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SQUAREFREE INTEGERS IN NON-LINEAR SEQUENCES

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The number of squarefree integers in sequences $[f(n+x)]$, $n = 1, 2, 3, \dots, N$ is $(6/\pi^2 + o(1))N$ for almost all $x \geq 0$, when f is a polynomial function of degree $k \geq 2$ or an exponential function.

Following a paper of Stux [4] Rieger has shown in [2] that for all real c with $1 < c < 3/2$ the equation

$$\sum_{\substack{1 \leq n \leq N \\ [n^c] \text{ squarefree}}} 1 = \frac{6}{\pi^2} N + O_c\left(N^{(2c+1)/4}\right)$$

is an immediate consequence from estimates of Deshouillers [1] concerning the distribution of the integer-sequences $[n^c]$ modulo k.

Using the same method and results of [3] we prove for functions f belonging to one of the following classes

$$\mathfrak{F}_{\text{pol}} = \left\{ ay^k + \sum_{i=1}^{m} a_i y^{k_i};\ a > 0,\ k > k_1 > \cdots > k_m \geq 0 \right\},$$

$$\mathfrak{F}_{\text{exp}} = \left\{ e^{ky+l} + f(y);\ k > 0,\ f \in \mathfrak{F}_{\text{pol}} \cup \{0\} \right\}:$$

THEOREM. *For $f \in \mathfrak{F}_{\text{pol}}$ of degree $k \geq 2$ or $f \in \mathfrak{F}_{\text{exp}}$,*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{\substack{1 \leq n \leq N \\ [f(n+x)] \text{ squarefree}}} 1 = \frac{6}{\pi^2}$$

holds for almost all real $x \geq 0$.

Proof. Let $\mathfrak{Q}$ denote the set of squarefree integers, $0 \leq x < 1$, and

$$S_N(\mathfrak{Q}; x) = \sum_{\substack{1 \leq n \leq N \\ [f(n+x)] \in \mathfrak{Q}}} 1,$$

$$S_N(d\mathbf{N}; x) = \sum_{\substack{1 \leq n \leq N \\ [f(n+x)] \in d\mathbf{N}}} 1.$$

Then

$$|\mu(n)| = \sum_{d > 0,\, d^2 | n} \mu(d)$$

gives

$$S_N(\mathfrak{Q}; x) = \sum_{d>0} \mu(d) S_N(d^2\mathbf{N}; x)$$

and therefore

$$(1)\quad \int_0^1 \left| S_N(\mathfrak{Q}; x) - \frac{6}{\pi^2} N \right| dx = \int_0^1 \left| \sum_{d>0} \mu(d) \left(S_N(d^2\mathbf{N}; x) - \frac{N}{d^2} \right) \right| dx$$
$$= O\left(\sum_{d>0} \int_0^1 \left| S_N(d^2\mathbf{N}; x) - \frac{N}{d^2} \right| dx \right).$$

To estimate the sum on the right side of (1) we use

$$\int_0^1 \left| S_N(d\mathbf{N}; x) - \frac{N}{d} \right| dx = d^{1/2} O_f(N^{3/4} \log N)$$

which for functions $f \in \mathfrak{F}_{\text{pol}}$ of degree $k \geq 2$ is Satz 9 in [**3**] and for $f \in \mathfrak{F}_{\exp}$ follows easily from the proof of Satz 6(ii) in [**3**]. Thus we have for small values of d

$$(2)\quad \sum_{0<d\leq M} \int_0^1 \left| S_N(d^2\mathbf{N}; x) - \frac{N}{d^2} \right| dx = O_f(M^2 N^{3/4} \log N).$$

The rest of the sum in (1) is estimated by

$$(3)\quad \sum_{d>M} \int_0^1 \left| S_N(d^2\mathbf{N}; x) - \frac{N}{d^2} \right| dx$$
$$\leq \sum_{d>M} \int_0^1 S_N(d^2\mathbf{N}; x)\, dx + N \sum_{d>M} \frac{1}{d^2}$$
$$\leq \sum_{d>M} \int_0^1 S_N(d^2\mathbf{N}; x)\, dx + \frac{N}{M}.$$

The explicit formula for the value of the integral $\int_0^1 S_N^{\text{nat}}(\mathfrak{P}; x)\, dx$ as a finite sum of interval-lengths

$$\sum_{1\leq n\leq N} \sum_{p\in\mathfrak{P}_n(0,1)} \mu\left(I_{p,n}^{0,1}\right)$$

as given at the end of part 1 in [**3**] shows in particular

$$(4)\quad \int_0^1 S_N(d^2\mathbf{N}; x)\, dx = O_f\left(\sum_{\substack{m\in d^2\mathbf{N} \\ f(0)\leq m\leq f(N+1)}} (f^{-1})'(m) \right).$$

For $f \in \mathfrak{F}_{\text{pol}}$, $\deg f = k \geq 2$, we have

$$(f^{-1})'(y) = O_f(y^{1/k-1})$$

and thus get from (4)

$$(5)\quad \int_0^1 S_N(d^2\mathbf{N};x)\,dx = O_f\Bigg(\sum_{1\le n\le f(N+1)/d^2} (d^2n)^{1/k-1}\Bigg)$$

$$= O_f\Bigg(d^{2/k-2}\sum_{1\le n\le N^k/d^2} n^{1/k-1}\Bigg) = O_f\left(\frac{N}{d^2}\right).$$

For $f \in \mathfrak{F}_{\exp}$, we even have

$$(f^{-1})'(y) = O_f(y^{-1})$$

and obtain (5) again.

Finally we choose $M = N^{1/12}$ and (1), (2), (3), (5) give

$$\int_0^1 \left|\frac{1}{N}S_N(\mathfrak{Q};x) - \frac{6}{\pi^2}\right| dx = O_f(N^{-1/12}\log N).$$

This implies first

$$\lim_{N\to\infty} \frac{1}{N^{13}} S_{N^{13}}(\mathfrak{Q};x) = \frac{6}{\pi^2}$$

for almost all x in the interval $[0,1[$ (with respect to the Lebesgue-measure on $\mathbf{R}$) and then by the principle of Hilfssatz 1 in [3]

$$\lim_{N\to\infty} \frac{1}{N} S_N(\mathfrak{Q};x) = \frac{6}{\pi^2}$$

for almost all $x \geq 0$.

References

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