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NEAR ISOMETRIES OF BOCHNER L^1 AND L^∞ SPACES

MICHAEL CAMBERN

Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces, $i = 1, 2$, and let E be a Hilbert space. If the Bochner spaces $L^p(\Omega_1, \Sigma_1, \mu_1, E)$ and $L^p(\Omega_2, \Sigma_2, \mu_2, E)$ are nearly isometric, for either $p = 1$ or $p = \infty$, then $L^1(\Omega_1, \Sigma_1, \mu_1, E)$ is isometric to $L^1(\Omega_2, \Sigma_2, \mu_2, E)$ and hence $L^\infty(\Omega_1, \Sigma_1, \mu_1, E)$ is isometric to $L^\infty(\Omega_2, \Sigma_2, \mu_2, E)$.

Throughout this paper the letter E will denote a Banach space which will often be taken to be Hilbert space. Interaction between elements of a Banach space and those of its dual will be denoted by $\langle \cdot , \cdot \rangle$. We will write $E_1 \cong E_2$ to indicate that the Banach spaces E_1 and E_2 are isometric.

Following Banach, [**2**, p. 242], we will call the Banach spaces E_1 and E_2 nearly isometric if $1 = \inf\{\|T\|\|T^{-1}\|\}$, where T runs through all isomorphisms of E_1 onto E_2. It is of course equivalent to suppose that $1 = \inf\{\|T\|\}$, where $\|T^{-1}\| = 1$, and hence T is a norm-increasing isomorphism of E_1 onto E_2. For if T is any continuous isomorphism of one Banach space onto another, we obtain an isomorphism $\hat{T}$ having the desired properties by defining $\hat{T}$ to be equal to $\|T^{-1}\|T$.

If (Ω, Σ, μ) is a positive measure space and E a Banach space, the Bochner spaces $L^p(\Omega, \Sigma, \mu, E)$ will be denoted by $L^p(\mu, E)$ when there is no danger of confusing the underlying measurable spaces involved, and by $L^p(\mu)$ when E is the scalar field. For the definitions and properties of these spaces we refer to [**8**].

It has been noted by Benyamini [**4**] that, as a consequence of known properties of spaces of continuous functions, if two spaces $L^p(\mu_1)$ and $L^p(\mu_2)$ are nearly isometric, for either $p = 1$ or $p = \infty$, then they are isometric. What we wish to show is that the same conclusion can be drawn for near isometries of certain Bochner spaces. We will prove the following:

THEOREM. *Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces, $i = 1, 2$, and E a Hilbert space. If there exists an isomorphism T, with $\|T^{-1}\| = 1$ and $\|T\| < 3/(2\sqrt{2})$, mapping $L^p(\Omega_1, \Sigma_1, \mu_1, E)$ onto $L^p(\Omega_2, \Sigma_2, \mu_2, E)$ for either $p = 1$ or $p = \infty$, then $L^1(\Omega_1, \Sigma_1, \mu_1, E) \cong L^1(\Omega_2, \Sigma_2, \mu_2, E)$ and $L^\infty(\Omega_1, \Sigma_1, \mu_1, E) \cong L^\infty(\Omega_2, \Sigma_2, \mu_2, E)$.*

In the scalar case, Banyamini's theorem follows from an analogous result for spaces of continuous functions obtained independently by D. Amir [1] and the author [5], [6]. And we note that if E is finite-dimensional with orthnormal basis $\{e_n: n = 1, \ldots, N\}$, and X_i denotes the maximal ideal space of $L^\infty(\mu_i)$, $i = 1, 2$, then it can be shown that $L^\infty(\mu_i, E)$ is isometrically isomorphic to $C(X_i, E)$, the space of continuous functions on X_i to E, under the map $\sum_{n=1}^N f_n e_n \to \sum_{n=1}^N \hat{f}_n e_n$, where $f \to \hat{f}$ is the Gelfand representation of $L^\infty(\mu_i)$. In this case the theorem of this article can be obtained from what is known about isomorphisms of continuous vector-valued functions [7], the result for vectorial L^∞ following directly from [7] and that for L^1 then following by arguments analogous to those given here in the proof of Lemma 8. However when E is infinite dimensional, the continuity on X_i of the coordinate functions $\hat{f}_n$ no longer implies continuity for $\sum_n \hat{f}_n e_n$, even in the presence of separability, and thus the problem requires different methods of approach.

