so by Lemma 1.2 it has an essential cyclic submodule C. Since A is α-critical, K.dim$(A^n/C) < \alpha$. Hence, R contains a nonzero ideal I such that $r_P(A^n/C) < 1$ for all prime ideals $P \not\supseteq I$. For such P, we have $r_P(C) \leq 1$ because C is cyclic, whence $r_P(A^n) < 2$. Therefore $r_P(A) < 2/n < \varepsilon$ for all prime ideals $P \not\supseteq I$. □

A. V. Jategaonkar has pointed out that the use of Krull dimension in proving Proposition 1.3 is not essential, for monoform modules can be used in place of critical modules. A proof along such lines is to be incorporated in [**6**].

THEOREM 1.4. *Let A be a finitely generated right module over a right noetherian ring R. Then the rule $P \mapsto r_P(A)$ defines a patch-continuous function from* Spec(R) *to* $\mathbf{Q}$.

Proof. Given a prime ideal T in R and a positive real number ε, we must find an ideal I properly containing T such that

$$|r_P(A) - r_T(A)| < \varepsilon$$

for all prime ideals $P \in V(T) \cap W(I)$. As $r_P(A) = r_{P/T}(A/AT)$ for all $P \in V(T)$, we may reduce to the case that $T = 0$, without loss of generality.

We first find a nonzero ideal I_1 such that $r_P(A) < r_0(A) + \varepsilon$ for all $P \in W(I_1)$. Write $r_0(A) = k/n$ for some $k \in \mathbf{Z}^+$ and $n \in \mathbf{N}$. Then $r_0(A^n) = k$, and hence $A^n \otimes_R Q_0$ is a free right Q_0-module of rank k. Choose a basis $\{x_1, \ldots, x_k\}$ for $A^n \otimes_R Q_0$, and write each $x_i = a_i \otimes c^{-1}$ where $a_1, \ldots, a_k \in A^n$ and c is a regular element in R. Set $B = a_1R + \cdots + a_kR$, and observe that $(A^n/B) \otimes_R Q_0 = 0$, whence A^n/B is a torsion module. By Proposition 1.3, there exists a nonzero ideal I_1 in R such that $r_P(A^n/B) < n\varepsilon$ for all $P \in W(I_1)$. Since B is generated by k elements, $r_P(B) \leq k$ for all prime ideals P. For $P \in W(I_1)$, we thus have $r_P(A^n) < k + n\varepsilon$, and hence

$$r_P(A) < (k/n) + \varepsilon = r_0(A) + \varepsilon.$$

Choose an exact sequence $0 \to K \to F \to A \to 0$ of right R-modules such that F is free of finite rank m, and note that

$$m = r_0(F) = r_0(K) + r_0(A).$$

Applying the argument of the previous paragraph to the module K, we obtain a nonzero ideal I_2 in R such that $r_P(K) < r_0(K) + \varepsilon$ for all $P \in W(I_2)$. As

$$m = r_P(F) \leq r_P(K) + r_P(A)$$

for any such P, we conclude that

$$r_P(A) \geq m - r_P(K) > m - r_0(K) - \varepsilon = r_0(A) - \varepsilon$$

for all $P \in W(I_2)$.

Therefore, $|r_P(A) - r_0(A)| < \varepsilon$ for all $P \in W(I_1 I_2)$. □

2. Generic regularity. In those cases where the continuity theorem can be improved to say that the maps $P \mapsto r_P(A)$ are locally constant with respect to the patch topology, regularity modulo prime ideals defines open sets in the patch topology. Namely, consider a right noetherian ring R, a prime ideal T of R, and an element $x \in \mathscr{C}(T)$, that is, $x \in R$ and $x + T$ is a regular element of R/T. Then $(xR + T)/T$ is an essential right ideal of R/T and $R/(xR + T)$ is a torsion right (R/T)-module, so that $r_T(R/xR) = 0$. If the map $P \mapsto r_P(R/xR)$ is locally constant in the patch topology, the set

$$X = \{P \in \operatorname{Spec}(R) | r_P(R/xR) = 0\}$$

is patch-open. For all $P \in X$, observe that $R/(xR + P)$ is a torsion right (R/P)-module and so $(xR + P)/P$ is an essential right ideal of R/P, whence $x \in \mathscr{C}(P)$. Since x is regular modulo all the prime ideals in the patch-open set X, we may say that x is "generically" regular (with respect to the patch topology).

