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CONTINUA OF CONSTANT DISTANCES IN SPAN THEORY

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It is proved that, for each non-negative number β not exceeding the span of a mapping $f\colon X \to Y$, where X and Y are compact metric spaces, there exists a non-empty continuum $K_\beta \subset X \times X$ with identical two projections and such that the distances between $f(x)$ and $f(x')$ are all equal to β for $(x, x') \in K_\beta$. Similar results hold for other types of spans.

1. Preliminaries. All spaces are assumed to be non-empty metric spaces, and all mappings to be continuous functions. Let $f\colon X \to Y$ be a mapping. If X is connected, the *surjective span* $\sigma^*(f)$ of f is defined to be the least upper bound of the set of real numbers α with the following property: there exist non-empty connected sets $C_\alpha \subset X \times X$ such that $\operatorname{dist}[f(x), f(x')] \geq \alpha$ for $(x, x') \in C_\alpha$, and

$$(\sigma^*) \qquad p_1(C_\alpha) = p_2(C_\alpha) = X,$$

where p_1 and p_2 denote the standard projections of the product, that is, $p_1(x, x') = x$ and $p_2(x, x') = x'$. The *span* $\sigma(f)$, the *semispan* $\sigma_0(f)$, both for mappings f with the domains X not necessarily connected, and the *surjective semispan* $\sigma_0^*(f)$ in the case of connected domains, are defined similarly with condition (σ^*) relaxed to conditions

$$(\sigma) \qquad p_1(C_\alpha) = p_2(C_\alpha),$$

$$(\sigma_0) \qquad p_1(C_\alpha) \supset p_2(C_\alpha),$$

$$(\sigma_0^*) \qquad p_1(C_\alpha) = X,$$

respectively. The following inequalities and formulae are direct consequences of the definitions:

$$(1) \qquad 0 \leq \sigma^*(f) \leq \sigma(f) \leq \sigma_0(f) \leq \operatorname{diam} Y,$$

$$(2) \qquad 0 \leq \sigma^*(f) \leq \sigma_0^*(f) \leq \sigma_0(f) \leq \operatorname{diam} Y,$$

$$(3) \qquad \sigma(f) = \operatorname{Sup}\{\sigma^*(f \mid A)\colon A \subset X,\ A \neq \varnothing \text{ connected}\},$$

$$(4) \qquad \sigma_0(f) = \operatorname{Sup}\{\sigma_0^*(f \mid A)\colon A \subset X,\ A \neq \varnothing \text{ connected}\}.$$

For $\tau = \sigma, \sigma^*, \sigma_0, \sigma_0^*$, the corresponding spans $\tau(X)$ of a space X are the spans $\tau(\mathrm{id}_X)$ of the identity mapping on X. The span $\sigma(f)$ of a mapping f was originally defined by Ingram [3], while the present author earlier introduced the span $\sigma(X)$ and, subsequently, the other types of these quantities for metric spaces (see [6] and [7]). It is known that, for some particular spaces, neither two of these four types of spans need to be equal (see [7] and [8]).

1.1. *If* $f\colon X \to Y$ *is a mapping and* $\tau = \sigma, \sigma_0$, *then*

$$0 \le \tau(f) \le \tau(Y) \le \operatorname{diam} Y.$$

Moreover, if X *is connected and* $f(X) = Y$, *the same conclusion also holds for* $\tau = \sigma^*, \sigma_0^*$. *Consequently,* $\tau(Y) = 0$ *implies* $\tau(f) = 0$.

Proof. For a set $C_\alpha \subset X \times X$, let $D_\alpha = (f \times f)(C_\alpha)$. Since $f \times f$ is continuous, the connectedness of C_α implies that of D_α. But, since the standard projections p_i $(i = 1, 2)$ commute with other mappings (see [8], p. 39), conditions (τ) involving $p_i(C_\alpha)$ in the definition of $\tau(f)$, where τ stands for σ, σ^*, σ_0, or σ_0^*, imply the analogous conditions involving $p_i(D_\alpha)$ in the definition of $\tau(\mathrm{id}_Y)$, respectively. This yields $\tau(f) \le \tau(Y)$.

We show (see 1.3) that, in the case of compact spaces, $\tau(X) = 0$ also implies $\tau(f) = 0$. By a *continuum* we understand a connected compact metric space. If A and B are non-empty subsets of a metric space, we write $\rho(A, B)$ to mean

$$\rho(A, B) = \operatorname{Inf}\{\operatorname{dist}(a, b)\colon a \in A, b \in B\}.$$

1.2. *If* $f\colon X \to Y$ *is a mapping,* X *is a compact metric space, and* $\tau = \sigma, \sigma_0$, *then*

$$\operatorname{Inf}\{\rho[f^{-1}(y), f^{-1}(y')]\colon y, y' \in f(X), \operatorname{dist}(y, y') \ge \tau(f)\} \le \tau(X).$$

Moreover, if X *is a continuum, the same conclusion also holds for* $\tau = \sigma^*, \sigma_0^*$.

