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# DERIVATIONS WITH INVERTIBLE VALUES IN RINGS WITH INVOLUTION

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**Let $R$ be a semiprime 2-torsion free ring with involution * and let $S = \{x \in R \mid x = x^*\}$ be the set of symmetric elements. We prove that if $R$ has a derivation $d$, non-zero on $S$, such that for all $s \in S$ either $d(s) = 0$ or $d(s)$ is invertible, then $R$ must be one of the following: (1) a division ring, (2) $2 \times 2$ matrices over a division ring, (3) the direct sum of a division ring and its opposite with exchange involution, (4) the direct sum of $2 \times 2$ matrices over a division ring and its opposite with exchange involution, (5) $4 \times 4$ matrices over a field with symplectic involution.**

Recently Bergen, Herstein and Lanski studied the structure of a ring $R$ with a derivation $d \neq 0$ such that, for each $x \in R$, $d(x) = 0$ or $d(x)$ is invertible. They proved that, except for a special case which occurs when $2R = 0$, such a ring must be either a division ring $D$ or the ring $D_2$ of $2 \times 2$ matrices over a division ring.

In this paper we address ourselves to a similar problem in the setting of rings with involution, namely: let $R$ be a 2-torsion free semiprime ring with involution and let $S$ be the set of symmetric elements. If $d \neq 0$ is a derivation of $R$ such that the non-zero elements of $d(S)$ are invertible, what can we conclude about $R$?

We shall prove that $R$ must be rather special. In fact we shall show the following:

THEOREM. *Let $R$ be a 2-torsion free semiprime ring with involution. Let $d$ be a derivation of $R$ such that $d(S) \neq 0$ and the non-zero elements of $d(S)$ are invertible in $R$. Then $R$ is either*:

1. *a division ring $D$, or*
2. *$D_2$, the ring of $2 \times 2$ matrices over $D$, or*
3. *$D \oplus D^{\mathrm{op}}$, the direct sum of a division ring and its opposite relative to the exchange involution, or*
4. *$D_2 \oplus D_2^{\mathrm{op}}$ with the exchange involution, or*
5. *$F_4$, the ring of $4 \times 4$ matrices over a field $F$ with symplectic involution.*

In case $R = F_4$ with * symplectic we shall prove that $d$ is inner. As Herstein has pointed out, an easy example of such a ring is given by taking $F$ to be a field in which $-1$ is not a square and $d$ the inner derivation in $F_4$ induced by $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where $I$ is the identity matrix in $F_2$.

Now, if $R = D \oplus D^{\text{op}}$ or $R = D_2 \oplus D_2^{\text{op}}$ then $S \cong D$ or $S \cong D_2$ respectively. Thus both cases come naturally from [**1**].

We remark that if $d(S) = 0$ then $d(\bar{S}) = 0$, where $\bar{S}$ is the subring generated by $S$; hence, if $R$ is semiprime, by [**3**, Theorem 2.1.5] either $S$ lies in the center of $R$ (and $R$ satisfies the standard identity of degree 4) or $d(J) = 0$ for some non-zero ideal $J$ of $R$.

Let $R$ be a ring with involution; we denote by $Z$ the center of $R$ and by $S$ and $K$ the sets of symmetric and skew elements of $R$ respectively. Throughout this paper, unless otherwise stated, *$R$ will be a 2-torsion free semiprime ring with an involution $*$ and $d$ will be a derivation of $R$ such that $d(S) \neq 0$ and the non-zero elements of $d(S)$ are invertible.*

We begin with the following

LEMMA 1. *If $I = I^*$ is a non-zero ideal of $R$ then $d(I \cap S) \neq 0$.*

*Proof.* Suppose, by contradiction, that $d(I \cap S) = 0$ and let $t \in S$ be such that $d(t) \neq 0$. For all $S \in I \cap S$ the elements $sts$ and $st + ts$ lie in $I \cap S$, hence

$$0 = d(sts) = sd(t)s$$
$$0 = d(st + ts) = sd(t) + d(t)s$$

Multiplying the second equality from the left by $s$, we obtain $s^2d(t) = 0$. Now, from our basic hypothesis on $R$, $d(t)$ is invertible; hence $s^2 = 0$, for all $s \in I \cap S$.

Now let $x \in R$, $s \in I \cap S$. Then the element $sx + x^*s$ lies in $I \cap S$ and, so, it must be square-zero. Therefore, since $s^2 = 0$,

$$0 = (sx + x^*s)sx = (sx)^3,$$

that is, every element in the right ideal $sR$ is nilpotent of index $\leq 3$. By Levitski's Theorem [**2**, Lemma 1] we must have $sR = 0$ and, so, $s = 0$. This proves that $I \cap S = 0$.

