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NON-ASSOCIATIVE L^p - SPACES

BRUNO IOCHUM

L^p-spaces associated to Jordan algebras with traces are defined. They have the usual properties of their equivalents on a measure space, but the product is non-associative.

1. Introduction. The Banach lattices $L^p(Z, \nu)$ where (Z, ν) is a measure space can be extended in a non commutative algebraic context then it remains a commutative notion under the form of a trace on a von Neumann algebra ([**12**], [**34**], [**29**]). Here we show that it is possible to use the same approach in the non associative case of Jordan-Banach algebras with predual (J.B.W. algebras). The Jordan algebras appeared in the thirties as a formalism of quantum mechanics and are useful in this context (see for instance the references in [**19**]). There are many connections between operator-algebras and Jordan Banach algebras and this explains why we use ideas from von Neumann algebra theory, especially those in Dixmier's paper [**12**]. Actually it is possible to prove part of the results of this paper using the paper [**2**] by Ajupov and a structure theorem on J.B.W. algebras (see [**35**], [**16**]) which reduces the problem to the study of all possible cases. However we prefer a global and direct approach since in our opinion the close relations between the non associative but commutative product of Jordan algebras and the associative but not commutative product of operator algebras are not sufficiently well understood.

The paper is organized as follows: In section 2 we recall the necessary details about Jordan Banach algebras and semifinite traces. In section 3 the L^p-spaces are defined and we prove that $(L^p)^*$ is isomorphic to L^q for $p > 1$, where $1/p + 1/q = 1$. It follows from Clarkson's inequalities that these spaces are uniformly convex and uniformly smooth for $p > 1$. The case $0 < p < 1$ is also investigated. Section 4 contains related results.

2. Notations and basic properties. A Jordan-Banach (J.B.) algebra M is a real Banach space and a real Jordan algebra such that

$$\|xy\| \leq \|x\| \, \|y\|$$

$$\|x^2\| = \|x\|^2$$

$$\|x^2\| \leq \|x^2 + y^2\| \qquad x, y \text{ in } M$$

(see [4]). Let L_x be the multiplication operator by x: $L_x y = xy$ and $U_x = 2(L_x)^2 - L_{x^2}$ the triple product. For instance, if M is the selfadjoint part of a C^*-algebra, and $L_x y = 2^{-1}(x \cdot y + y \cdot x)$ where $\cdot$ is the operator product, then $U_x y = x \cdot y \cdot x$. If M is the dual of a (necessarily unique) Banach space M_* then M is called a J.B.W. algebra ([35], [16]).

Note that if $M^+ = \{a^2 | a \in M\}$ then M^+ is a closed convex cone such that $M = M^+ - M^+$. In particular $|x| = (x^2)^{1/2} \in M^+$. If S is the set of symmetries (i.e. $s^2 = \mathbf{1}$) and M is a J.B.W. algebra then each x in M has the decomposition $x = s|x|$ where $s \in S$. A trace φ on a J.B. algebra M is an application from M^+ to $[0, \infty]$ satisfying the following:

$$\varphi(x + y) = \varphi(x) + \varphi(y),$$
$$\varphi(\lambda x) = \lambda \varphi(x),$$
$$\varphi(U_x y^2) = \varphi(U_y x^2), \qquad x, y \in M, \ \lambda \in \mathbf{R}^+.$$

Define $M_1^+ = \{x \in M^+ | \varphi(x) < \infty\}$ and $\leq$ the order in M given by M^+.

φ is said to be *faithful* if $\varphi(x) = 0$ yields $x = 0$, *semifinite* if $\varphi(x) = \sup\{\varphi(y) | y \in M_1^+, \ y \leq x\}$, *normal* if $\varphi(x_\alpha) \uparrow \varphi(x)$ for every increasing net $x_\alpha \uparrow x$ x_α, x in M^+ (M is a J.B.W. algebra).

Recall some basic facts on traces ([19] V.1.2, V.1.4, [20], [31] and [3]) where M now as in the following denotes a J.B.W. algebra and φ a semifinite faithful normal trace.

LEMMA 1.

(i) $M_1 = M_1^+ - M_1^+$ *is a J.B. ideal in* M *and* φ *can be extended by linearity to* M_1.

(ii) $\varphi(x(yz)) = \varphi((xy)z) = \varphi(y(xz))$, $x, y \in M$, $z \in M_1$.

(iii) $\varphi(U_s x) = \varphi(x)$, $s \in S$, $x \in M^+$.

(iv) $\varphi(U_x z + U_y z) = \varphi(U_{(x^2+y^2)^{1/2}} z)$, $x, y \in M$, $z \in M_1$.

(v) $\varphi(U_x y) = \varphi(x^2 y)$, $x \in M$, $y \in M_1$.

In particular $\varphi(xy) \geq 0$ *if* $x \in M^+$, $y \in M_1^+$.

(vi) $\varphi(e^{t[L_x, L_y]} z) = \varphi(z)$ $t \in \mathbf{R}$, $x, y \in M$, $z \in M_1$.

(vii) $\varphi(x(U_z y)) = \varphi(y(U_z x))$, $x, z \in M$, $y \in M_1$.

(viii) $\varphi(|xy|) \leq \|x\| \varphi(|y|)$, $x \in M$, $y \in M_1$.

(ix) *The* $\sigma(M, M_*)$ *closure of* M_1 *is* M.

DEFINITION 2. For $x \in M$ and $0 < p < \infty$ define

$$\|x\|_p = \varphi(|x|^p)^{1/p} \in [0, \infty].$$

We adopt the convention $\|x\|_\infty = \|x\|$.

Note that $\|x\|_p < \infty$ for all $p \in [1, \infty]$ if $x \in M_1$.

LEMMA 3.

(i) *Hölder inequality*: *If* $p, q \in [1, \infty]$ *such that* $1/p + 1/q = 1$, *then* $\|xy\|_1 \leq \|x\|_p \|y\|_q$, $x, y \in M_1$.