Proof. Let l be the number Inf forming the left-hand side of the last inequality. Notice that l is well-defined since, by the definition of $\tau(f)$ and the compactness of $f(X)$, there always exist points y, $y' \in f(X)$ satisfying $\operatorname{dist}(y, y') \ge \tau(f)$. From the definition of $\tau(f)$, we get an infinite sequence $\alpha_1, \alpha_2, \ldots$ of real numbers converging to $\tau(f)$, and non-empty connected sets $C_{\alpha_n} \subset X \times X$ such that

$$\operatorname{dist}[f(x), f(x')] \ge \alpha_n \qquad \big((x, x') \in C_{\alpha_n}; n = 1, 2, \ldots\big) \tag{5}$$

and appropriate conditions (τ) involving $p_i(C_{\alpha_n})$ $(i = 1, 2)$ are satisfied. Since $X \times X$ is compact, the same conditions are then fulfilled for the closures of C_{α_n} which are non-empty continua. Without loss of generality, we can assume that $C_{\alpha_1}, C_{\alpha_2}, \ldots$ is already a sequence of continua. By the compactness of $X \times X$, it has a convergent subsequence whose limit is a continuum (see [5], pp. 45, 49 and 139), and, again, we can assume that the sequence itself converges to a non-empty continuum $C \subset X \times X$. Since

$$p_i(C) = p_i\Big(\operatorname*{Lim}_{n\to\infty} C_{\alpha_n}\Big) = \operatorname*{Lim}_{n\to\infty} p_i(C_{\alpha_n}) \qquad (i = 1,2),$$

by the compactness of $X \times X$, and the inclusions as well as equalities are preserved in the process of taking a limit, we see that same conditions (τ) involving $p_i(C_{\alpha_n})$ are also satisfied by $p_i(C)$. Now, let $(x, x') \in C$. There exist points $(x_n, x'_n) \in C_{\alpha_n}$ $(n = 1,2,\ldots)$ converging to (x, x'). Let $y = f(x)$ and $y' = f(x')$. Thus

$$\begin{aligned}\operatorname{dist}(y, y') &= \operatorname{dist}[f(x), f(x')] = \operatorname{dist}\Big[f\Big(\lim_{n\to\infty} x_n\Big), f\Big(\lim_{n\to\infty} x'_n\Big)\Big] \\ &= \lim_{n\to\infty} \operatorname{dist}[f(x_n), f(x'_n)] \geq \lim_{n\to\infty} \alpha_n = \tau(f),\end{aligned}$$

by (5). Moreover, since $x \in f^{-1}(y)$ and $x' \in f^{-1}(y')$, we obtain

$$\operatorname{dist}(x, x') \geq \rho[f^{-1}(y), f^{-1}(y')] \geq l,$$

where l is one of the α's in the definition of $\tau(\mathrm{id}_X)$. We conclude that $l \leq \tau(\mathrm{id}_X) = \tau(X)$.

1.3. *If f: $X \to Y$ is a mapping, X is a compact metric space, and $\tau = \sigma, \sigma_0$, then $\tau(X) = 0$ implies $\tau(f) = 0$. Moreover, if X is a continuum, the same conclusion also holds for $\tau = \sigma^*, \sigma_0^*$.*

Proof. Observe that if $\varepsilon > 0$ and there exist points y, $y' \in f(X)$ satisfying $\operatorname{dist}(y, y') \geq \varepsilon$, then

$$\operatorname{Inf}\{\rho[f^{-1}(y), f^{-1}(y')]: y, y' \in f(X), \operatorname{dist}(y, y') \geq \varepsilon\} > 0,$$

which means that $\tau(f) > 0$ would imply $\tau(X) > 0$, by 1.2.

