PACIFIC JOURNAL OF MATHEMATICS
Vol. 122, No. 2, 1986

COMMUTANTS OF QUASISIMILAR SUBNORMAL OPERATORS

MARC RAPHAEL

In this paper it is shown if two rationally cyclic subnormal operators are quasisimilar, then they have naturally isomorphic commutants.

1. Introduction. An operator S on a Hilbert space $\mathscr{H}$ is *subnormal* if there is a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a normal operator N on $\mathscr{K}$ such that $N\mathscr{H} \subset \mathscr{H}$ and $S = N|\mathscr{H}$ (the restriction of N to $\mathscr{H}$). The weak* topology on $\mathscr{B}(\mathscr{H})$ is the topology $\mathscr{B}(\mathscr{H})$ has as the Banach space dual of $\mathscr{B}_1(\mathscr{H})$, the trace class operators [**4**].

An operator S in $\mathscr{B}(\mathscr{H})$ is *rationally cyclic* if there is an e in $\mathscr{H}$ such that $\{r(S)e: r \in \operatorname{Rat} \sigma(S)\}$ is dense in $\mathscr{H}$. (Notation: $\operatorname{Rat} \sigma(S)$ is the set of rational functions with poles off $\sigma(S)$, the spectrum of S; and e is called a *rationally cyclic vector* for S.) The commutant of S is the weak* closed subalgebra of $\mathscr{B}(\mathscr{H})$ defined by: $\{S\}' = \{A \in \mathscr{B}(\mathscr{H}): AS = SA\}$.

A *measure* μ is always a compactly supported, positive, regular Borel measure on the complex plane, $\mathbf{C}$. If S is a rationally cyclic subnormal operator then there exist a measure μ and a compact set K containing the support of μ such that S is unitarily equivalent to $S(K, \mu)$, the operator of multiplication by z on $R^2(K, \mu) =$ the closure of $\operatorname{Rat} K$ in $L^2(\mu)$ [**4**]. Yoshino's Theorem [**4**] states that the map from $R^2(K,\mu) \cap L^\infty(\mu)$ onto $\{S(K,\mu)\}'$ given by $\phi \mapsto \phi(S(K,\mu)) =$ multiplication by ϕ is an isometric isomorphism and a weak* homeomorphism. For f in $L^\infty(\mu)$, $\|f\|_\mu$ denotes the μ-essential supremum of f.

If $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces, a bounded linear operator X: $\mathscr{H}_1 \to \mathscr{H}_2$ is said to be *quasi-invertible* if it is injective and has dense range. If $S_j \in \mathscr{B}(\mathscr{H}_j)$ $(j = 1, 2)$, then S_1 is *quasisimilar* to S_2 if there are quasi-invertible operators X_{21}: $\mathscr{H}_1 \to \mathscr{H}_2$ and X_{12}: $\mathscr{H}_2 \to \mathscr{H}_1$ such that $X_{21}S_1 = S_2X_{21}$ and $X_{12}S_2 = S_1X_{12}$. Unlike similarity, quasisimilar operators need not have equal spectra [**5**], though their spectra cannot be disjoint [**5**]. However, quasisimilar subnormal operators must have equal spectra [**1, 2, 3**], and rationally cyclic ones have the same approximate point spectra [**6**].

2. The main result. It is proved in [3] that the weak* algebras generated by quasisimilar subnormal operators are isometrically isomorphic and weak* homeomorphic via a natural map. Theorem 1 shows the existence of a similar map between properly larger algebras, but carries the additional hypothesis that the subnormal operators be rationally cyclic. Some restrictive hypothesis on the subnormal operators is necessary since there is an example of an irreducible subnormal operator similar to a reducible one (p. 276, [4]).

THEOREM 1. *If S_1 and S_2 are quasisimilar rationally cyclic subnormal operators, then there exists an isometric isomorphism and weak* homeomorphism $\Lambda: \{S_1\}' \to \{S_2\}'$ such that $\Lambda(r(S_1)) = r(S_2)$ for all r in* $\operatorname{Rat}\sigma(S_1)$.

Theorem 1 answers an open question posed on page 225 of [4]. Its proof follows from several preliminary results and is postposed until the end of this paper.

The following proposition is essentially the same as Proposition 4.1 of [3]. Its statement and proof for the rationally cyclic case are included here since there will be need for recourse to them later.