Consequently, in what follows, E will represent an infinite-dimensional Hilbert space. Although the proofs presented here require only that the dimension of E be greater than two, for all finite-dimensional Hilbert spaces E not only does our theorem follow from [7], but it follows with the bound $3/(2\sqrt{2})$ replaced by the better bound $\sqrt{2}$.

Our approach here will be to replace the measure spaces $(\Omega_i, \Sigma_i, \mu_i)$ by measure spaces in which we have a topology, and on which measurable vector-valued functions are very close to being continuous. For this we will require the notion of a perfect measure. Thus, following [3], if X is an extremally disconnected compact Hausdorff space we will call a nonnegative, extended real-valued measure μ defined on the Borel sets $\mathscr{B}(X)$ of X *perfect* if

(i) every nonempty clopen set has positive measure,
(ii) every nowhere dense Borel set has measure zero, and
(iii) every nonempty clopen set contains another nonempty clopen set with finite measure.

The proof of our theorem is now completed by means of a sequence of lemmas.

LEMMA 1. *Let* (Ω, Σ, μ) *be a* σ*-finite measure space, and let* X *be the Stonean space of the measure algebra* Σ/μ. (*Equivalently,* X *is the maximal ideal space of* $L^\infty(\mu)$.) *For* $A \in \Sigma$ *let* $\hat{A}$ *denote the clopen subset of* X *which represents the equivalence class of* A. *Then the measure* $\hat{\mu}$ *defined on the algebra* $\mathscr{A}$ *of clopen subsets of* X *by* $\hat{\mu}(\hat{A}) = \mu(A)$, $A \in \Sigma$, *can be extended to a perfect measure, also denoted by* $\hat{\mu}$, *on* $\mathscr{B}(X)$ *such that* $L^1(\Omega, \Sigma, \mu, E) \cong L^1(X, \mathscr{B}(X), \hat{\mu}, E)$, *and hence* $L^\infty(\Omega, \Sigma, \mu, E) \cong L^\infty(X, \mathscr{B}(X), \hat{\mu}, E)$.

Proof. The set function $\hat{\mu}$ defined above is, indeed, countably additive on $\mathscr{A}$, [**8**, p. 11]. Thus, by the Carathéodory extension theorem, $\hat{\mu}$ has a unique extension to the σ-algebra generated by $\mathscr{A}$. This σ-algebra clearly contains the Baire sets of X.

First suppose that μ is finite. Then, [**9**, p. 351], $\hat{\mu}$ can be further extended to a regular measure on $\mathscr{B}(X)$, which is clearly perfect. (The proof that every nowhere dense Borel set has measure zero is contained in [**10**, p. 18, Lemma 9.4].)

If μ is σ-finite but not finite, let Ω be the disjoint union $\Omega = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in \Sigma$ and $0 < \mu(A_n) < \infty$ for all n. Then define the finite measure μ_0 on Σ by $\mu_0(A) = \sum_{n=1}^{\infty} \mu(A \cap A_n)/(2^n \cdot \mu(A_n))$. Since the μ_0-null and μ-null sets of Σ coincide, the measure algebras Σ/μ and Σ/μ_0 have the same Stonean space X. The measure $\hat{\mu}_0$ defined as above on $\mathscr{A}$ extends to a perfect regular Borel measure on X. And since for sets $A \in \Sigma$ we have

$$\mu(A) = \sum_n \mu(A \cap A_n) = \sum_n 2^n \cdot \mu(A_n)\mu_0(A \cap A_n),$$

it follows that for $\hat{A} \in \mathscr{A}$,

$$\hat{\mu}(\hat{A}) = \sum_n \hat{\mu}(\hat{A} \cap \hat{A}_n) = \sum_n 2^n \cdot \hat{\mu}(\hat{A}_n)\hat{\mu}_0(\hat{A} \cap \hat{A}_n).$$

Thus if we define, for $B \in \mathscr{B}(X)$, $\hat{\mu}(B) = \sum_n 2^n \cdot \hat{\mu}(\hat{A}_n)\hat{\mu}_0(B \cap \hat{A}_n)$, the set function so defined is an extension of $\hat{\mu}$ to a perfect measure on $\mathscr{B}(X)$.