REMARKS. Let R, I, J, and S denote the real line, the unit interval $[0,1]$ in R, the open segment $I \setminus \{0,1\}$, and the unit circle $\{(u, v): u^2 + v^2 = 1\}$ in R^2, respectively. Consider any two "onto" mappings f_1: $I \to S$, f_2: $S \to I$, and a homeomorphism f_3: $J \to R$. Clearly, $\tau(I) = \tau(J) = 0$ for $\tau = \sigma, \sigma^*, \sigma_0, \sigma_0^*$ (see [8], p. 36). By 1.3, we have $\tau(f_1) = 0$, and it is not difficult to check that $\tau(S) = \operatorname{diam} S = 2$, whence the inequality $\tau(f) \leq \tau(Y)$ in 1.1 cannot be replaced by the equality. Also, by 1.1, we have $\tau(f_2) = 0$ which shows that the converse of 1.3 does not hold. Finally, an easy argument proves that $\tau(f_3) = \infty$. Hence the compactness of X in 1.3 cannot be omitted.

2. Main results. One of the consequences of results in this section is that, in the definition of the span $\sigma(f)$ (see §1) for compact domains, the inequality $\operatorname{dist}[f(x), f(x')] \geq \alpha$ can be replaced by the equality. The author wishes to thank M. B. de Castro and L. G. Oversteegen for helpful discussions concerning such a possibility. It can be derived directly from Theorem 2.1 below, and the same replacement can also be made in the definitions of other types of spans. The two kinds of definitions, one using the equality and another one using the inequality, are then equivalent, respectively, in each of the four cases of the concepts involved for mappings of compact metric spaces.

2.1. THEOREM. *If $f\colon X \to Y$ is a mapping, X is a compact metric space, $\tau = \sigma, \sigma_0$, and $0 \leq \beta \leq \tau(f)$, then there exists a non-empty continuum $K_\beta \subset X \times X$ such that*

$$\operatorname{dist}[f(x), f(x')] = \beta$$

for $(x, x') \in K_\beta$, and condition (τ) is satisfied for K_β in lieu of C_α, respectively. Moreover, if X is a continuum, the same conclusion also holds for $\tau = \sigma^*, \sigma_0^*$.

Proof. We distinguish four cases of which the first one is most significant (cf. [**8**], Lemmas 2.2 and 2.3).

Case 1. $\tau = \sigma^*$. In this case X is a continuum. If $\sigma^*(f) = 0$, then $\beta = 0$, and the theorem states a trivial fact. Indeed, the diagonal $K_0 = \{(x, x)\colon x \in X\}$ is a non-empty continuum, and condition (σ^*) is satisfied for K_0. Assume $\sigma^*(f) > 0$, and let α_n, β_n be real numbers such that

$$0 < \beta_n < \alpha_n < \sigma^*(f), \quad |\beta - \beta_n| < 1/n \qquad (n = 1, 2, \ldots). \tag{6}$$

Without loss of generality, it can be assumed that X is a subset of the Hilbert space R^ω, and $f(X)$ is a subset of the Hilbert cube I^ω. The continuum $f(X)$ has a metric inherited from the metric space Y, and there exists a metric d for I^ω which is an extension of that given metric in $f(X)$ (see [**2**]). We use I^ω equipped with this distance d; the metric in R^ω, denoted by ρ, is arbitrary. Since f can be regarded as a mapping $f\colon X \to I^\omega$ and I^ω is an absolute retract, there exists a continuous extension $\bar{f}\colon R^\omega \to I^\omega$ of f over R^ω. Since X is compact, there exist numbers $\delta_n > 0$ such that $\delta_n < 1/n$, and if $x \in X$, $z \in R^\omega$ and $\rho(x, z) < \delta_n$, then $d[f(x), \bar{f}(z)] < \frac{1}{2}(\alpha_n - \beta_n)$ $(n = 1, 2, \ldots)$. By (6) and the definition of $\sigma^*(f)$, there exist non-empty connected sets $C_n \subset X \times X$ such that $d[f(x), f(x')] \geq \alpha_n$ for $(x, x') \in C_n$ and condition (σ^*) is satisfied for C_n

$(n = 1, 2, \dots)$. If $(x, x') \in C_n$ and z, $z' \in R^\omega$ are points such that $\rho(x, z) < \delta_n$ and $\rho(x', z') < \delta_n$, then

$$\begin{aligned}\alpha_n &\le d[f(x), f(x')] \\ &\le d[f(x), \bar{f}(z)] + d[\bar{f}(z), \bar{f}(z')] + d[\bar{f}(z'), f(x')] \\ &< \frac{1}{2}(\alpha_n - \beta_n) + d[\bar{f}(z), \bar{f}(z')] + \frac{1}{2}(\alpha_n - \beta_n),\end{aligned}$$

whence

$$(7)\quad (x, x') \in C_n, \quad z, z' \in R^\omega, \quad \rho(x, z) < \delta_n > \rho(x', z')$$

$$\text{imply} \quad d[\bar{f}(z), \bar{f}(z')] > \beta_n \qquad (n = 1, 2, \dots).$$