(ii) $\|x\|_p = \sup\{|\varphi(xy)| \mid y \in M, \|y\|_q \leq 1\}$ *and the supremum is attained.*

(iii) $\|xy\|_p \leq \|x\| \|y\|_p$, $x \in M$, $y \in M_1$.

(iv) $\|\cdot\|_p$ *is a norm in* M_1.

Proof. (i) We first prove the inequality $\varphi(a(xy)) \leq \|x\|_p \|y\|_q \|a\|$, where $a \in M$, for $x(\underline{\lambda})$ and $y(\underline{\mu})$ of the form $\Sigma_{i=1}^n \lambda_i e_i$ (resp. $\Sigma_{j=1}^m \mu_j f_j$) where $\underline{\lambda} = (\lambda_i)_i$ (resp. $\underline{\mu} = (\mu_j)_j$ is a set of reals and $\{e_i\}_i$ (resp. $\{f_j\}_j$) is an orthogonal family of non zero idempotents in M_1.

Let $a \in M$ such that $\|a\| \leq 1$ and let A_a be the bilinear form defined by

$$A_a(\underline{\lambda}, \underline{\mu}) = \varphi\big(a\big(x(\underline{\lambda})\, y(\underline{\mu})\big)\big) = \sum_{i,j} \lambda_i \mu_j \varphi\big(a(e_i f_j)\big).$$

The Riesz's theorem (cf. [**32**], p. 472) asserts the convexity of $\log(M_a(p', q'))$ for (p', q') in the triangle $0 \leq p' \leq 1$, $0 \leq q' \leq 1$, $p' + q' \geq 1$ where $M_a(p', q') = \sup|A_a(\underline{\lambda}, \underline{\mu})|$ for $\underline{\lambda}$ and $\underline{\mu}$ such that

$$\|x(\underline{\lambda})\|_{1/p'} = \sum_{i=1}^n |\lambda_i|^{1/p'} \varphi(e_i) \leq 1$$

and

$$\|y(\underline{\mu})\|_{1/q'} = \sum_{j=1}^m |\mu_j|^{1/q'} \varphi(f_j) \leq 1$$

(take $\sup_i |\lambda_i| \leq 1$ if $p' = 0$ and $\sup_j |\mu_j| \leq 1$ if $q' = 0$). Thus, if $p' = 1$ and $q' = 0$, the condition $|\varphi(a(xy))| \leq 1$ for $\|x\|_1 \leq 1$, $\|y\|_\infty \leq 1$ (Lemma 1) yields $\log M_a(1, 0) \leq 0$. For the same reason $\log M_a(0, 1) \leq 0$, thus for $p' = 1/p$, $q' = 1/q$, $\log M_a(1/p, 1/q) \leq 0$, so that the conditions $\|x\|_p \leq 1$ and $\|y\|_q \leq 1$ yield $\varphi(a(xy)) \leq 1$ as claimed.

Let now x, y be arbitrary in M_1. By spectral theory, there exists $\{x_n\}_{n \in \mathbf{N}}$ such that x_n is in the J.B.W. algebra generated by x, $0 \leq x_n \leq \mathbf{1}$, x_n tends to $\mathbf{1}$ in the $s(M, M_*)$-topology and xx_n has the previous form. Let $\{y_n\}_n$ be analogous sequence for y. The application: $z \in M \to \varphi((x(a(yy_m)))z)$ is $\sigma(M, M_*)$ continuous, hence by Lemma 1,

$$\varphi\big(x(a(yy_m))\big) = \lim_n \varphi\big((x(a(yy_m)))x_n\big) = \lim_n \varphi\big((xx_n)(a(yy_m))\big)$$

$$\leq \|yy_m\|_q \|a\| \lim_n \|xx_n\|_p.$$

Since

$$\|xx_n\|_p = \varphi(|x|^p x_n^p)^{1/p} \leq \|x_n^p\|^{1/p} \varphi(|x|^p)^{1/p} \leq \|x\|_p$$

we obtain the result by taking the limit in m and afterwards choose $a = s$ where s is the symmetry given by $s(xy) = |xy|$.

(ii) For $x \in M$, $|x| = sx$, where $s \in S$, thus for $p = 1$,

$$\varphi(|x|) \leq \sup\{|\varphi(xy)| \mid \|y\| \leq 1\} \leq \varphi(|x|).$$

For $p > 1$, take $z = \|x\|_p^{-p/q} s|x|^{p-1}$. Clearly $\|y\|_q = 1$ and $\varphi(xz) = \|x\|_p$.

(iii) Follows from (ii) and Lemma 1.

(iv) $\|\cdot\|_p$ is a seminorm as sup-limit of seminorms and a norm by the faithfulness of φ. □

3. L^p-spaces.

DEFINITION 4. For $p \in [1, \infty[$, $L^p = \overline{M_1}^{\|\,\|_p}$ is a Banach space. For $p = \infty$ we adopt the convention $L^\infty = M$.

Note that it is also possible to define $L^p = \overline{M_{1/p}}^{\|\,\|_p}$, where $M_{1/p} = \{x \in M \mid \|x\|_p < \infty\}$, but this definition is of interest only if it is known that $M_{1/p}$ is an ideal of M such that $(M_{1/p})^{1/p'} = M_{1/pp'}$ and $M_{1/p}M_{1/p'} = M_{1/p+1/p'}$.

REMARK. Suppose that M is the space $C_{\mathbf{R}}(X)$ of real valued continuous functions on a compact hyperstonean space X. Then M is a J.B.W. algebra [35] with the usual product of functions. Let $\{\mu_\alpha\}_{\alpha\in\Gamma}$ be a maximal family of positive normal measures on X with disjoint supports S_α. If $\Omega = \bigcup_\alpha S_\alpha$, $\mu = \Sigma_\alpha \mu_\alpha$ then Ω is a locally compact dense set in X, μ is a positive Radon measure on Ω and M is isomorphic to the space $L^\infty(\Omega, \mu)$ of real valued essentially bounded μ-measurable functions over Ω. The L^p-spaces defined above are exactly $L^p(\Omega, \mu)$. This justifies the above convention. In this case, the following theorem is well known.