Finally, the map $\sum_{j=1}^{n} e_j \chi_{A_j} \to \sum_{j=1}^{n} e_j \chi_{\hat{A}_j}$ carries the dense subspace of $L^1(\Omega, \Sigma, \mu, E)$ consisting of simple functions isometrically into the corresponding subspace of $L^1(X, \mathscr{B}(X), \hat{\mu}, E)$. Since every $B \in \mathscr{B}(X)$ differs from a clopen set by a set of $\hat{\mu}$-measure zero [**3**, p. 1], the map is actually onto the subspace of simple functions in $L^1(X, \mathscr{B}(X), \hat{\mu}, E)$ and thus extends to an isometry of $L^1(\Omega, \Sigma, \mu, E)$ onto $L^1(X, \mathscr{B}(X), \hat{\mu}, E)$.

LEMMA 2. *Let X and $\hat{\mu}$ be as in Lemma* 1. *Then given a measurable E-valued function F on X there exists an open dense subset U_F of X such that $F|_{U_F}$ is continuous, and $\hat{\mu}(X - U_F) = 0$.*

Proof. First assume that $\hat{\mu}$ is finite. Here we follow the argument given by Peter Greim in [**11**, p. 124]. Take a sequence $\{F_n\}$ of simple functions converging a.e. to F. Again using the fact that each set in $\mathscr{B}(X)$ differs from a clopen set by a set of measure zero, we may suppose that each F_n is continuous. Then Egoroff's theorem shows that F is the almost uniform limit of continuous functions. Hence for each $\varepsilon > 0$ there is a measurable set U_ε such that the restriction of F to U_ε is continuous and $\hat{\mu}(X - U_\varepsilon) < \varepsilon$. Using the facts that $\hat{\mu}$ is regular and that an open set and its closure have the same measure, we may assume that U_ε is clopen. If then U_F is the union of all the U_ε's it has the required properties, for its complement is closed and has measure zero, and thus can contain no non-void open set.

If $\hat{\mu}$ is σ-finite but not finite let $\hat{\mu}_0$ be the finite measure that appears in the proof of Lemma 1. The argument of the preceding paragraph with $\hat{\mu}$ replaced by $\hat{\mu}_0$ then shows that F is continuous on a dense open set U_F with $\hat{\mu}_0(X - U_F) = 0$. Since $\hat{\mu}$ and $\hat{\mu}_0$ have the same null sets, the proof is complete.

As a consequence of Lemma 1 it suffices to prove our theorem for two σ-finite perfect Borel measures defined on extremally disconnected compact Hausdorff spaces. Accordingly, we shall henceforth assume that X and Y are extremally disconnected compact Hausdorff spaces and that μ (resp. ν) is a σ-finite perfect measure on $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$). Until further notice, T will denote a norm-increasing isomorphism of $L^\infty(X, \mathscr{B}(X), \mu, E)$ onto $L^\infty(Y, \mathscr{B}(Y), \nu, E)$ with $\|T\| < 3/(2\sqrt{2})$ and $\|T^{-1}\| = 1$.

LEMMA 3. *If $F \in L^\infty(\mu, E)$ and $\|F(x)\| = 1$ for almost all $x \in X$, then, for almost all $y \in Y$, $(63/64)^{1/2} \le \|T(F)(y)\|$.*

Proof. Suppose, to the contrary, that there exists a set $A \in \mathscr{B}(Y)$ with $\nu(A) > 0$ such that $\|T(F)(y)\| < (63/64)^{1/2}$ for $y \in A$. Again using [**3**, p. 1], $A = B \triangle C$ with B clopen and C of first category. We may assume that $T(F) = 0$ on the ν-null set $B \cap C$ and hence that $\|T(F)(y)\| < (63/64)^{1/2}$ on the clopen set B with $\nu(B) = \nu(A) > 0$. Let $U_{T(F)}$ be an open dense subset of Y on which $T(F)$ is continuous, and whose complement has ν-measure zero. Then $\nu(B \cap U_{T(F)}) = \nu(B) > 0$, $B \cap U_{T(F)}$ is open and $T(F)$ is continuous on this set.

Let $k = \|T(F)\|_\infty$. Choose $y_0 \in B \cap U_{T(F)}$ and take $e \in E$ with $\|e\| = 1$ perpendicular to $T(F)(y_0)$. Then for all scalars α with $|\alpha| = 1$,

$$\left\|T(F)(y) + \alpha(k^2 - 63/64)^{1/2} \cdot e\right\|^2$$

$$\le \left\|T(F)(y)\right\|^2 + 2(k^2 - 63/64)^{1/2} |\langle e, T(F)(y)\rangle| + k^2 - (63/64).$$

For $y = y_0$ the expression on the right is less than k^2, and since it is continuous on $B \cap U_{T(F)}$, there exists a clopen set D containing y_0 such that for all $y \in D$ we have $\|T(F)(y) + \alpha(k^2 - 63/64)^{1/2} \cdot e\|^2 < k^2$.