Conversely, we shall see that generic regularity implies patch-local-constantness of the maps $P \mapsto r_P(A)$.

DEFINITION. Let R be a ring. A subset X of $\operatorname{Spec}(R)$ satisfies the *generic regularity condition* (or is *sparse*, in the terminology of **[6]**) provided that for any prime ideal T of R (not necessarily in X) and any element $x \in \mathscr{C}(T)$, there is a patch-neighborhood U of T such that $x \in \mathscr{C}(P)$ for all $P \in X \cap U$. In other words, there must exist an ideal I properly containing T such that $x \in \mathscr{C}(P)$ for those $P \in X$ such that $P \supseteq T$ but $P \not\supseteq I$.

Two easy cases in which generic regularity occurs were observed in [**6**, Chapter 6] and in [**13**, Lemma 1], namely any set of completely prime ideals in any right noetherian ring, and any set of prime ideals in any right fully bounded right noetherian ring satisfy generic regularity. Another easy case is given in the following proposition.

PROPOSITION 2.1. *Let R be a right noetherian ring with* r.K.dim$(R) \leq 1$. *Then any subset of* Spec(R) *satisfies the generic regularity condition.*

Proof. It suffices to show that generic regularity holds for Spec(R). Consider $T \in \text{Spec}(R)$ and $x \in \mathscr{C}(T)$, and note that $(xR + T)/T$ is an essential right ideal of R/T. If T is a maximal ideal of R, then $\{T\}$ is a patch-neighborhood of T and we are done. Suppose then that T is not a maximal ideal. As r.K.dim$(R/T) \leq 1$, all proper prime factor rings of T and all finitely generated torsion right (R/T)-modules are artinian. In particular, $R/(xR + T)$ must have finite length. Hence, there are at most finitely many maximal ideals $M_i \supset T$ such that the (unique) simple right (R/M_i)-module appears as a composition factor of $R/(xR + T)$. There is an ideal I properly containing T such that each $M_i \supseteq I$.

Let $P \in V(T) \cap W(I)$. Since $x \in \mathscr{C}(T)$ to begin with, suppose that $P \neq T$. Then P is a maximal ideal of R. If A is any composition factor of $R/(xR + T)$, the annihilator of A is either T or one of the M_i. In either case, $P \neq \text{ann}_R(A)$ and so $P \not\subseteq \text{ann}_R(A)$, whence $AP = A$ and $r_P(A) = 0$. As a result, $r_P(R/(xR + T)) = 0$ and hence $R/(xR + P)$ is a torsion right (R/P)-module. Therefore $x \in \mathscr{C}(P)$. □

As mentioned above, any set X of completely prime ideals in a right noetherian ring R (in other words, the right Goldie rank of R/P is 1 for each $P \in X$) is easily seen to satisfy the generic regularity condition. More generally, any set X of prime ideals of R such that there is a bound on the Goldie ranks of the prime factor rings R/P for $P \in X$ must satisfy generic regularity. In case R is noetherian on both sides, this follows from Stafford's continuity theorem [**10**, first proof of Corollary 3.9; **13**, Lemma 1]. Stafford and Snider devised an ultraproduct argument to prove this result for right noetherian rings [**10**, second proof of Corollary 3.9; **6**, Chapter 6], but with the continuity theorem available for right noetherian rings, it is much more direct to use that, as follows.

PROPOSITION 2.2. *Let R be a right noetherian ring and X a subset of* Spec(R). *If there exists a positive integer n such that R/P has right Goldie rank at most n for each $P \in X$, then X satisfies the generic regularity condition.*

Proof. For any $P \in X$, the quotient ring Q_P of R/P has length at most n, and so $\lambda(A) \geq 1/n$ for all nonzero finitely generated right Q_P-modules A. Consequently, $r_P(B)$, for any finitely generated right R-module B, is either 0 or at least $1/n$.