THEOREM 5. *The application*: $x \in M_1 \to \varphi(x\,\cdot\,) \in M_*$ *can be extended to an isometrically isomorphism from* L^1 *onto* M_* (*i.e.*: $(L^1)^* = M$). *In the same way, for* $p \in\,]1, \infty[$ *the Banach space* L^p *is isometrically isomorphic to* $(L^q)^*$ *with* $q = p/p - 1$.

In the general case, several preliminaries are necessary.

LEMMA 6. *Hansen inequality. Let* A *be a J.B. algebra and* $x \in A^+$. *For every* x *in* A, $\|x\| \leq 1$ *and all operator monotone functions* f (*i.e.* $x \leq y$,

$x, y \in A \Rightarrow f(x) \leq f(y)$),

$$U_a f(x) \leq f(U_a x).$$

In particular $U_a x^p \leq (U_a x)^p$ *if* $p \in]0,1]$ ([**30**], 1.3.8).

Proof. This results from [**17**] and [**39**] pp. 2.1. □

Note that a non spatial proof of the Hansen inequality can be carried out with [**5**] using Löwner's theory (see [**14**], Th. 5).

Now we follow [**38**].

LEMMA 7. *Let* $x, y \in M_1^+$

(i) *if* $x \leq y$ *then* $\varphi(x^p) \leq \varphi(y^p)$ *for* $p \in \ ]0, \infty[$

(ii) $\varphi(x^{np}) \leq \varphi([U_{x^{1/2}}(x+y)^{p-1}]^n)$ *for* $p \in \ ]1, \infty[$, $n \in \mathbf{N} \setminus \{0\}$.

Proof. First we prove

$$\varphi\big((U_{x^{1/2}} y)^{2n}\big) = \varphi\big((U_{y^{1/2}} x)^{2n}\big), \qquad 0 \neq n \in \mathbf{N}. \tag{1}$$

Recall that $(U_a b)^2 = U_a U_b a^2$ and $U_{a^2} = U_a U_a$. The J.B.W. algebra generated by x, y is special (Shirshov-Cohn's theorem [**16**]) so we easily check that

$$(U_{x^{1/2}} y)^{2n} = U_{x^{1/2}} U_{y^{1/2}} (U_{y^{1/2}} x)^{2n-1}.$$

Thus by Lemma 1

$$\begin{aligned}\varphi\big((U_{x^{1/2}} y)^{2n}\big) &= \varphi\big(x U_{y^{1/2}} (U_{y^{1/2}} x)^{2n-1}\big)\\ &= \varphi\big((U_{y^{1/2}} x)(U_{y^{1/2}} x)^{2n-1}\big) = \varphi\big((U_{y^{1/2}} x)^{2n}\big).\end{aligned}$$

(i) We first prove the result by induction for $p = 2^n$, $n \in \mathbf{N}$. If $n = 1$,

$$\varphi(y^2 - x^2) = \varphi((x+y)(x-y)) = \varphi\big(U_{(x+y)^{1/2}}(y-x)\big) \geq 0$$

(Lemma 1). Assume now the result for $p = 2^n$. Then

$$\begin{aligned}\varphi\big(y^{2^{n+1}}\big) &= \varphi\big((U_{y^{1/2}} y)^{2^n}\big)\\ &\geq \varphi\big((U_{y^{1/2}} x)^{2^n}\big) \quad \text{by hypothesis because } U_{y^{1/2}} y \geq U_{y^{1/2}} x\\ &= \varphi\big((U_{x^{1/2}} y)^{2^n}\big) \quad (1)\\ &\geq \varphi\big((U_{x^{1/2}} x)^{2^n}\big) \quad \text{by hypothesis}\\ &= \varphi\big(x^{2^{n+1}}\big).\end{aligned}$$

For an arbitrary p, it is possible to choose n such that $q = p2^{-n} < 1$. Since $x^q \leq y^q$ (cf. [30], 1.3.8),

$$\varphi(y^p) = \varphi\left((y^q)^{2^n}\right) \geq \varphi\left((x^q)^{2^n}\right) = \varphi(x^p).$$

(ii) By induction: If $p \in \,]1, 2]$, $x^{p-1} \leq (x+y)^{p-1}$ and

$$x^p = U_{x^{1/2}}(x^{p-1}) \leq U_{x^{1/2}}\left((x+y)^{p-1}\right).$$

Using part (i),

$$\varphi(x^{np}) \leq \varphi\left(\left[U_{x^{1/2}}(x+y)^{p-1}\right]^n\right), \qquad 0 \neq n \in \mathbf{N}.$$

Suppose now that (ii) is satisfied for $p \in]1, m]$ where $2 \leq m \in \mathbf{N}$. If $q = m + p'$ where $p' \in]0, 1]$, we have

$$U_{x^{1/2}}\left(U_{(x+y)^{q/2-1}}x\right) \leq U_{x^{1/2}}\left((x+y)^{q-1}\right).$$

Using (i)

$$\begin{aligned}\varphi\left(\left(U_{x^{1/2}}(x+y)^{q-1}\right)^n\right) &\geq \varphi\left(\left(U_{x^{1/2}}U_{(x+y)^{q/2-1}}x\right)^n\right)\\ &= \varphi\left(\left(U_{x^{1/2}}(x+y)^{q/2-1}\right)^{2n}\right)\\ &\geq \varphi\left(\left(U_{x^{1/2}}x^{q/2-1}\right)^{2n}\right) \quad \text{since } q/2 \in]1, m]\\ &= \varphi(x^{nq})\end{aligned}$$

where in the second step we used $(U_a b)^2 = U_a U_b a^2$. □

Parts of the following propositions were proved in [8] for the L^p-spaces on a measure space and in [6], [11], [12], [13], [15], [18], [22], [25], [26], [27], [28], [33], [34], [36], [37], [38], [40], [43] in the associative context of operator algebras.