For $x \in I$, $x + x^* \in I \cap S$; hence $x = -x^*$ and $x^2 \in I \cap S = 0$. This $I$ is a nilideal of index $\leq 2$. This forces $I = 0$, a contradiction. □

At this stage we are able to prove our result in case $R$ is not simple; in fact we have

PROPOSITION 1. *If $R$ is not a simple ring then either $R \cong D \oplus D^{\text{op}}$, $D$ a division ring, or $R \cong D_2 \oplus D_2^{\text{op}}$ and $*$ is the exchange involution.*

*Proof*. Let $I \neq R$ be an ideal of $R$ such that $I = I^*$.

Since $d(I^2 \cap S) \subset d(I^2) \subset I$, Lemma 1 shows that $I^2 = 0$ and the semiprimeness of $R$ forces $I = 0$. We have proved that $R$ does not contain proper *-ideals.

If $R$ is not simple, then there exists a proper ideal $I \neq I^*$. Since $I + I^*$ is a non-zero *-ideal of $R$, $I + I^* = R$. Also $I \cap I^* \neq R$ is a *-ideal of $R$, hence $I \cap I^* = 0$. Thus we have that $R = I \oplus I^*$. Moreover since $I^2 \neq I^{*2}$ we also get $R = I^2 \oplus I^{*2}$ and, so, $I = I^2$ and $I^* = I^{*2}$; hence, they are both invariant under $d$. Clearly $S = \{x + x^* \mid x \in I\}$ and so $d(x)$ and $d(x^*)$ are both 0 or both units in $I$ and $I^*$ respectively.

By [**1**, Theorem 1] $I$, and hence also $I^*$, is either a division ring $D$ or $D_2$. If $d(I) = 0$, then $d(I^*) \neq 0$ and the argument above leads to the same conclusion. Clearly the involution in $R$ is the exchange involution. □

If $R$ is a prime ring we denote by $C$ the extended centroid of $R$ and by $Q = RC$ the central closure of $R$ (see [**3**, pg. 22]). The next lemma holds for arbitrary rings with involution, with a derivation $d \neq 0$.

LEMMA 2. *Let $R$ be a prime ring with involution, with a derivation $d \neq 0$. Let $x \in R$ be such that for all $s \in S$*

$$xsx^*d(R)xsx^* = 0.$$

*Then either $x^*d(R)x = 0$ or $Q = RC$ has a minimal right ideal.*

*Proof*. For $y \in R$ let $u = x^*d(y)x$. Then if $s \in S$, $ususu = ususu^* = 0$; now, if $r \in R$, $su^*r^* + rus \in S$ and, so,

$$0 = usu(su^*r^* + rus)u(su^*r^* + rus)u = usurusurusu.$$

This says that every element in the right ideal $usuR$ is nilpotent of index $\leq 3$. By Levitski's theorem [**2**, Lemma 1.1], $usuR = 0$ and so $usu = 0$ for all $s \in S$. By [**5**, Lemma 3], if $u \neq 0$, $Q = RC$ has a minimal right ideal. □

In light of Proposition 1 we now make a first reduction: from now on, unless otherwise stated, *we will always assume that $R$ is a simple ring with* 1. In this case clearly $R$ coincides with its own central closure.

The next lemmas give us some information about the nature of the symmetric elements in the kernel of $d$.

LEMMA 3. *Let $a \in S$. If for all $s \in S$ we have that $asa = \lambda a$, for some $\lambda = \lambda(s) \in z$, then $R$ has a minimal right ideal.*

*Proof*. Let $x \in R$. Then $a(x + x^*)a = \lambda a$, for some $\lambda \in Z$, that is $ax^*a = \lambda a - axa$. Let $\mu \in Z$ be such that $a(xax + x^*ax^*)a = \mu a$. Playing these off against each other we get

$$0 = axaxa + ax^*ax^*a - \mu a = 2axaxa - 2\lambda axa + (\lambda^2 - \mu)a.$$

Therefore $2(ax)^3 - 2\lambda(ax)^2 + (\lambda^2 - \mu)ax = 0$ and, since $\operatorname{char} R \neq 2$, $ax$ is algebraic over $Z$ of degree at most 3. This proves that $aR$ is an algebraic algebra of bounded degree. Thus $aR$ satisfies a polynomial identity; hence $R$ satisfies a generalized polynomial identity. Since $R$ coincides with its own central closure, by a theorem of Martindale [**3**, Theorem 1.3.2.] $R$ has a minimal right ideal. □

LEMMA 4. *Suppose $R$ does not contain minimal right ideals. If $a \in S$ is such that $d(a) = 0$ then either $a$ is invertible or $ad(R)a = 0$.*

*Proof*. Suppose $a \neq 0$ and $a$ is not invertible. Since $d(a) = 0$ then, for all $s \in S$, $d(asa) = ad(s)a$ and it is not invertible. Hence $ad(s)a = 0$.