Since X is a continuum, it has finite open covers whose elements have arbitrarily small diameters and whose nerves are connected polyhedra. It follows that, for each $n = 1, 2, \dots,$ there exist a connected polyhedron $P_n \subset R^\omega$ contained in the $(\frac{1}{2}\delta_n)$-neighborhood of X in R^ω and a mapping g_n: $X \to P_n$ such that g_n is a $(\frac{1}{2}\delta_n)$-translation, i.e., $\rho[x, g_n(x)] < \frac{1}{2}\delta_n$ for $x \in X$ (see **[4]**, pp. 319, 324 and 330). Since P_n is semilocally 1-connected (see **[11]**, p. 217), it possesses a covering space U_n which is connected and simply connected (see **[12]**, p. 83). Let h_n: $U_n \to P_n$ be the covering projection $(n = 1, 2, \dots)$. Then the product $h_n \times h_n$: $U_n \times U_n \to P_n \times P_n$ is also a covering projection and $U_n \times U_n$ is simply connected (see **[11]**, pp. 76 and 146). The set

$$D_n = (g_n \times g_n)(C_n) \qquad (n = 1, 2, \dots)$$

is a connected subset of $P_n \times P_n$. We select a point $(c_n, c_n') \in C_n$, a point $v_n = (u_n, u_n') \in U_n \times U_n$ such that $(h_n \times h_n)(v_n) = (g_n \times g_n)(c_n, c_n')$, and a point $w_n = (u_n'', u_n'')$ in the diagonal of $U_n \times U_n$. Since

$$\rho[c_n, g_n(c_n)] < \frac{1}{2}\delta_n < \delta_n, \quad \rho[c_n', g_n(c_n')] < \frac{1}{2}\delta_n < \delta_n,$$

we get $d[\bar{f}h_n(u_n), \bar{f}h_n(u_n')] = d[\bar{f}g_n(c_n), \bar{f}g_n(c_n')] > \beta_n$, by (7). On the other hand, $d[\bar{f}h_n(u_n''), \bar{f}h_n(u_n'')] = 0 < \beta_n$, by (6). It follows that the closed set

$$(8)\qquad N_n = \{(u, u') \in U_n \times U_n\colon d[\bar{f}h_n(u), \bar{f}h_n(u')] = \beta_n\}$$

$$(n = 1, 2, \dots)$$

cuts the space $U_n \times U_n$ between the points v_n and w_n. But $U_n \times U_n$ is an arcwise connected and locally arcwise connected space. Being simply connected, it is also unicoherent (see **[5]**, pp. 439–441). Thus there exists a connected set $M_n \subset N_n$ which cuts $U_n \times U_n$ between v_n and w_n (see **[5]**, pp. 438–439). In particular, $M_n \neq \varnothing$. The non-empty connected set

$$L_n = (h_n \times h_n)(M_n) \qquad (n = 1, 2, \dots)$$

is contained in $P_n \times P_n \subset R^\omega \times R^\omega$. Since $M_n \subset N_n$, it follows from (8) that

$$(9) \qquad d\left[\bar{f}(z), \bar{f}(z')\right] = \beta_n \qquad ((z, z') \in L_n; n = 1, 2, \dots).$$

Extending the notation used in condition (σ^*), we denote by p_i: $R^\omega \times R^\omega \to R^\omega$ the standard projection ($i = 1, 2$). We need to show that

$$(10) \qquad p_i(D_n) \subset p_i(L_n) \qquad (i = 1, 2; n = 1, 2, \dots).$$

Let $z_0 \in p_i(D_n)$ be any point. Let $d_0 \in D_n$ be a point such that $p_i(d_0) = z_0$. Denoting $d_n = (g_n \times g_n)(c_n, c_n')$, we also have $d_n \in D_n$. Let Σ be a simplicial subdivision of the polyhedron P_n such that $\operatorname{diam} \Delta \leq \frac{1}{2}\delta_n$ for $\Delta \in \Sigma$. The sets $\Delta \times \Delta'$, where Δ, $\Delta' \in \Sigma$, form a finite cover of $P_n \times P_n$. Let E be the union of all sets $\Delta \times \Delta'$ ($\Delta, \Delta' \in \Sigma$) which intersect D_n. Then E is a locally connected continuum, and d_0, $d_n \in D_n \subset E \subset P_n \times P_n$. Consequently, there exists an arc $A \subset E$ joining d_0 and d_n. The arc A can be regarded as the range of a homotopy of a mapping of a singleton into d_n. Notice that $(h_n \times h_n)(v_n) = d_n$. Since $h_n \times h_n$ is a covering projection, the arc A can be lifted to an arc $B \subset U_n \times U_n$ such that $v_n \in B$ and $(h_n \times h_n)(B) = A$ (see **[11]**, p. 156). Since $d_0 \in A$, there exists a point $v_0 \in B$ such that $(h_n \times h_n)(v_0) = d_0$. We use the same symbol p_i to denote the standard projection p_i: $U_n \times U_n \to U_n$ ($i = 1, 2$). Let $q = p_i(v_0) \in U_n$. The set $p_i^{-1}(q)$ is a copy of U_n, whence it is connected, and $v_0 \in B \cap p_i^{-1}(q)$. Let W be the diagonal in $U_n \times U_n$. Then W is connected and $(q, q) \in p_i^{-1}(q) \cap W$. It follows that the set