PROPOSITION 8. *If* $p \in]0, 1]$ *then*
(i) $\|x + y\|_p^p \leq \|x\|_p^p + \|y\|_p^p$, $x, y \in M^+$
(ii) $\|x^p - y^p\|_1 \leq \|x - y\|_p^p$, $x, y \in M_1^+$.

Proof. We can suppose $p < 1$. We first prove the inequality (i) for $x, y \in M^+$. For integers n, m define $x_m = x + 1/m$, $y_m = y + 1/m$ and $z_n = x + y + 1/n$. Let $\{e_\alpha\}_{\alpha \in \Gamma}$ be an increasing net in M_1^+ such that $e_\alpha \uparrow \mathbf{1}$ with respect to Γ (see [19], Appendix 5 and Lemma 1).

$$\begin{aligned}\varphi(x_m^p e_\alpha) + \varphi(y_m^p e_\alpha) &= \varphi\left(\left(U_{x_m^{1/2}}U_{z_n^{-1/2}}z_n\right)^p e_\alpha\right) + \varphi\left(\left(U_{y_m^{1/2}}U_{z_n^{-1/2}}z_n\right)^p e_\alpha\right)\\ &\geq \varphi\left(\left(U_{x_m^{1/2}}U_{z_n^{-1/2}}z_n^p\right)e_\alpha\right) + \varphi\left(\left(U_{y_m^{1/2}}U_{z_n^{-1/2}}z_n^p\right)e_\alpha\right)\end{aligned}$$

(Lemma 6 and Lemma 1(v)).

Note that $U_{x_n^{1/2}}a$ tends in norm $\| \|_\infty$ to $U_{x^{1/2}}a$ for all a in M. (In fact,

$$\|U_{x_n^{1/2}}a - U_{x^{1/2}}a\| \le 2\|x_n^{1/2}(x_n^{1/2}a) - x^{1/2}(x^{1/2} - a)\| + \|(x_n - x)a\|$$

and

$$\begin{aligned}
&\|x_n^{1/2}(x_n^{1/2}a) - x^{1/2}(x^{1/2} - a)\| \\
&\qquad \le \|(x_n^{1/2} - x^{1/2})(x_n^{1/2} - a)\| + \|x^{1/2}((x_n^{1/2} - x^{1/2})a)\| \\
&\qquad \le \|x_n^{1/2} - x^{1/2}\|(\|x_n^{1/2}\| + \|x^{1/2}\|)\|a\|.)
\end{aligned}$$

Then, taking the limit in m in the previous inequality and using Lemma 1(v) and (vii)

$$\begin{aligned}
\varphi(x^p e_\alpha) + \varphi(y^p e_\alpha) &\ge \varphi\big([(U_{x^{1/2}} + U_{y^{1/2}})(z_n^{p-1})]\, e_\alpha\big) \\
&= \varphi\big(U_{z_n^{(p-1)/2}}(U_{x^{1/2}} + U_{y^{1/2}})e_\alpha\big).
\end{aligned}$$

Using $\varphi(a^2 e_\alpha) = \varphi(U_a e_\alpha)$ for $a \in M$ and the normality of φ, we obtain in the α-limit

$$\begin{aligned}
\varphi(x^p) + \varphi(y^p) &\ge \varphi\big(U_{z_n^{(p-1)/2}}(U_{x^{1/2}} + U_{y^{1/2}})\mathbf{1}\big) = \varphi\big(z_n^{p-1}(x + y)\big) \\
&= \varphi\big(U_{[z_n^{p-1}(x+y)]^{1/2}}\mathbf{1}\big)
\end{aligned}$$

because z_n and $(x + y)$ operator commute (cf. [4]) and $z_n^{p-1}(x + y)$ is positive

$$\ge \varphi\big(U^{1/2}_{[z_n^{p-1}(x+y)]}e_\alpha\big) = \varphi\big((z_n^{p-1}(x + y))e_\alpha\big).$$

Hence

$$\begin{aligned}
\varphi(x^p) + \varphi(y^p) &\ge \lim_n \varphi\big((z_n^{p-1}(x + y))e_\alpha\big) = \varphi\big((x + y)^p e_\alpha\big) \\
&= \varphi\big(U_{(x+y)^{p/2}}e_\alpha\big).
\end{aligned}$$

As before, the limit in α gives

$$\|x + y\|_p^p \le \|x\|_p^p + \|y\|_p^p \qquad \text{for } x, y \in M^+. \tag{2}$$

(ii) [24] Suppose $0 \le y \le x$. Since $y^p \le x^p$ ([30] 1.3.8.)

$$\begin{aligned}
\|x - y\|_p^p - \|x^p - y^p\|_1 &= \varphi\big((x - y)^p\big) - \varphi\big(((x - y) + y)^p\big) + \varphi(y^p) \\
&\ge 0 \quad \text{by part (i).}
\end{aligned}$$

Let now x, y be arbitrary. If $x - y = (x-y)_+ - (x-y)_-$ is the Jordan decomposition of $x - y$ in M,

$$\begin{aligned}\|x^p - y^p\|_1 &\le \|x^p - (y + (x-y)_+)^p\|_1 + \|(y+(x-y)_+)^p - y^p\|_1 \\ &\le \|(x-y)_-\|_p^p + \|(x-y)_+\|_p^p \quad \text{using the previous result} \\ &= \|x-y\|_p^p. \qquad \square\end{aligned}$$