PROPOSITION 2 (*Conway*, [4]). *Suppose S_1 and S_2 are rationally cyclic quasisimilar subnormal operators. Then, for $i = 1, 2$, there exist measures μ_i such that S_i is unitarily equivalent to $S(\sigma(S_i), \mu_i)$, constants c_i, and ϕ in $R^2(\sigma(S_1), \mu_1) \cap L^\infty(\mu_1)$ such that*:

(a) *$\{\phi r\colon r \in \operatorname{Rat}\sigma(S_1)\}$ is dense in $R^2(\sigma(S_1), \mu_1)$*;

(b) *for every r in* $\operatorname{Rat} K$,

$$c_2 \int |\phi|^2 |r|^2 \, d\mu_1 \le \int |r|^2 \, d\mu_2 \le c_1 \int |r|^2 \, d\mu_1.$$

Proof. Let $Y_{ij}\colon \mathscr{H}_j \to \mathscr{H}_i$ be quasi-invertible operators such that $Y_{ij} S_j = S_i Y_{ij}$. If e_1 is a rationally cyclic vector for S_1, then it follows that $e_2 = Y_{21} e_1$ is a rationally cyclic vector for S_2. Choose measure μ_j and isomorphisms $U_j\colon \mathscr{H}_j \to R^2(\sigma(S_j), \mu_j)$ with $U_j e_j = 1$ and $U_j S_j U_j^{-1} = S(\sigma(S_j), \mu_j)$ (p. 146 [4]). Let $X_{ij} = U_i Y_{ij} U_j^{-1}$. So $X_{ij}\colon R^2(\sigma(S_j), \mu_j) \to R^2(\sigma(S_i), \mu_i)$ is quasi-invertible. Moreover, it is straightforward to verify that $X_{ij} S(\sigma(S_j), \mu_j) = S(\sigma(S_i), \mu_i) X_{ij}$.

If $r \in \operatorname{Rat}\sigma(S_1)$, then

$$\begin{aligned} X_{21} r &= X_{21} r(S(\sigma(S_1), \mu_1))1 = r(S(\sigma(S_2), \mu_2)) X_{21} 1 \\ &= r(S(\sigma(S_2), \mu_2)) U_2 Y_{21} e_1 = r(S(\sigma(S_2), \mu_2))1 = r. \end{aligned}$$

If $c_1 = \|X_{21}\|^2$, this shows that $\int |r|^2 \, d\mu_2 \le c_1 \int |r|^2 \, d\mu_1$.

To find the constant c_2, notice that $X_{12}X_{21}$ commutes with $S(\sigma(S_1), \mu_1)$. By Yoshino's Theorem there exists ϕ in $R^2(\sigma(S_1), \mu_1) \cap L^\infty(\mu_1)$ such that $X_{12}X_{21}f = \phi f$ for every f in $R^2(\sigma(S_1), \mu_1)$. Hence for r in $\operatorname{Rat} \sigma(S_2)$, $\phi r = X_{12}X_{21}r = X_{12}r$. Let $c_2 = \|X_{12}\|^{-2}$. □

The next two results are the keys to establishing Theorem 1.

PROPOSITION 3. *If S_1 and S_2 are similar rationally cyclic subnormal operators, then the conclusion of Theorem* 1 *is valid.*

Proof. Let $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ be an invertible bounded linear operator such that $S_2 = XS_1X^{-1}$. If $A \in \{S_1\}'$ it is easy to see that $XAX^{-1} \in \{S_2\}'$. Define $\Lambda\colon \{S_1\}' \to \{S_2\}'$ by $\Lambda A = XAX^{-1}$. It is easy to verify that Λ is onto $\{S_2\}'$ and an algebra isomorphism. By Yoshino's Theorem A and ΛA are subnormal operators. Since A and ΛA are similar, they have the same spectra; hence $\|A\| = \|\Lambda A\|$.