$$Q = B \cup p_i^{-1}(q) \cup W$$

is a connected subset of $U_n \times U_n$, and $v_n \in B \subset Q$, $w_n \in W \subset Q$. But since M_n cuts the space $U_n \times U_n$ between v_n and w_n, we obtain $M_n \cap Q \neq \varnothing$. Now, let $(u, u') \in B$ be any point. We have

$$(h_n(u), h_n(u')) = (h_n \times h_n)(u, u') \in A \subset E,$$

and thus there exist simplexes Δ, $\Delta' \in \Sigma$ such that $\Delta \times \Delta'$ contains $(h_n(u), h_n(u'))$ and intersects D_n. Hence there exists a point $(x, x') \in C_n$ with $(g_n \times g_n)(x, x') \in \Delta \times \Delta'$. This means that $\Delta \times \Delta'$ also contains $(g_n(x), g_n(x'))$, and therefore $g_n(x)$, $h_n(u) \in \Delta$ and $g_n(x'), h_n(u') \in \Delta'$. Consequently, we get

$$\rho[x, h_n(u)] \leq \rho[x, g_n(x)] + \rho[g_n(x), h_n(u)] < \frac{1}{2}\delta_n + \operatorname{diam} \Delta \leq \delta_n,$$

$$\rho[x', h_n(u')] \leq \rho[x', g_n(x')] + \rho[g_n(x'), h_n(u')]$$

$$< \frac{1}{2}\delta_n + \operatorname{diam} \Delta' \leq \delta_n,$$

which implies $d[\bar{f}\bar{h}_n(u), \bar{f}\bar{h}_n(u')] > \beta_n$, by (7). Since $M_n \subset N_n$, we conclude that $M_n \cap B = \varnothing$, by (8). On the other hand, if (u, u) is any point of the diagonal W, then $d[\bar{f}\bar{h}_n(u), \bar{f}\bar{h}_n(u)] = 0 < \beta_n$, by (6), and $M_n \cap W = \varnothing$, by (8). Since $M_n \cap Q \neq \varnothing$, as a result we have $M_n \cap p_i^{-1}(q) \neq \varnothing$, that is, $p_i(v_0) = q \in p_i(M_n)$. This yields

$$\begin{aligned} z_0 &= p_i(d_0) = p_i(h_n \times h_n)(v_0) \\ &= h_n p_i(v_0) \in h_n p_i(M_n) = p_i(h_n \times h_n)(M_n) = p_i(L_n), \end{aligned}$$

and the proof of (10) is completed.

In the product $R^\omega \times R^\omega$ metrized by the usual Pythagorean metric $([\rho(\cdot,\cdot)]^2 + [\rho(\cdot,\cdot)]^2)^{1/2}$, the distance between two points whose coordinates have distances less than $\frac{1}{2}\delta_n$ from one another is less than $2^{1/2}\frac{1}{2}\delta_n < \delta_n < 1/n$. Thus the set $P_n \times P_n$ is contained in the $(1/n)$-neighborhood of $X \times X$ in $R^\omega \times R^\omega$ and $g_n \times g_n$: $X \times X \to P_n \times P_n$ is a $(1/n)$-translation $(n = 1, 2, \ldots)$. Since $X \times X$ and $P_n \times P_n$ are compact sets in $R^\omega \times R^\omega$, this implies that the union

$$Z = (X \times X) \cup \bigcup_{n=1}^{\infty} (P_n \times P_n)$$

is also a compact set. We have $L_n \subset P_n \times P_n \subset Z$ for $n = 1, 2, \ldots$ The closures $\operatorname{cl} L_n$ of the connected sets L_n form an infinite sequence of non-empty continua contained in Z. By the compactness of Z, it has a convergent subsequence, and, without loss of generality, we can assume that the sequence itself converges to a non-empty continuum in Z. We define K_β to be this limit, that is,