Given $T \in \operatorname{Spec}(R)$ and $x \in \mathscr{C}(T)$, we have $r_T(R/xR) = 0$. By Theorem 1.4, there exists a patch-neighborhood U of T such that $r_P(R/xR) < 1/n$ for all $P \in U$. Thus for $P \in X \cap U$, we have $r_P(R/xR) = 0$, and therefore $x \in \mathscr{C}(P)$. □

Without a bound on the Goldie ranks of prime factor rings, generic regularity need not hold. For an example, consider again

$$R = A_1(\mathbf{Z}) = \mathbf{Z}[x][\theta; d/dx].$$

Since R is a domain, $x \in \mathscr{C}(0)$, yet R has prime ideals $P(q)$ for all prime integers q such that $\bigcap P(q) = 0$ and $x \notin \mathscr{C}(P(q))$ [**8**, Proposition 7.3]. Thus $\operatorname{Spec}(R)$ does not satisfy the generic regularity condition.

That generic regularity implies patch-local-constantness of the maps $P \mapsto r_P(A)$ over right and left noetherian rings was shown by Warfield in [**13**, Lemma 3]. For the right noetherian case, a modification of the proof of Theorem 1.4 may be used, as follows.

THEOREM 2.3. *Let A be a finitely generated right module over a right noetherian ring R, and let X be a subset of* $\operatorname{Spec}(R)$ *that satisfies the generic regularity condition. Then any prime ideal T of R has a patch-neighborhood U such that $r_P(A) = r_T(A)$ for all $P \in X \cap U$.*

Proof. We first prove the analog of Proposition 1.3, namely the case that $r_T(A) = 0$.

Choose generators $a_1, \ldots, a_n$ for A. Since $r_T(A) = 0$, the module A/AT is torsion as a right (R/T)-module, and hence there exists $x \in \mathscr{C}(T)$ such that $a_i x \in AT$ for $i = 1, \ldots, n$. By generic regularity, T has a patch-neighborhood U such that $x \in \mathscr{C}(P)$ for all $P \in X \cap U$. Since U may be intersected with $V(T)$, there is no loss of generality in assuming that $U \subseteq V(T)$. Now for any $P \in X \cap U$ we have $x \in \mathscr{C}(P)$ and $a_i x \in AT \subseteq AP$ for $i = 1, \ldots, n$, whence A/AP is a torsion right (R/P)-module, and thus $r_P(A) = 0$.

With this case in hand, we follow the proof of Theorem 1.4, replacing $[< \varepsilon]$ everywhere by $[= 0]$, and ignoring prime ideals not in X. □

Theorem 2.3 simultaneously generalizes [**2**, Propositions 4.4 and 4.10].

3. Applications. The main applications of the original patch-local-constantness and patch-continuity results concerned estimates for numbers of generators of modules over noetherian rings [**11**, **12**, **8**, **9**] and the identification of extremal states on K_0 of noetherian rings [**2**, **8**]. To spotlight the uses of local constantness and continuity arguments, we briefly indicate their occurrence in some of these results.

(a) By analogy with the Forster-Swan Theorem, one wishes to estimate the number of generators required for a finitely generated right module A over a right noetherian ring R, in terms of the minimal number of generators $g(P, A)$ for the Q_P-module $A \otimes_R Q_P$, for each prime ideal P, or for each J-prime ideal P. Observe that $g(P, A)$ is the smallest integer greater than or equal to $r_P(A)$. If $g(P, A) = 0$, set $b(P, A) = 0$, while if $g(P, A) \neq 0$, set

$$b(P, A) = g(P, A) + J\text{-dim}(R/P).$$

In case R is right fully bounded, Warfield proved that A can be generated by

$$b = \sup\{b(P, A) \mid P \in J\text{-Spec}(R)\}$$

elements [**11**, Theorem B; **12**, Theorems C, 2].

A key reduction step in Warfield's proof requires that, assuming b to be nonzero and finite, only finitely many J-prime ideals P satisfy $b(P, A) = b$ [**11**, Lemma 1; **12**, Lemma 5]. The proofs of these lemmas used patch-local-constantness together with an induction on J-dimension. There is a more direct argument using compactness, which was communicated to the author by Warfield, as follows.