Now let $x \in R$. Then $ad(x + x^*)a = 0$ implies $ad(x)a = -ad(x^*)a$. Therefore for all $s \in S$, recalling that $d(a) = ad(s)a = 0$ we get

$$asad(x)a = ad(sax)a = -ad(x^*as)a = -ad(x^*)asa = ad(x)asa.$$

We have proved that for all $x \in R$, $s \in S$,

$$asa\,d(x)a = ad(x)asa \tag{1}$$

Since $d(a) = 0$, $d(aR) \subset aR$; moreover if $\rho_R(a)$ is the left annihilator of $a$ in $R$, $d(\rho_R(a)) \subset \rho_R(a)$; this says that $d$ induces a derivation (which we will still denote by $d$) in the prime ring $R_1 = aR/\rho_R(a) \cap aR$. Moreover, for $s \in S$, if $\overline{as}$ is the image of $as$ in $R_1$, from (1) we get

$$\overline{as}\,d(\overline{ax}) = d(\overline{ax})\overline{as}, \quad \text{for all } \overline{ax} \in R_1.$$

By [**4**] since $\operatorname{char} R \neq 2$ either $d = 0$ in $R_1$ or $\overline{as} \in Z(R_1)$, the center of $R_1$. That is, either $ad(R)a = 0$ or $asaxa = axasa$ for all $x \in R$.

If $ad(R)a = 0$ we are done; therefore we may assume that $asaxa = axasa$, for all $x \in R$, $s \in S$. But then, by [**3**, Lemma 1.3.2.], $asa = \lambda a$, for some $\lambda \in Z$ and, by Lemma 3, $R$ has a minimal right ideal, a contradiction. □

We remark that since $R$ is simple with 1 then it must be a primitive ring. Now, through a repeated application of the density theorem we will be able to prove that $R$ is artinian.

PROPOSITION 2. *R is a simple artinian ring.*

*Proof*. Since $R$ is primitive it is a dense ring of linear transformations on a vector space $V$ over a division ring $D$. By [3, Lemma 1.1.2.] to prove that $R$ is artinian it is enough to prove that $R$ has a minimal right ideal or equivalently that $R$ contains a non-zero transformation of finite rank. Suppose, by contradiction, that this is not the case.

Let $s \in S$ be such that $d(s) \neq 0$ and suppose that there exist linearly independent vectors $v, w \in V$ such that

$$vs = ws = 0.$$

Since $d(s)$ is invertible, the vectors $vd(s)$ and $wd(s)$ are linearly independent over $D$. Moreover, since $R$ doesn't contain non-zero transformations of finite rank, there exists a vector $u \in V$ such that $us \notin vd(s)D + wd(s)D$, i.e., $us$, $vd(s)$, $wd(s)$ are linearly independent over $D$.

By the density of the action of $R$ on $V$, there exists $x \in R$ such that

$$usx \neq 0$$

$$vd(s)x = 0$$

$$wd(s)x \neq 0.$$

Let $t \in S$. Since $vd(s)x = vs = 0$ then $vd(sxtx^*s) = 0$; hence, since $sxtx^*s \in S$ and $d(sxtx^*s)$ is not invertible, we must have $d(sxtx^*s) = 0$. Moreover $s$, and so $sxtx^*s$, is not invertible. Since $R$ has no minimal right ideals, by applying Lemma 4 to the element $sxtx^*s$, we get $sxtx^*sd(R)sxtx^*s = 0$, for all $t \in S$. Hence Lemma 2 implies $x^*sd(R)sx = 0$.

Now let $y, z \in R$. Since $x^*sd(y)sx = 0$ we have

$$0 = x^*sd(ysxz)sx = x^*syd(sxz)sx.$$

Hence $x^*sRd(sxR)sx = 0$ and, since $x^*s \neq 0$, the primeness of $R$ forces $d(sxR)sx = 0$. If $y \in R$ we get

$$0 = d(sxy)sx = d(s)xysx + sd(xy)sx;$$

hence, since $ws = 0$, $0 = wd(sxy)sx = wd(s)xysx$. But $wd(s)x \neq 0$, and, by the density of the action of $R$ on $V$, $wd(s)xR = V$; thus $0 = wd(s)xRsx = Vsx$ implying $sx = 0$, a contradiction.

We have proved that for every $s \in S$ with $d(s) \neq 0$, $\dim_D \ker s \leq 1$.