$$K_\beta = \operatorname*{Lim}_{n \to \infty} \operatorname{cl} L_n.$$

Since $\operatorname{cl} L_n \subset P_n \times P_n$ and $P_n \times P_n$ lies in the $(1/n)$-neighborhood of the closed set $X \times X$ in $R^\omega \times R^\omega$ $(n = 1, 2, \ldots)$, we conclude that $K_\beta \subset X \times X$. If $(x, x') \in K_\beta$ is any point, there exist points $(z_n, z'_n) \in L_n$ $(n = 1, 2, \ldots)$ converging to (x, x'). Hence

$$\begin{aligned} d[f(x), f(x')] &= d[\bar{f}(x), \bar{f}(x')] = d\left[\bar{f}\left(\lim_{n\to\infty} z_n\right), \bar{f}\left(\lim_{n\to\infty} z'_n\right)\right] \\ &= \lim_{n\to\infty} d[\bar{f}(z_n), \bar{f}(z'_n)] = \lim_{n\to\infty} \beta_n = \beta, \end{aligned}$$

by (6) and (9). Applying (σ^*) for C_n, and (10), we obtain

$$g_n(X) = g_n p_i(C_n) = p_i(g_n \times g_n)(C_n) = p_i(D_n) \subset p_i(L_n) \subset p_i(\operatorname{cl} L_n)$$

for $i = 1, 2$ and $n = 1, 2, \ldots$ Since $\delta_n < 1/n$ and g_n is a $(\frac{1}{2}\delta_n)$-translation of the closed set X in R^ω, the sets $g_n(X)$ $(n = 1, 2, \ldots)$ converge to X. On the other hand, the sets $p_i(\operatorname{cl} L_n)$ $(n = 1, 2, \ldots)$ are images under the mapping p_i of subsets of the compact set Z which form a convergent sequence. Thus they themselves make a convergent sequence in $p_i(Z)$, and their limit is $p_i(K_\beta)$. It follows that

$$X = \operatorname*{Lim}_{n\to\infty} g_n(X) \subset \operatorname*{Lim}_{n\to\infty} p_i(\mathrm{cl}\, L_n) = p_i(K_\beta) \subset X$$

for $i = 1, 2$; and condition (σ^*) is satisfied for K_β.

Case 2. $\tau = \sigma_0^*$. The proof is similar to the argument presented above in Case 1, the only difference being that now we have $i = 1$ instead of $i = 1, 2$, and condition (σ_0^*) replaces condition (σ^*).

Case 3. $\tau = \sigma$. Since $0 \leq \beta \leq \sigma(f)$, we use (3) to get an infinite sequence $\beta_1, \beta_2, \ldots$ of real numbers converging to β, and non-empty connected sets $A_n \subset X$ such that $0 \leq \beta_n \leq \sigma^*(f \mid A_n)$ for $n = 1, 2, \ldots$ Suppose n is fixed, α is a real number and $C_\alpha \subset A_n \times A_n$ is a non-empty connected set such that $\mathrm{dist}[f(x), f(x')] \geq \alpha$ for $(x, x') \in C_\alpha$ and $p_i(C_\alpha) = A_n$ $(i = 1, 2)$. Then $\mathrm{cl}\, C_\alpha \subset \mathrm{cl}\, A_n \times \mathrm{cl}\, A_n \subset X \times X$ and $\mathrm{dist}[f(x), f(x')] \geq \alpha$ for $(x, x') \in \mathrm{cl}\, C_\alpha$. Since $X \times X$ is compact, we also have $p_i(\mathrm{cl}\, C_\alpha) = \mathrm{cl}\, p_i(C_\alpha) = \mathrm{cl}\, A_n$ $(i = 1, 2)$. This means that if α belongs to the set of numbers appearing in the definition of $\sigma^*(f \mid A_n)$ (see §1), it also does in the definition of $\sigma^*(f|\mathrm{cl}\, A_n)$. Hence $\sigma^*(f|A_n) \leq \sigma^*(f|\mathrm{cl}\, A_n)$. But $\mathrm{cl}\, A_n$ is a non-empty continuum and $0 \leq \beta_n \leq \sigma^*(f|\mathrm{cl}\, A_n)$. According to what was proved in Case 1, there exists a non-empty continuum $K_{\beta_n} \subset \mathrm{cl}\, A_n \times \mathrm{cl}\, A_n$ such that $\mathrm{dist}[f(x), f(x')] = \beta_n$ for $(x, x') \in K_{\beta_n}$ and $p_i(K_{\beta_n}) = \mathrm{cl}\, A_n$ $(i = 1, 2)$.