Suppose $\{A_\alpha\} \subset \{S_1\}'$ is a net converging weak* to A in $\{S_1\}'$. Let $T \in \mathscr{B}_1(\mathscr{H}_2)$. Then

$$\begin{aligned}\operatorname{tr}(T\Lambda A_\alpha) &= \operatorname{tr}\left(TXA_\alpha X^{-1}\right) = \operatorname{tr}\left(X^{-1}TXA_\alpha\right)\\ &\to \operatorname{tr}(X^{-1}TXA) = \operatorname{tr}(TXAX^{-1}) = \operatorname{tr}(T\Lambda A).\end{aligned}$$

This shows Λ is weak* continuous. Since $\Lambda(r(S_1)) = r(S_2)$ for all r in $\operatorname{Rat} \sigma(S_1)$, the proof is complete. □

SIMPLIFICATION THEOREM 4. *For $i = 1, 2$ let μ_i be a measure and K be a compact subset of* $\mathbf{C}$ *such that the support of μ_i is contained in K. Suppose $\phi \in R^2(K, \mu_1) \cap L^\infty(\mu_1)$ and $\{\phi r\colon r \in \operatorname{Rat} K\}$ is dense in $R^2(K, \mu_1)$.*

If the bounded linear operators $X_{21}^1\colon R^2(K, \mu_1) \to R^2(K, \mu_2)$ and $X_{12}^\phi\colon R^2(K, \mu_2) \to R^2(K, \mu_1)$ are quasi-invertible and satisfy

(a) *$X_{21}^1 r = r$ for all r in* $\operatorname{Rat} K$

(b) *$X_{12}^\phi r = \phi r$ for all r in* $\operatorname{Rat} K$,

then $\Lambda\colon R^2(K, \mu_1) \cap L^\infty(\mu_1) \to R^2(K, \mu_2) \cap L^\infty(\mu_2)$, defined by

$$\Lambda f = X_{21}^1 f,$$

is an isometric isomorphism and a weak homeomorphism.*

The author does not know if the map in Theorem 1 is uniquely determined by the condition that $r(S_1) \mapsto r(S_2)$ for all r in $\operatorname{Rat} \sigma(S_1)$. Couched in terms of the Simplification Theorem, it is easy to see that the aforementioned map is unique if the following query has an affirmative answer.

If Λ: $R^2(K,\mu) \cap L^\infty(\mu) \to R^2(K,\mu) \cap L^\infty(\mu)$ is an isometric isomorphism and a weak* homeomorphism that is the identity map when restricted to $\operatorname{Rat} K$, then must Λ be the identity map on $R^2(K,\mu) \cap L^\infty(\mu)$?

Two lemmas are now stated and proved. They are used in the proof of the Simplification Theorem.

LEMMA 5. *Assume the hypothesis of Simplification Theorem* 4 *with the exception that* X_{21}^1 *need not be bounded on* $\operatorname{Rat} K$. *Then for all* f *in* $R^2(K,\mu_2) \cap L^\infty(\mu_2)$ *and all non-negative integers* n,

$$X_{12}^\phi f^n = \left(X_{12}^\phi f\right)^n \cdot \phi^{1-n} \quad \mu_1 - a.e. \tag{6}$$

Consequently, X_{12}^ϕ *maps* $R^2(K,\mu_2) \cap L^\infty(\mu_2)$ *into* $R^2(K,\mu_1) \cap L^\infty(\mu_1)$ *and* $\|X_{12}^\phi f\|_{\mu_1} \le \|\phi\|_{\mu_1} \|f\|_{\mu_2}$.

Proof. Equation (6) will be shown by induction on n. If $n = 1$ then (6) certainly holds. Suppose (6) is true for $1 \le k \le n$. To show that (6) holds for $k = n + 1$, let $f \in R^2(K,\mu_2) \cap L^\infty(\mu_2)$ and let $\{r_s\}$ and $\{q_m\}$ be sequences in $\operatorname{Rat} K$ such that $r_s \to f$ in $L^2(\mu_2)$ and $q_m \to f^n$ in $L^2(\mu_2)$. Then

$$\begin{aligned} X_{12}^\phi f^{n+1} &= \lim_{m\to\infty} X_{12}^\phi (f q_m) = \lim_{m\to\infty} \left\{ X_{12}^\phi \left(\lim_{s\to\infty} r_s q_m \right) \right\} \\ &= \lim_{m\to\infty} \left\{ q_m \left(\lim_{s\to\infty} X_{12}^\phi r_s \right) \right\} = \phi^{-1} \left(\lim_{m\to\infty} X_{12}^\phi q_m \right) \left(X_{12}^\phi f \right) \\ &= \phi^{-1} \left(X_{12}^\phi f^n \right) \left(X_{12}^\phi f \right) = \left(X_{12}^\phi f \right)^{n+1} \cdot \phi^{-n}, \end{aligned}$$

where all limits are taken in the appropriate L^2-space and the induction hypothesis is used to obtain the next to last equality. This substantiates (6).