PROPOSITION 9. *Clarkson inequalities.*

(i) *Let* $p \in [1, \infty[$. *Then*

$$2^{1-p}\|x+y\|_p^p \le \|x\|_p^p + \|y\|_p^p, \qquad x, y \in M_1.$$

(ii) *Let* $p \in [1, \infty[$. *Then*

$$\|x\|_p^p + \|y\|_p^p \le \|x+y\|_p^p, \qquad x, y \in M_1^+.$$
$$\|x^{1/p} - y^{1/p}\|_p^p \le \|x-y\|_1$$

(iii) *Let* $p \in [2, \infty[$. *Then*

$$\|x+y\|_p^p + \|x-y\|_p^p \le 2^{p-1}\left(\|x\|_p^p + \|y\|_p^p\right), \qquad x, y \in M_1.$$

(iv) *Let* $p \in]1, 2]$ *and* $1/p + 1/q = 1$. *Then*

$$\|x+y\|_p^q + \|x-y\|_p^q \le 2\left(\|x\|_p^p + \|y\|_p^p\right)^{q/p}, \qquad x, y \in M_1.$$

Proof. (i) follows from the convexity of $s \in \mathbf{R} \to s^p$ and the Minkowski inequality $\|x+y\|_p \le \|x\|_p + \|y\|_p$.

(ii) The previous lemma yields for $p > 1$

$$\begin{aligned}\|x\|_p^p + \|y\|_p^p &= \varphi(x^p) + \varphi(y^p) \\ &\le \varphi\left(U_{x^{1/2}}(x+y)^{p-1} + U_{y^{1/2}}(x+y)^{p-1}\right) \\ &\le \varphi\left((x+y)(x+y)^{p-1}\right) \quad \text{(Lemma 1(v))} \\ &= \|x+y\|_p^p.\end{aligned}$$

Second estimate: Suppose first that $x \ge y \in M_1^+$. Then using $x^{1/p} \ge y^{1/p}$ and the previous inequality extended to L^1,

$$\begin{aligned}\|x-y\|_1 - \|x^{1/p} - y^{1/p}\|_p^p &= \varphi(x) - \varphi\left(y + (x^{1/p} - y^{1/p})^p\right) \\ &= \varphi\left((u+v)^p\right) - \varphi(u^p + v^p) \ge 0\end{aligned}$$

where $u = x^{1/p} - y^{1/p} \in M^+$ and $v = y^{1/p} \in {}^+$. For general x, y in M_1^+, let $(x-y)_+ - (x-y)_-$ be the Jordan decomposition of $x - y$ and e the support of $(x^{1/p} - y^{1/p})_+$. Since $x \le y + (x-y)_+$, $x^{1/p} \le (y + (x-y)_+)^{1/p}$ hence

$$(x^{1/p} - y^{1/p})_+ = U_e(x^{1/p} - y^{1/p}) \le U_e([y + (x-y)_+]^{1/p} - y^{1/p}).$$

Thus

$$\begin{aligned}\|(x^{1/p} - y^{1/p})_+\|_p^p &\le \|[y + (x-y)_+]^{1/p} - y^{1/p}\|_p^p \quad \text{(Lemmas 1 and 3)}\\ &\le \|(x-y)_+\|_1 \quad \text{by the first half of the proof.}\end{aligned}$$

Switching x and y, we get $\|(x^{1/p} - y^{1/p})_-\|_p^p \le \|(x-y)_-\|_1$ and we are done by adding the last two estimates.

REMARK 10. Notice that for $p = 2$, this reduces to the Powers-Størmer inequality.

Case $p = 1$ is trivial.

(iii)

$$\begin{aligned}\|x + y\|_p^p + \|x - y\|_p^p &= \|(x+y)^2\|_{p/2}^{p/2} + \|(x-y)^2\|_{p/2}^{p/2}\\ &\le \|(x+y)^2 + (x-y)^2\|_{p/2}^{p/2} \quad \text{by part (ii)}\\ &= 2^{p/2}\|x^2 + y^2\|_{p/2}^{p/2}\\ &\le 2^{p-1}\left(\|x^2\|_{p/2}^{p/2} + \|y^2\|_{p/2}^{p/2}\right) \quad \text{by part (i)}\\ &= 2^{p-1}\left(\|x\|_p^p + \|y\|_p^p\right).\end{aligned}$$

(iv) We use now an idea from [22], also exploited by H. Kosaki.

The inequality follows from

$$\begin{aligned}&|\varphi((x_1 + x_2)x_3 + (x_1 - x_2)x_4)|\\ &\qquad \le 2^{1/q}\left(\|x_1\|_p^p + \|x_2\|_p^p\right)^{1/p}\left(\|x_3\|_q^p + \|x_4\|_q^p\right)^{1/p}\end{aligned} \tag{3}$$

valid for $x_i \in M_1$. In fact, if

$$x_1 = x, \quad x_2 = y$$

$$x_3 = \|x + y\|_p^{q-p} s|x + y|^{p-1} \quad \text{where } s \in S \text{ and } x + y = s|x + y|$$

$$x_4 = \|x - y\|_p^{q-p} t|x - y|^{p-1} \quad \text{where } t \in S \text{ and } x - y = t|x - y|$$

then it is easy to check that

$$\varphi\big((x_1 + x_2)x_3\big) = \|x + y\|_p^q = \|x_3\|_q^p$$
$$\varphi\big((x_1 - x_2)x_4\big) = \|x - y\|_p^q = \|x_4\|_q^p.$$