By the compactness of $X \times X$, the infinite sequence $K_{\beta_1}, K_{\beta_2}, \ldots$ has a convergent subsequence whose limit $K_\beta \subset X \times X$ is a non-empty continuum, and, without loss of generality, we can assume that the sequence itself converges to K_β. If $(x, x') \in K_\beta$ is any point, there exist points $(x_n, x'_n) \in K_{\beta_n}$ $(n = 1, 2, \ldots)$ converging to (x, x'). Thus

$$\begin{aligned} \mathrm{dist}[f(x), f(x')] &= \mathrm{dist}\Big[f\Big(\lim_{n\to\infty} x_n\Big), f\Big(\lim_{n\to\infty} x'_n\Big)\Big] \\ &= \lim_{n\to\infty} \mathrm{dist}\big[f(x_n), f(x'_n)\big] = \lim_{n\to\infty} \beta_n = \beta. \end{aligned}$$

Since

$$p_i(K_\beta) = p_i\Big(\operatorname*{Lim}_{n\to\infty} K_{\beta_n}\Big) = \operatorname*{Lim}_{n\to\infty} p_i(K_{\beta_n}) \qquad (i = 1, 2),$$

by the compactness of $X \times X$, and $p_1(K_{\beta_n}) = p_2(K_{\beta_n})$ for $n = 1, 2, \ldots$, we obtain $p_1(K_\beta) = p_2(K_\beta)$.

Case 4. $\tau = \sigma_0$. The proof in this case is almost identical with that in Case 3. We now use (4) instead of (3), and have $i = 1$ instead of $i = 1, 2$. The surjective semispan σ_0^* replaces the surjective span σ^*. The result of Case 2 is utilized instead of Case 1 to find a non-empty continuum $K_{\beta_n} \subset \mathrm{cl}\, A_n \times \mathrm{cl}\, A_n$ which satisfies the condition $p_1(K_{\beta_n}) = \mathrm{cl}\, A_n \supset p_2(K_{\beta_n})$. The formula in the last sentence of the argument for Case 3 is still true for both $i = 1$ and $i = 2$, but the last two equalities there should be replaced by the inclusions $p_1(K_{\beta_n}) \supset p_2(K_{\beta_n})$ and $p_1(K_\beta) \supset p_2(K_\beta)$, respectively.

2.2. COROLLARY. *If f: $X \to Y$ is a mapping, X is a compact metric space, $\tau = \sigma$, σ_0, and*

$$\alpha(\beta) = \mathrm{Inf}\{\rho[f^{-1}(y), f^{-1}(y')]: y,\ y' \in f(X), \mathrm{dist}(y, y') = \beta\}$$

for $0 \leq \beta \leq \tau(f)$, then

$$\mathrm{Sup}\{\alpha(\beta): 0 \leq \beta \leq \tau(f)\} \leq \tau(X).$$

Moreover, if X is a continuum, the same conclusion also holds for $\tau = \sigma^*$, σ_0^*.

2.3. COROLLARY. *If X is a compact metric space, $\tau = \sigma$, σ_0, and $0 \leq \beta \leq \tau(X)$, then there exists a non-empty continuum $K_\beta \subset X \times X$ such that* $\mathrm{dist}(x, x') = \beta$ *for $(x, x') \in K_\beta$, and condition (τ) is satisfied for K_β in lieu of C_α, respectively. Moreover, if X is a continuum, the same conclusion also holds for $\tau = \sigma^*$, σ_0^*.*

REMARKS. A special case of 2.2 as well as some other results of this paper were announced earlier in **[9]** and **[10]**. Note that

$$\mathrm{Inf}\{\rho[f^{-1}(y), f^{-1}(y')]: y,\ y' \in f(X), \mathrm{dist}(y, y') \geq \tau(f)\}$$
$$\leq \alpha[\tau(f)] \leq \mathrm{Sup}\{\alpha(\beta): 0 \leq \beta \leq \tau(f)\},$$

so that 2.2 is stronger than 1.2. Also, 2.3 is stronger than some previous results (see **[8]**, Lemmas 2.2 and 2.3).