Thus if we define $G \in L^\infty(\nu, E)$ by $G = (k^2 - 63/64)^{1/2} \cdot e \cdot \chi_D$, then G is a nonzero element of $L^\infty(\nu, E)$ such that for all scalars α with $|\alpha| = 1$ $\|T(F) + \alpha G\|_\infty = k$.

We can suppose that $\|F(x)\| = 1$ for all $x \in X$. We must have

$$\|T^{-1}(G)\|_\infty \geq (1/\|T\|)(k^2 - 63/64)^{1/2} > ((2\sqrt{2})/3)(k^2 - 63/64)^{1/2}.$$

And since the complement of $U_F \cap U_{T^{-1}(G)}$ has μ-measure zero, we can choose $x_0 \in U_F \cap U_{T^{-1}(G)}$ with

$$\|T^{-1}(G)(x_0)\| > ((2\sqrt{2})/3)(k^2 - 63/64)^{1/2}.$$

Next note that if α is a scalar with $|\alpha| = 1$ such that $\operatorname{Re} \alpha\langle T^{-1}(G)(x_0), F(x_0)\rangle \geq 0$, then

$$\|F(x_0) + \alpha T^{-1}(G)(x_0)\|^2 > 1 + (8/9)(k^2 - 63/64).$$

Since $\|F(x) + \alpha T^{-1}(G)(x)\|$ is continuous on $U_F \cap U_{T^{-1}(G)}$, there is a clopen set W containing x_0 such that

$$\|F(x) + \alpha T^{-1}(G)(x)\|^2 > 1 + (8/9)(k^2 - 63/64) \quad \text{on } W.$$

Thus

$$\|F + \alpha T^{-1}(G)\|_\infty^2 > 1 + (8/9)(k^2 - 63/64),$$

and we will have obtained a contradiction to the fact that T^{-1} is norm-decreasing if the quantity on the right is greater than k^2-equivalently if $63/64 < (9 - k^2)/8$. But since $k^2 \leq \|T\|^2 < 9/8$, we indeed have $63/64 < (9 - k^2)/8$ and this contradiction completes the proof of the lemma.

LEMMA 4. *Let $F \in L^\infty(\mu, E)$ with $(63/64)^{1/2} \leq \|F(x)\| \leq \|T\|$ a.e. For $A \in \mathscr{B}(X)$ define $\phi(A) \in \mathscr{B}(Y)$ by $\phi(A) = \{y \in Y\colon \|T(\chi_A F)(y)\| \geq 31/32\}$.*

(i) *If A and B are disjoint measurable subsets of X then $\phi(A) \cap \phi(B)$ is a ν-null set and, modulo a ν-null set, $\phi(A') = [\phi(A)]'$ (where for any set A, A' denotes its complement).*

(ii) *If we furthermore assume that $\|F\|_\infty \leq 1$ then $\|T(\chi_A F)(y)\| < .44$ a.e. on $\phi(A')$.*

Proof. (i). If $\phi(A) \cap \phi(B)$ had positive measure then, proceeding as in the proof of the previous lemma we could find a nonempty clopen set $C \subseteq Y$ on which $\|T(\chi_A F)(y)\| > 15/16$ and $\|T(\chi_B F)(y)\| > 15/16$, and on which both $T(\chi_A F)$ and $T(\chi_B F)$ are continuous. By choosing first a point $y_0 \in C$ and then a scalar α such that

$$\operatorname{Re} \alpha \langle T(\chi_B F)(y_0), T(\chi_A F)(y_0) \rangle \geq 0,$$

it would then follow that $\|T(\chi_A F) + \alpha T(\chi_B F)\|_\infty > (15\sqrt{2})/16 > 1.3$. But since for all scalars α with $|\alpha| = 1$ we have $\|\chi_A F + \alpha \chi_B F\|_\infty \leq \|T\|$, $\|T(\chi_A F) + \alpha T(\chi_B F)\|_\infty$ must be less than $\|T\|^2 < 1.2$, and thus $\phi(A)$ and $\phi(B)$ must be a.e. disjoint.