It is routine using spectral theory to verify that each x in M_1^+ is a $\|\ \|_p$-limit of elements of the form $\sum_{i=1}^n \lambda_i e_i$ where $\lambda_i \in \mathbf{R}$ and $\{e_i\}_i$ is a finite set of orthogonal idempotents in M_1^+. It is sufficient to prove (3) for

$$x_k = \sum_{i=1}^{n(k)} \lambda_{k,i} e_{k,i}, \qquad k \in \{1,2,3,4\}.$$

Denote also by φ the complex linear extension of φ on the complex Jordan extension $M_1^{\mathbf{C}} = M_1 + iM_1$. (Actually $M^{\mathbf{C}} = M + iM$ is a JB*-algebra for the natural involution but we do not use this fact.) We also use the notation $\|x\|_2 = \varphi(x^*x)^{1/2}$ for $x \in M_1^{\mathbf{C}}$. Define

$$y_k(z) = \sum_{i=1}^{n(k)} \operatorname{sgn}(\lambda_{k,i})|\lambda_{k,i}|^{pz} e_{k,i} \quad \text{for } k = 1,2$$
$$y_k(z) = \|x_k\|_q^{pz - q(1-z)} \sum_{i=1}^{n(k)} \operatorname{sgn}(\lambda_{k,i})|\lambda_{k,i}|^{q(1-z)} e_{k,i} \quad \text{for } k = 3,4.$$

If $g(z) = \varphi((y_1(z) + y_2(z))y_3(z) + (y_1(z) - y_2(z))y_4(z))$ then g is an analytic function bounded in the strip $\frac{1}{2} \le \operatorname{Re} z \le 1$. For $\operatorname{Re} z = 1$

$$\begin{aligned} |g(z)| &\le |\varphi(y_1(z)y_3(z))| + |\varphi(y_2(z)y_3(z))| \\ &\quad + |\varphi(y_1(z)y_4(z))| + |\varphi(y_2(z)y_4(z))| \\ &\le \big(\|x_1\|_p^p + \|x_2\|_p^p\big)\big(\|x_3\|_q^p + \|x_4\|_q^p\big). \end{aligned}$$

For $\operatorname{Re} z = \frac{1}{2}$,

$$|g(z)| \le |\varphi((y_1(x) + y_2(z))y_3(z))| + |\varphi((y_1(z) - y_2(z))y_4(z))|$$
$$\le \|y_1(z) + y_2(z)\|_2 \|y_3(z)\|_2 + \|y_1(z) - y_2(z)\|_2 \|y_4(z)\|_2$$

Cauchy-Schwarz inequality for φ

$$\le \big(\|y_1(z) + y_2(z)\|_2^2 + \|y_1(z) - y_2(z)\|_2^2\big)^{1/2}\big(\|y_3(z)\|_2^2 + \|y_4(z)\|_2^2\big)^{1/2}$$
$$= 2^{1/2}\big(\|y_1(z)\|_2^2 + \|y_2(z)\|_2^2\big)^{1/2}\big(\|y_3(z)\|_2^2 + \|y_4(z)\|_2^2\big)^{1/2}.$$

Since

$$g\left(\frac{1}{p}\right) = \varphi((x_1 + x_2)x_3 + (x_1 - x_2)x_4)$$

we obtain the desired inequality by the case of the Phragmén-Lindelöf's principle known as the three lines theorem [44] p. 93,

$$g\left(\frac{1}{p}\right) \leq \sup\left\{|g(z)|\,\middle|\,\mathrm{Re}(z) = \tfrac{1}{2}\right\}^{2(1-1/p)} \sup\left\{|g(z)|\,\middle|\,\mathrm{Re}(z) = 1\right\}^{2(1/p-1/2)}$$

$$\leq 2^{1/q}\left(\|x_1\|_p^p + \|x_2\|_p^p\right)^{1/p}\left(\|x_3\|_q^p + \|x_4\|_q^p\right)^{1/p}. \qquad \square$$

REMARK 11. It is possible to prove the two last inequalities of Proposition 9 appealing again to Riesz's convexity theorem and the reduction to simple elements used in Lemma 3. For instance, as in [8] Theorem 1, we obtain the following generalization

$$(4) \qquad \left(\|x+y\|_p^r + \|x-y\|_p^r\right)^{1/r} \leq 2^{1-1/s}\left(\|x\|_p^s + \|y\|_p^s\right)^{1/s}$$

where $x, y \in M_1$, $r \geq p \geq s > 1$ and $s \geq r/(r-1)$. In fact, the inequality asserts the truth of Proposition 9–(iii) for $r = s = p$ and of (iv) for $r = q$ and $s = p$.

COROLLARY 12. *L^p is uniformly convex for* $p \in]1, \infty[$.

Proof. Recall that a Banach space X is uniformly convex if its modulus of convexity $\delta_X(\varepsilon) = \inf\{1 - 2^{-1}\|x+y\| \mid x, y \in X,\ \|x\| = \|y\| = 1 \text{ and } \|x-y\| = \varepsilon\}$ is strictly positive for $0 < \varepsilon \leq 2$ [21]. In fact, the inequalities (iii) and (iv) in Proposition 9 yield

$$\delta_{L^p}(\varepsilon) \geq 1 - (1 - 2^{-p}\varepsilon^p)^{1/p} \quad \text{for } p \geq 2$$

$$\geq 1 - (1 - 2^{-q}\varepsilon^q)^{1/q} \quad \text{for } p > 1.$$

Proof of Theorem 5. Suppose $p > 1$.

The map: $x \in L^p \to \varphi(x\,\cdot\,) \in (L^q)^*$ is a linear isometry extending the application with $x \in M_1$ endowed with the norm $\|\cdot\|_p$. By a Milman's theorem, L^q is reflexive being uniformly convex. If the previous application is not surjective, there exists $y \in L^q$ with $y \neq 0$ such that $\varphi(xy) = 0$ for all $x \in L^p$, in contradiction with $\|y\|_q = \sup\{|\varphi(xy)| \mid x \in M_1,\ \|x\|_p \leq 1\}$.

Suppose $p = 1$.