3. Applications. Given a mapping f: $X \to Y$, we consider the set Δ_f of real numbers defined by

$$\Delta_f = \{\mathrm{dist}(x, x'): x,\ x' \in X,\ f(x) = f(x')\}.$$

3.1. THEOREM. *If f: $X \to Y$, g: $Y \to Z$ are mappings, X is a compact metric space, $\tau = \sigma$, σ_0, and $0 \leq \beta \leq \tau(f)$, then there exist points x_0, $x_0' \in X$ such that*

$$\mathrm{dist}[f(x_0), f(x_0')] = \beta, \qquad \mathrm{dist}[gf(x_0), gf(x_0')] \leq \tau(gf).$$

Moreover, if X is a continuum, the same conclusion also holds for $\tau = \sigma^*, \sigma_0^*$. Consequently, $\tau(gf) = 0$ implies $\beta \in \Delta_g$, and thus $[0, \tau(f)] \subset \Delta_g$.

Proof. Applying 2.1, we obtain a non-empty continuum $K_\beta \subset X \times X$ which satisfies the conclusion of 2.1. Suppose, on the contrary, that 3.1 is not true. Then

$$\text{dist}[gf(x), gf(x')] > \tau(gf)$$

for $(x, x') \in K_\beta$. Since K_β is compact, there exists a number $\alpha_0 > \tau(gf)$ such that $\text{dist}[gf(x), gf(x')] \geq \alpha_0$ for $(x, x') \in K_\beta$. This means that α_0 is one of the numbers taken in the definition of $\tau(gf)$ (see §1). Hence $\alpha_0 \leq \tau(gf)$, a contradiction.

3.2. COROLLARY. *If f: $X \to Y$ is a mapping, X is a compact metric space, $\tau = \sigma, \sigma_0$, and $\tau(f) = 0$, then $[0, \tau(X)] \subset \Delta_f$. Moreover, if X is a continuum, the same conclusion also holds for $\tau = \sigma^*, \sigma_0^*$.*

3.3. COROLLARY. *If f: $X \to Y$ is a mapping, X is a compact metric space, $\tau = \sigma, \sigma_0$, and $\tau(Y) = 0$, then $[0, \tau(X)] \subset \Delta_f$. Moreover, if X is a continuum and $f(X) = Y$, the same conclusion also holds for $\tau = \sigma^*, \sigma_0^*$.*

REMARKS. Substituting the identity mapping for f in 3.1 and switching g into f produces 3.2 which, when combined with 1.1, implies 3.3. Again, 3.1 and 3.3 strengthen and generalize some earlier results (see **[8]**, Theorem 2.4 and Corollary 2.5). We can apply 3.3 to the special case of real-valued continuous functions since bounded subsets of the real line have all spans equal to zero (cf. **[8]**, p. 36). Therefore, if X is a continuum, then the interval $[0, \tau(X)]$ is always contained in Δ_f for each real-valued continuous function f defined on X, and $\tau = \sigma, \sigma^*, \sigma_0, \sigma_0^*$. By (1) and (2), the longest of these four intervals is $[0, \sigma_0(X)]$, so that the best result is the inclusion $[0, \sigma_0(X)] \subset \Delta_f$. On the other hand, estimating the semi-span or the span of spaces by means of other quantities is also possible. For a mapping g: $X \to S$ of X onto the unit circle S, we consider the number

$$\zeta(g) = \text{Inf}\{\rho[g^{-1}(z), g^{-1}(-z)]: z \in S\},$$

where $-z$ is the point of S antipodal to z. If X is compact, then $\zeta(g) > 0$. If X is compact and g is homotopically essential, then $\zeta(g) \leq \sigma(X)$ (see **[6]**, p. 211). Many objects, such as simple closed curves, solenoids, or the pseudo-circle, admit homotopically essential mappings onto S. If X is any of such compact metric spaces with a mapping $g\colon X \to S$ homotopically essential, and $f\colon X \to R$ is any real-valued continuous function, then

$$[0, \zeta(g)] \subset [0, \sigma(X)] \subset \Delta_f,$$

by Corollary 3.3. For simple closed curves and some other locally connected continua, a "chord sliding" theorem was proved by Fenn [**1**]. (I am indebted to Morton Brown for this reference.) If one allows a "generalized chord sliding" in which the parameter space is not necessarily I or S but is permitted to be an arbitrary continuum, then the main results of the present paper can be interpreted as complementing those of [**1**]. For example, it follows from our Corollary 2.3 that if X is a continuum, then each chord of a length belonging to the interval $[0, \sigma^*(X)]$ can be slid completely around X in such a generalized way. It is rather easy to show that, for each simple closed curve C, we have $\sigma(C) = \sigma^*(C)$. Hence if C is a simple closed curve, then all chords of lengths from the interval $[0, \sigma(C)]$ admit generalized slidings around C. Thus, in particular, the same thing is true for all chords of lengths from the interval $[0, \zeta(h)]$, where $h\colon C \to S$ is any homeomorphism.

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