We wish next to show that the union of $\phi(A)$ and $\phi(A')$ is almost all of Y. Suppose, to the contrary, that on some Borel set $D \subseteq Y$ with $\nu(D) > 0$ we had $\|T(\chi_A F)(y)\| < 31/32$ and $\|T(\chi_{A'} F)(y)\| < 31/32$. We may suppose that D is clopen and that both $T(\chi_A F)$ and $T(\chi_{A'} F)$ are continuous on D. Let $k_1 = \|T(\chi_A F)\|_\infty$, $k_2 = \|T(\chi_{A'} F)\|_\infty$ and $k = \max\{k_1, k_2\}$. Then arguing as in the second paragraph of the proof of Lemma 3, we could find a $G \in L^\infty(\nu, E)$ with $\|G\|_\infty = (k^2 - (31/32)^2)^{1/2}$ and such that $\|T(\chi_A F) + \alpha G\|_\infty \leq k$ and $\|T(\chi_{A'} F) + \alpha G\|_\infty \leq k$ for all scalars α with $|\alpha| = 1$.

Then $\|T^{-1}(G)\|_\infty > ((2\sqrt{2})/3)(k^2 - (31/32)^2)^{1/2}$ so that by an argument analogous to that given in the third paragraph of the proof of Lemma 3, we can find a scalar α with $|\alpha| = 1$ such that

$$\|F + \alpha T^{-1}(G)\|_\infty^2 > 63/64 + (8/9)\left(k^2 - (31/32)^2\right).$$

This latter quantity will be greater than k^2 iff $(9 \cdot 63 - 64 \cdot k^2)/8 \cdot 64 > (31/32)^2$ an inequality which in fact holds since here $\|F\|_\infty \leq \|T\|$ gives $k \leq \|T\|^2$ and hence $k^2 \leq \|T\|^4 < 81/64$. Thus $\|F + \alpha T^{-1}(G)\|_\infty > k$.

But since $\|T(\chi_A F) + \alpha G\|_\infty \leq k$ and $\|T(\chi_{A'} F) + \alpha G\|_\infty \leq k$ and T^{-1} is norm-decreasing, we must have $\|\chi_A F + \alpha T^{-1}(G)\|_\infty \leq k$ and $\|\chi_{A'} F + \alpha T^{-1}(G)\|_\infty \leq k$. Since, for any $x \in X$, $F(x) + \alpha T^{-1}(G)(x)$ is equal either to $\chi_A(x) F(x) + \alpha T^{-1}(G)(x)$ or to $\chi_{A'}(x) F(x) + \alpha T^{-1}(G)(x)$ we have a contradiction and thus, modulo a null set, $\phi(A') = [\phi(A)]'$.

(ii): We know that $\|T(\chi_{A'} F)(x)\| \geq 31/32$ on $\phi(A')$ and thus on this set we must have $\|T(\chi_A F)(x)\|^2 < 9/8 - (31/32)^2 < .19$ so that $\|T(\chi_A F)(x)\| < .44$ a.e. on $\phi(A')$. Otherwise an argument analogous to that of the first paragraph of this proof would provide a contradiction. This concludes the proof of the lemma.

Now fix an $F \in L^\infty(\mu, E)$ with $\|F(x)\| = 1$ a.e. $[\mu]$. Then by Lemma 4(i) we obtain a map ϕ, defined modulo null sets, from $\mathscr{B}(X)$ to $\mathscr{B}(Y)$ determined, for $A \in \mathscr{B}(X)$, by $\phi(A) = \{y \in Y: \|T(\chi_A F)(y)\| \geq 31/32\}$

and satisfying $\phi(A') = [\phi(A)]'$. Next note that $R = \|T\|T^{-1}$ is a norm-increasing isomorphism of $L^\infty(\nu, E)$ onto $L^\infty(\mu, E)$ satisfying $\|R\| < 3/(2\sqrt{2})$ and $\|R^{-1}\| = 1$, and that by Lemma 3,

$$(63/64)^{1/2} \le \|T(F)(y)\| \le \|T\| = \|R\| \quad \text{a.e. } [\nu].$$

Thus, interchanging the roles of T and R, of F and $T(F)$, and those of $\mathscr{B}(X)$ and $\mathscr{B}(Y)$, by Lemma 4(i) we obtain a map ψ from $\mathscr{B}(Y)$ to $\mathscr{B}(X)$ satisfying $\psi(B') = [\psi(B)]'$, modulo null sets, for $B \in \mathscr{B}(Y)$ and determined by $\psi(B) = \{x \in X\colon \|R(\chi_B \cdot T(F))(x)\| \ge 31/32\}$.