The map: $x \in M_1 \to \varphi(x\,\cdot\,) \in M_*$ is again a linear isometry for the norm $\|\ \|_1$ on M_1 and can be extended to L^1. We now show that the image of M_1 is dense in M_*: Let C be the $\|\cdot\|_{M_*}$-closure of the image, so that C is a closed convex cone. Thus by Hahn-Banach theorem for every non zero $\omega \in M_* \setminus C$, there exists a non zero $y \in M$ such that $\varphi(yx) = 0$ for all $x \in M_1$ and $\omega(y) < 0$. Using **[19]** Appendix 5 and Lemma 1, there exists an increasing net $\{x_\alpha\}_{\alpha \in \Gamma}$ in M_1^+ $\sigma(M, M_*)$-convergent to $\mathbf{1}$ which respect to Γ. Let $s \in S$ be defined by $sy = |y|$. We have

$$\begin{aligned}\varphi(|y|) &= \varphi(U_{|y|^{1/2}}\mathbf{1}) = \lim_\alpha \varphi(U_{|y|^{1/2}}x_\alpha)\\ &= \lim_\alpha \varphi(|y|x_\alpha) = \lim_\alpha \varphi(y(sx_\alpha)) = 0.\end{aligned}$$

The fidelity of φ yields a contradiction. □

Using **[21]** §26.10 (6) and (9) we obtain immediately

COROLLARY 13. *The L^p-spaces are uniformly smooth and the norms $\|\cdot\|_p$ are uniformly strongly differentiable (Frechet differentiable) except at* 0 *for* $p \in]1, \infty[$.

As an application of this corollary we obtain as in **[23]** some related results without analytic proof (see **[38]**) in our real context.

We define $(L^p)^+$ as the $\|\cdot\|$-closure of M_1^+.

LEMMA 14. *If the map $f\colon\ t \in \mathbf{R} \to f(t) \in (L^p)^+$, $p \in]1, \infty[$ is differentiable for the norm $\|\ \|_p$ at t_0 such that $f(t_0) \neq 0$ then $t \to \varphi(f(t)^p)$ is differentiable at t_0 and*

$$\frac{d}{dt}\varphi(f(t)^p)\Big|_{t=t_0} = p\varphi\left(f(t_0)^{p-1}\frac{d}{dt}f(t)\Big|_{t=t_0}\right).$$

Proof. The strong derivative of $\|\cdot\|_p$ at $f(t_0)$ is the linear form

$$u = \|f(t_0)\|_p^{1-p}\varphi(f(t_0)^{p-1}\,\cdot)$$

because the supporting hyperplane through $f(t_0)$ of the ball of radius $\|f(t_0)\|_p$ is given by

$$\left\{x \in L^p \,\Big|\, \|f(t_0)\|_p^{1-p}\varphi(f(t_0)^{p-1}x) = \|f(t_0)\|_p\right\}$$

and thus one can apply **[21]**, (12) p. 349 and (4) p. 364. By the chain rule property the strong derivative of $\|\cdot\|_p^p$ at $f(t_0)$ is $v = p\varphi(f(t_0)^{p-1}\cdot)$. By assumption for small $\varepsilon \in \mathbf{R}^+$, $x_\varepsilon = f(t_0 + \varepsilon) - f(t_0) \in L^p$ and

$$\frac{d}{dt}f(t)\Big|_{t=t_0} = \|\cdot\|_p - \lim \varepsilon^{-1}x_\varepsilon \in L^p.$$

Consequently,

$$\varepsilon^{-1}\big[\varphi\big(f(t_0+\varepsilon)^p\big)-\varphi\big(f(t_0)^p\big)\big]=p\varphi\big(f(t_0)\varepsilon^{-1}x_\varepsilon\big)+\varepsilon^{-1}\delta_\varepsilon$$

where $\delta_\varepsilon=\|f(t_0+x_\varepsilon)\|_p^p-\|f(t_0)\|_p^p-v(x_\varepsilon)$and $\|x_\varepsilon\|_p^{-1}|\delta_\varepsilon|$ tends to 0 as $\|x_\varepsilon\|_p$ tends to 0 as we have seen. □

We are now in position to look at the case of equality in Proposition 9.

LEMMA 15.
(i) $\|x\|_p^p+\|y\|_p^p=\|x+y\|_p^p$ *for* $x,y\in M_1^+$, $p\in\,]1,\infty[$ *iff* $xy=0$
(ii) $2^{1-p}\|x+y\|_p^p=\|x\|_p^p+\|y\|_p^p$ *for* $x,y\in M_1^+$, $p\in\,]0,\infty[\,\setminus\{1\}$ *iff* $x=y$
(iii) $\|x+y\|_p^p+\|x-y\|_p^p=2(\|x\|_p^p+\|y\|_p^p)$ *for* $x,y\in M_1$, $p\in[1,\infty[\,\setminus\{2\}$ *iff* $xy=0$.

Proof. We can assume that M is a J.W. algebra ([**39**] Prop. 2.1) by restricting to the algebra generated by x and y.

(i) If $0=xy=2^{-1}(x\cdot y+y\cdot x)$ where $\cdot$ is the usual operator product, $x^2\cdot y=-x\cdot y\cdot x=y.x^2$. Since $[x^2,y]=0$, $[x,y]=0$ because $x=(x^2)^{1/2}=\lim_n p_n(x)$ where p_n is a polynomial of order n and $x\cdot y=0$. The equality $(x+y)^p=x^p+y^p$ follows as $\|x+y\|_p^p=\|x\|_p^p+\|y\|_p^p$.