In order to obtain the norm estimate involving X_{12}^ϕ let $f \in R^2(K,\mu_2) \cap L^\infty(\mu_2)$ such that $\|f\|_{\mu_2} < 1$. Since $f^n \to 0$ in $L^2(\mu_2)$ and X_{12}^ϕ is bounded, it follows from (6) that

$$\left\| X_{12}^\phi f^n \right\|_2^2 = \int \left| X_{12}^\phi f \right|^{2n} |\phi|^{2-2n} \, d\mu_1 \to 0$$

as $n \to \infty$. Hence $|X_{12}^\phi f| \le |\phi|$ μ_1-a.e.; so $\|X_{12}^\phi f\|_{\mu_1} \le \|\phi\|_{\mu_1}$. Since $\|f\|_{\mu_2} < 1$, it follow easily that $\|X_{12}^\phi f\|_{\mu_1} \le \|\phi\|_{\mu_1} \|f\|_{\mu_2}$ for all f in $R^2(K,\mu_2) \cap L^\infty(\mu_2)$. □

LEMMA 7. *Assume the hypothesis and notation of Simplification Theorem 4. Then* Λ *is an isometry on* $R^2(K,\mu_1)\cap L^\infty(\mu_1)$, *and* $X_{21}^1(fg) = (X_{21}^1 f)(X_{21}^1 g)$ *whenever* $f\in R^2(K,\mu_1)\cap L^\infty(\mu_1)$ *and* $g\in R^2(K,\mu_1)$.

Proof. Let $\{g_n\}$ be a sequence in Rat K such that $g_n\to g$ in $L^2(\mu_1)$. Then

$$X_{21}^1(fg_n) = X_{21}^1 g_n(S(K,\mu_1))f = g_n(S(K,\mu_2))X_{21}^1 f = (X_{21}^1 g_n)(X_{21}^1 f).$$

Letting $n\to\infty$ shows that $X_{21}^1(fg) = (X_{21}^1 f)(X_{21}^1 g)$.

It follows from Lemma 5 with $\phi = 1$ that $\|X_{21}^1 f\|_{\mu_2}\le \|f\|_{\mu_1}$ when $f\in R^2(K,\mu_1)\cap L^\infty(\mu_1)$. To show the desired reverse inequality, it may be assumed that $\|X_{21}^1 f\|_{\mu_2} < 1$ and $\|\phi\|_{\mu_1} = 1$.

Let $\{r_s\}$ be a sequence in Rat K such that $r_s\to f$ in $L^2(\mu_1)$. Then

$$(X_{12}^\phi X_{21}^1)(f) = \lim_{s\to\infty}(X_{12}^\phi X_{21}^1)(r_s) = \lim_{s\to\infty}\phi r_s = \phi f,$$

where all limits are in $L^2(\mu_1)$. So, for $n\ge 1$, it follows easily from (6) that

$$X_{12}^\phi(X_{21}^1 f)^n = \phi f^n \quad \mu_1\text{-a.e.}$$

Since, by the last part of Lemma 5, $\|X_{12}^\phi(X_{21}^1 f)^n\|_{\mu_1}\le\|(X_{21}^1 f)^n\|_{\mu_2}\to 0$ as $n\to\infty$, it must be that $\|f\|_{\mu_1}\le 1$. This completes the proof of Lemma 7. □

Proof of Simplification Theorem 4. By Lemma 7 and the Krein-Smulian Theorem, it suffices to show Λ is onto $R^2(K,\mu_2)\cap L^\infty(\mu_2)$ and weak* continuous. By the proof of Proposition 2 it is possible to do the following: choose measures ν_i $(i = 1, 2)$ so that if $T_i = S(K,\nu_i)$ and $S_i = S(K,\mu_i)$, then S_i and T_i are unitarily equivalent; and the operators defined by

$$Y_{12}^1\colon R^2(K,\nu_2)\to R^2(K,\nu_1)$$
$$r\mapsto r,\quad\text{and}$$
$$Y_{21}^\lambda\colon R^2(K,\nu_1)\to R^2(K,\nu_2)$$
$$r\mapsto \lambda r$$

are quasi-invertible. (Here λ is a fixed member of $R^2(K,\nu_2)\cap L^\infty(\nu_2)$ and $r\in$ Rat K.)