LEMMA 5. $\|T^{-1}(\chi_{B'} \cdot T(F))(x)\| < .44$ *a.e. on* $\psi(B)$.

Proof. For $B \in \mathscr{B}(Y)$ we have $\|R(\chi_B \cdot T(F))(x)\| \ge 31/32$ on $\psi(B)$ and thus

$$\|T^{-1}(\chi_B \cdot T(F))(x)\| = \|R(\chi_B \cdot T(F))(x)\| / \|T\| \ge .9 \quad \text{on } \psi(B).$$

If we let

$$P = \operatorname*{ess\,sup}_{x \in \psi(B)} \|T^{-1}(\chi_{B'} \cdot T(F))(x)\|$$

then since $F = T^{-1}(\chi_B \cdot T(F)) + T^{-1}(\chi_{B'} \cdot T(F))$ we must have $(.9)^2 + P^2 \le 1 = \|F\|_\infty$ and hence $P < .44$ as claimed.

LEMMA 6. *If* $B \in \mathscr{B}(Y)$ *then, modulo a* ν*-null set,* $\phi(\psi(B)) = B$. *Hence* ϕ *is a mapping, defined modulo null sets, of* $\mathscr{B}(X)$ *onto* $\mathscr{B}(Y)$.

Proof. Recall that $\phi(\psi(B))$ is the set on which $\|T(\chi_{\psi(B)} \cdot F)(y)\| \ge 31/32$. We have

$$\chi_{\psi(B)} \cdot F = \chi_{\psi(B)} \cdot T^{-1}(\chi_B \cdot T(F)) + \chi_{\psi(B)} \cdot T^{-1}(\chi_{B'} \cdot T(F)).$$

Thus for $x \in \psi(B)$, $\chi_{\psi(B)}(x) \cdot F(x)$ differs from $T^{-1}(\chi_B \cdot T(F))(x)$ by $\chi_{\psi(B)}(x) \cdot T^{-1}(\chi_{B'} \cdot T(F))(x)$ which, by Lemma 5, has norm $< .44$ for almost all x. And for $x \in \psi(B')$, $\chi_{\psi(B)}(x) \cdot F(x) = 0$ and so can differ from $T^{-1}(\chi_B \cdot T(F))(x)$ by this latter function itself which, again by Lemma 5, has norm a.e. $< .44$ on $\psi(B')$. Hence

$$\|\chi_{\psi(B)} \cdot F - T^{-1}(\chi_B \cdot T(F))\|_\infty \le .44$$

and thus

$$(*) \qquad \|T(\chi_{\psi(B)} \cdot F) - \chi_B \cdot T(F)\|_\infty \le .44\|T\| < .47.$$

If we suppose that $\phi(\psi(B)) - B$ has positive ν-measure, we have, for $x \in \phi(\psi(B)) - B$, $\|T(\chi_{\psi(B)} \cdot F)(x)\| \ge 31/32$ and $\chi_B(x)T(F)(x) = 0$, which contradicts $(*)$ above. And if we suppose that $B - \phi(\psi(B))$ has positive ν-measure then, by Lemma 3, $\chi_B(x) \cdot T(F)(x)$ has norm $\geq (63/64)^{1/2} > .99$ a.e. on this set, while by Lemma 4(ii) $T(\chi_{\psi(B)} \cdot F)(x)$ has norm $< .44$ a.e. on $B - \phi(\psi(B)) \subseteq \phi(\psi(B'))$. This again contradicts $(*)$ and so completes the proof of the lemma.

Recall that a mapping ϕ, defined modulo null sets, of $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ is called a *regular set isomorphism* if it satisfies the properties

$$\phi(A') = [\phi(A)]'$$

$$\phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \phi(A_n)$$

and

$$\nu[\phi(A)] = 0 \quad \text{if, and only if, } \mu(A) = 0,$$

for all sets A, A_n in $\mathscr{B}(X)$, **[12]**.