Conversely suppose $\varphi((x+y)^p)=\varphi(x^p)+\varphi(y^p)$. For every $a,b\in M$, $t\in\mathbf{R}$ we have by Proposition 9(ii)

$$f(t)=\varphi\big(\big(x+e^{t[L_a,L_b]}y\big)^p\big)\tag{5}$$

$$\geq\varphi(x^p)+\varphi\big(\big(e^{t[L_a,L_b]}y\big)^p\big).$$

The fact that $e^{t[L_a,L_b]}$ is an automorphism of M leaving the trace invariant (Lemma 1) implies $f(t)\geq\varphi(x^p)+\varphi(y^p)=f(0)$. Thanks to the previous lemma,

$$0=f'(0)=p\varphi\big(z\big([L_a,L_b]y\big)\big)\quad\text{where } z=(x+y)^{p-1}$$

and

$$\varphi\big(z\big(a(by)\big)\big)=\varphi\big(z\big(b(ay)\big)\big)\quad\forall a,b\in M$$

that is

$$\varphi\big(\big(z(by)\big)a\big)=\varphi\big(\big((zb)y\big)a\big).$$

Since $(L^1)^*=M$ (Theorem 5), $z(by)=(zb)y$ $\forall b\in M$. In particular for $b=y$, $zy^2=y(yz)$ and $U_yz=y^2z$.

Theorem 5 of [**42**] asserts the associativity of the J.B. algebra generated by z and y hence by x and y, thus this algebra is isometrically isomorphic to $C(X)$ ([**4**] Proposition 2.3). If μ is the positive measure on X associated to φ, the equality

$$\int_X (x(\xi) + y(\xi))^p \, d\mu(\xi) = \int_X \left(x(\xi)^p + y(\xi)^p\right) d\mu(\xi)$$

yields $x(\xi)y(\xi) = 0$ a.e.

Thus $xy = 0$.

(ii) Suppose $2^{1-p}\varphi((x+y)^p) = \varphi(x^p) + \varphi(y^p)$ for $p > 1$. The function $2^{1-p}f$ in (5) attains its maximum at $t = 0$ (Proposition 9(i)). The same method as before yields $x = y$.

For $p < 1$ the concavity of: $x \in M \to x^p \in M$ ([**30**]) implies that $x^p + y^p \leq 2^{1-p}(x+y)^p$ thus the function $2^{1-p}f$ in (5) attains its minimum at $t = 0$ and we have still $x = y$.

(iii) Suppose $\|x+y\|_p^p + \|x-y\|_p^p = 2(\|x\|_p^p + \|y\|_p^p)$.
Then for $q = p/2 \neq 1$

$$\left\|(x+y)^2\right\|_q^q + \left\|(x-y)^2\right\|_q^q = 2\left(\|x^2\|_q^q + \|y^2\|_q^q\right)$$

and by Proposition 9(i) and (ii) this is greater than

$$2\|x^2 + y^2\|_q^q \geq 2\left(\|x^2\|_q^q + \|y^2\|_q^q\right) \quad \text{for } q > 1$$

and for $q < 1$, this is less than

$$\begin{aligned} &2^{1-q}\left\|(x+y)^2 + (x-y)^2\right\|_q^q \quad (\text{concavity } z \to z^q \text{ for } z \in M^+) \\ &\leq 2\left(\|x^2\|_q^q + \|y^2\|_q^q\right) \quad (\text{Proposition } 8). \end{aligned}$$

Thus

$$\left\|(x+y)^2\right\|_q^q + \left\|(x-y)^2\right\|_q^q = 2\|x^2 + y^2\|_q^q \quad \text{for } q \in]0, \infty[\,\setminus\{1\}.$$

The application of (ii) gives $(x+y)^2 = (x-y)^2$ and $xy = 0$.

Conversely, suppose $xy = 0$. The first part of the proof of (i) gives us

$$2\varphi\left((x^2)^q + (y^2)^q\right) = 2\varphi\left((x^2+y^2)^q\right) = \varphi\left((x+y)^{2q} + (x-y)^{2q}\right)$$

that is $2(\|x\|_p^p + \|y\|_p^p) = \|x+y\|_p^p + \|x-y\|_p^p$. □

The uniform convexity of L^p has a useful application. For instance, the following is standard ([**36**] Theorem 1.24).

LEMMA 16. *Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence in M_1, $x \in M_1$ and $p \in\,]1, \infty[$. If x_n tends to x for the $\sigma(L^p, (L^p)^*)$-topology and $\|x_n\|_p$ tends to $\|x\|_p$ then $\|x - x_n\|_p$ tends to zero.*

REMARK 17. If we replace the $\sigma(L^p,(L^p)^*)$-topology by the $\sigma(M, M_*)$-topology, the same result shows (Grümms' theorem, cf. [36] Theorem 2.21).

If $p = 1$ and x_n, $x \in M_1^+$, then the previous lemma holds for the $\sigma(M, M_*)$ topology (see [9], Appendix).

4. Miscellaneous results. The space L^2 has a natural structure of Hilbert space. For more details see [3] if the trace is finite and [19] for the semifinite case.

It is possible to give a short proof of the weak Hölder inequality $|\varphi(xy)| \leq \|x\|_p\|y\|_q$: Restricting to simple elements in Lemma 3, we can see that the map $f \in C_{\mathbf{R}}(X \times Y) \to \sum_{i,j} f(\lambda_i, \mu_j)\varphi(e_i f_j)$ where $X =$ spectrum (x), $Y =$ spectrum (y) and $C_{\mathbf{R}}(X \times Y)$ is the space of real valued continuous functions on $X \times Y$, defines a positive Borel measure reducing the problem to the Hölder's inequality on a measure space. With the same trick it is possible to prove for $x, y \in M_1^+$ that

$$\begin{aligned}\varphi(x^{p-1}y) &\leq \varphi(xy^{p-1})^{p-1}\varphi(y^p)^{2-p}, \qquad p \in]1,2[,\\ &\leq \varphi(xy^{p-1})^{q-1}\varphi(x^p)^{2-q}, \qquad p \in]2,\infty[,\ \frac{1}{p}+\frac{1}{q}=1\end{aligned}$$

since these inequalities are true on measure spaces [22].

All inequalities on measure spaces involving integrals of product of positive elements can be extended by this method to our L^p spaces.

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C.N.R.S. Luminy Case 907
Centre de Physique Théorique
F-13288 Marseille Cedex 9
France