By [**2**, Proposition 4.3], J-Spec(R) is patch-closed in Spec(R) and so is patch-compact. Using local constantness, each $P \in J$-Spec(R) has a patch-neighborhood $U(P)$ of the form $V(P) \cap W(I_P)$ such that $r_Q(A) = r_P(A)$ for all $Q \in U(P)$. Then $g(Q, A) = g(P, A)$ for such Q. If $Q \neq P$, then Q properly contains P, and so if $Q \in J$-Spec(R) we obtain $J\text{-dim}(R/Q) < J\text{-dim}(R/P)$ and hence $b(Q, A) < b$. By compactness, some $U(P_1), \ldots, U(P_n)$ cover J-Spec(R), and any J-prime ideals P which satisfy $b(P, A) = b$ must be among $P_1, \ldots, P_n$.

(b) Warfield also proved that if R is right fully bounded and noetherian on both sides, A can be generated by

$$\max\{g(M, A) \mid M \text{ is a maximal ideal of } R\} + J\text{-dim}(R)$$

elements [**12**, Theorems A, 5]. This estimate follows from the previous one provided that for each J-prime ideal P there is a maximal ideal M such that $g(P, A) \leq g(M, A)$ [**12**, Lemma 6]. This is an easy consequence of local constantness, as follows.

There is an ideal I properly containing P such that $r_Q(A) = r_P(A)$ for all Q in $V(P) \cap W(I)$. Since P is J-prime, there is a right primitive ideal $M \supseteq P$ that does not contain I. Then $r_M(A) = r_P(A)$ and so $g(P, A) = g(M, A)$. As R is right fully bounded, M is a maximal ideal. (Because the prime spectrum of any right fully bounded right noetherian ring satisfies generic regularity, the left noetherian hypothesis for Warfield's second estimate is not needed. Both estimates can be proved assuming only that R is right noetherian, all right primitive factor rings of R are artinian, and J-Spec(R) satisfies generic regularity.)

(c) Stafford's number of generator estimates apply to any finitely generated right module A over any right and left noetherian ring R. In these estimates the value $b(P,A)$ is either 0 or $g(P, A) + \text{r.K.dim}(R/P)$, and A can be generated by

$$\max\{\text{K.dim}(A) + 1, \sup\{b(P, A) | P \in J\text{-Spec}(R)\}\}$$

elements [**8**, Theorem 3.1]. Stafford also proved that A can be generated by

$$\max\{g(P, A) | P \text{ is a right primitive ideal of } R\} + \text{r.K.dim}(R)$$

elements [**8**, Corollary 4.6]. The second estimate follows from the first provided that for each J-prime ideal P there is a right primitive ideal M such that $g(P, A) \leq g(M, A)$, and Stafford obtained this from patch-continuity, as follows.

Since $g(P, A)$ is the smallest integer greater than or equal to $r_P(A)$, we have $r_P(A) > g(P, A) - 1$. By patch-continuity, there is an ideal I properly containing P such that $r_Q(A) > g(P, A) - 1$ for all Q in $V(P) \cap W(I)$. Since P is J-prime, there is a right primitive ideal $M \supseteq P$ that does not contain I. Then $g(M, A) \geq r_M(A) > g(P, A) - 1$, and hence $g(M, A) \geq g(P, A)$, because these are integers.

(d) The *state space* of K_0 of a ring R is a compact convex subset $St(R)$ of the product space of all real-valued functions on $K_0(R)$, consisting of all group homomorphisms $s\colon K_0(R) \to \mathbf{R}$ such that $s([R]) = 1$ and $s([A]) \geq 0$ for all finitely generated projective right R-modules A. In case R is right noetherian, each of the normalized rank functions r_P (for $P \in \text{Spec}(R)$) induces a state $s_P \in \text{St}(R)$, where $s_P([A]) = r_P(A)$ for all A.

Warfield and the author proved that for a right noetherian ring R such that either R is right fully bounded with finite J-dimension, or r.K.dim$(R) \leq 1$, every extreme point of $St(R)$ has the form s_P for some $P \in J$-Spec(R) [2, Theorems 4.5, 4.11]. Stafford derived the same conclusion when R is right and left noetherian with finite right Krull dimension [8, Theorem 6.4]. A key step in these proofs is to show that $\{s_p \mid P \in J\text{-Spec}(R)\}$ is a compact subset of St(R). This follows from the patch-compactness of J-Spec(R), together with the patch-continuity of the map $P \mapsto s_P$, which is an immediate consequence of the patch-continuity of the maps $P \mapsto r_P(A)$.

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