LEMMA 7. *ϕ is a regular set isomorphism of $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$.*

Proof. We have seen that ϕ is a mapping, defined modulo null sets, of $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfying

$$\phi(A') = [\phi(A)]', \qquad A \in \mathscr{B}(X).$$

Note that for $A \in \mathscr{B}(X)$, $\mu(A) \neq 0$ iff $\chi_A \cdot F \neq 0$ in $L^\infty(\mu, E)$ which is true iff $T(\chi_A \cdot F) \neq 0$ in $L^\infty(\nu, E)$ which holds (since T is norm-increasing) iff $\nu[\phi(A)] = \nu(\{y \in Y\colon \|T(\chi_A \cdot F)(y)\| \geq 31/32\}) > 0$. Thus

$$\nu[\psi(A)] = 0 \quad \text{if } \mu(A) = 0.$$

Now suppose that A and B are disjoint set in $\mathscr{B}(X)$. Then by Lemma 4(i) $\phi(A)$ and $\phi(B)$ are a.e. disjoint. Thus if B is a measurable subset of the measurable set A, then B and A' are disjoint so that $\phi(B)$ and $\phi(A')$ are disjoint. Hence $B \subseteq A$ implies that $\phi(B) \subseteq \phi(A)$. The sentence before last also implies that A and B are disjoint iff $\phi(A)$ and $\phi(B)$ are disjoint.

Next assume that $\{A_1, A_2, \dots\}$ is a sequence of measurable subsets of X and let $A = \bigcup_{n=1}^{\infty} A_n$. Then since $A_n \subseteq A$ for all n we have $\phi(A_n) \subseteq \phi(A)$ for all n so that $\bigcup_{n=1}^{\infty} \phi(A_n) \subseteq \phi(A)$. Set $B = \phi(A) - \bigcup_{n=1}^{\infty} \phi(A_n)$. We would like to show that $\nu(B) = 0$.

By Lemma 6 there exists $C \in \mathscr{B}(X)$ with $\phi(C) = B$. By what we established in the paragraph before last, we must have $C \subseteq A$ in this instance. Thus if we suppose that B, hence C, has positive measure then, for some n, C meets A_n in a set of positive measure. But $\phi(A_n)$ and $\phi(C)$ are disjoint, and this contradiction shows that we must have $\nu(B) = 0$. Thus

$$\phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \phi(A_n),$$

completing the proof of the lemma.

The proof of our Theorem is now completed by the following:

LEMMA 8. *If there exists an isomorphism T of $L^p(\mu, E)$ onto $L^p(\nu, E)$ with $\|T^{-1}\| = 1$ and $\|T\| < 3/(2\sqrt{2})$ for $p = 1$ or $p = \infty$ then $L^1(\mu, E) \cong L^1(\nu, E)$ and $L^\infty(\mu, E) \cong L^\infty(\nu, E)$.*

Proof. First suppose that T is such a mapping of $L^\infty(\mu, E)$ onto $L^\infty(\nu, E)$. We have seen that there then exists a regular set isomorphism ϕ of $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$. Then for $B \in \mathscr{B}(Y)$ define $\lambda(B) = \mu[\phi^{-1}(B)]$. If $A \in \mathscr{B}(X)$ we have $\mu(A) = \lambda[\phi(A)] = \int_{\phi(A)} d\lambda$ so that the map $\sum_{j=1}^n e_j \chi_{A_j} \to \sum_{j=1}^n e_j \chi_{\phi(A_j)}$ carries the dense subspace of simple functions in $L^1(X, \mathscr{B}(X), \mu, E)$ isometrically onto the corresponding subspace of $L^1(Y, \mathscr{B}(Y), \lambda, E)$ and can thus be extended to an isometry of $L^1(X, \mathscr{B}(X), \mu, E)$ onto $L^1(Y, \mathscr{B}(Y), \lambda, E)$. Then multiplication by the scalar function $d\lambda/d\nu$ carries this latter space isometrically onto $L^1(Y, \mathscr{B}(Y), \nu, E)$. Hence $L^1(\mu, E) \cong L^1(\nu, E)$ and consequently $L^\infty(\mu, E) \cong L^\infty(\nu, E)$.

If we start with a map T of $L^1(\mu, E)$ onto $L^1(\nu, E)$ satisfying the conditions of the lemma, then T^* is an isomorphism of $L^\infty(\nu, E)$ onto $L^\infty(\mu, E)$ with $\|T^{*-1}\| = 1$ and $\|T^*\| < 3/(2\sqrt{2})$, and the proof then follows as above.

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UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CA 93106