In order to prove that Λ is onto $R^2(K,\mu_2)\cap L^\infty(\mu_2)$ it is shown that $X_{21}^1 Yf = f$ for all f in $R^2(K,\mu_2)\cap L^\infty(\mu_2)$, where $Y = I_1 Y_{12}^1 I_2$ and I_1: $R^2(K,\nu_1)\cap L^\infty(\nu_1)\to R^2(K,\mu_1)\cap L^\infty(\mu_1)$, I_2: $R^2(K,\mu_2)\cap L^\infty(\mu_2)\to R^2(K,\nu_2)\cap L^\infty(\nu_2)$ are the "identity" maps in the proof of Proposition 3.

Since S_1 and T_1 are unitarily equivalent operators there exist an isomorphism V: $R^2(K,\nu_1)\to R^2(K,\mu_1)$ and $\Psi\in R^2(K,\mu_1)$ such that

$Vr = \Psi r$ for all r in Rat K. Given f in $R^2(K,\mu_2) \cap L^\infty(\mu_2)$ let $\{r_s\}$ be a sequence in Rat K such that

$$(8) \qquad \begin{aligned} &r_s \to f = I_2 f \text{ in } L^2(\nu_2) \quad \text{and} \quad \nu_2\text{-a.e., and} \\ &r_s \to Y_{12}^1 I_2 f \text{ in } L^2(\nu_1) \quad \text{and} \quad \nu_1\text{-a.e.} \end{aligned}$$

Then,

$$\begin{aligned} [X_{21}^1\Psi][X_{21}^1(I_1Y_{12}^1I_2f)] &= X_{21}^1[\Psi(I_1Y_{12}^1I_2f)] \\ &= \lim_{s\to\infty}[X_{21}^1Vr_s] = \lim_{s\to\infty} X_{21}^1(\Psi r_s) \\ &= \lim_{s\to\infty}(X_{21}^1\Psi)(X_{21}^1r_s) = (X_{21}^1\Psi)f. \end{aligned}$$

In the above equation, Lemma 7 is used in the first and fourth equality. All limits are in $L^2(\mu_2)$ and are justifiable by (8). Since $X_{21}^1\Psi \neq 0$ μ_2-a.e., it follows that $X_{21}^1Yf = f$.

To finish the proof note that S_1 and $T = S(K,\mu_1+\mu_2)$ are similar via W: $R^2(K,\mu_1) \to R^2(K,\mu_1+\mu_2)$, where $Wr = r$ for all r in Rat K. Since Z: $R^2(K,\mu_1+\mu_2) \cap L^\infty(\mu_1+\mu_2) \to R^2(K,\mu_2) \cap L^\infty(\mu_2)$ defined by $Zf = f$ is weak* continuous, it follows from Proposition 3 that $\Lambda = ZW|R^2(K,\mu_1) \cap L^\infty(\mu_1)$ is also weak* continuous. □

The proof of Theorem 1 *can now be given*. By Proposition 2 and the Simplification Theorem, there exist measures μ_i $(i = 1, 2)$ such that S_i and $S(\sigma(S_i),\mu_i)$ are unitarily equivalent and $\{S(\sigma(S_i),\mu_i)\}'$ satisfy the conclusion of the Simplification Theorem. An application of Proposition 3 completes the proof. □

References

[1] W. S. Clary, *Equality of spectra of quasi-similar hyponormal operators*, Proc. Amer. Math. Soc., **53** (1975), 88–90.

[2] ———, *Quasi-similarity and subnormal operators*, Ph.D. thesis, University of Michigan, 1973.

[3] J. B. Conway, *On quasisimilarity for subnormal operators*, Illinois J. Math., **24** (1980), 689–702.

[4] ———, *Subnormal Operators*, Pitman Publishing Inc., Marshfield, Mass., 1981.

[5] T. B. Hoover, *Quasisimilarity of operators*, Illinois J. Math., **16** (1972), 678–686.

[6] M. Raphael, *Quasisimilarity and essential spectra for subnormal operators*, Indiana Univ. Math. J., **31**, No. 2 (1982), 243–246.

Received June 1, 1984.

University of Missouri
Rolla, MO 65401