Now let $W$ be a finite dimensional subspace of $V$ such that $\dim_D W > 1$ and let $\rho = \rho_w = \{x \in R \mid Wx = 0\}$; $\rho$ is a right ideal of $R$.

We claim that there exists $s \in \rho \cap S$ such that $s^2 \neq 0$. In fact, suppose not and let $x \in \rho$, $s \in \rho \cap S$. Then, since $(xs + sx^*) \in \rho \cap S$ and $(xs + sx^*)^2 = S^2 = 0$; we get $0 = s(xs + sx^*)^2 = s(xs)^2$, i.e., $s\rho$ is a right ideal nil of bounded index. By Levitski's theorem $s\rho = 0$; hence $(\rho \cap S)\rho = 0$. Now, since $R$ has no minimal right ideals, by [3, Lemma 5.1.2.], for $v \notin W$, there exists $x \in \rho$ such that $x^* \in \rho$, $vx^* = 0$ and $v(x + x^*) = vx \notin W + Dv$. But then, by density, there exists $y \in \rho$ such that $v(x + x^*)y \neq 0$, contradicting the fact that $(x + x^*)y \in (\rho \cap S)\rho = 0$. This establishes the claim.

Then set $s \in \rho \cap S$ such that $s^2 \neq 0$. Since $\rho$ is a proper right ideal of $R$, $s$ is not invertible; moreover, since $\dim_D \ker s \geq \dim W > 1$, $d(s) = 0$. Hence, by Lemma 4, $sd(R)s = 0$.

Now, if $x \in \rho$ then $sx^* + xs \in \rho \cap S$ and $d(s) = 0$ implies $0 = d(sx^* + xs) = sd(x^*) + d(x)s$. Since $sd(x^*)s = 0$, multiplying by $s$ from the right we get $d(x)s^2 = 0$. Thus $d(\rho)s^2 = 0$. Now, for $x$, $y \in \rho$, $0 = d(xy)s^2 = d(x)ys^2$ forces $d(\rho)\rho s^2 = 0$ and, since $R$ is prime and $s^2 \neq 0$, $d(\rho)\rho = 0$. Clearly $d(\rho) \neq 0$; so, let $x \in \rho$ be such that $d(x) \neq 0$. If $vd(x) \notin W$ for some $v \in V$, then by density there exists $r \in \rho$ such that $vd(x)r \neq 0$, contradicting the fact that $d(x)r \in d(\rho)\rho = 0$. Thus $Vd(x) \subset W$ and $d(x)$ is a tranformation of finite rank, a contradiction. □

We are now in a position to prove the Theorem:

*Proof of the Theorem*. By Proposition 1 and Proposition 2 we may assume that $R$ is a simple artinian ring. Hence, $R = D_n$, the ring of $n \times n$ matrices over a division ring $D$.

Suppose first that * on $D_n$ is of transpose type and assume $n > 2$. Let $e_{ij}$ be the usual matrix units. For $i = 1, \ldots, n$ $e_{ii} = e_{ii}^* \in S$ implies $d(e_{ii}) = e_{ii}d(e_{ii}) + d(e_{ii})e_{ii}$. Thus, since rank $e_{ii} = 1$, rank $d(e_{ii}) \leq 2$ and, being $n > 2$, $d(e_{ii})$ cannot be invertible. Hence $d(e_{ii}) = 0$, $i = 1, \ldots, n$.

Now, if $i \neq j$, for a suitable $0 \neq c \in D$, $e_{ij} + ce_{ji} = e_{ij} + e_{ij}^* \in S$. Thus

$$\begin{aligned} d(e_{ij} + ce_{ji}) &= d\big(e_{ii}(e_{ij} + ce_{ji}) + (e_{ij} + ce_{ji})e_{ii}\big) \\ &= e_{ii}d(e_{ij} + ce_{ji}) + d(e_{ij} + ce_{ji})e_{ii}; \end{aligned}$$

and so, rank $d(e_{ij} + ce_{ji}) \leq 2$. It follows $d(e_{ij} + ce_{ji}) = 0$ which implies $0 = d(e_{ii}(e_{ij} + ce_{ji})) = d(e_{ij})$.

We have proved that $d(e_{ij}) = 0$ for $i$, $j = 1, \ldots, n$. Now let $x \in D$.

If $i \neq j$, $S \ni xe_{ij} + (xe_{ij})^* = xe_{ij} + c_1x^*c_2e_{ji}$ for suitable $c_1$, $c_2 \in D \cap S$. We have:

$$\operatorname{rank}\left(d\left(xe_{ij} + c_1x^*c_2e_{ji}\right)\right) = \operatorname{rank}\left(d(x)e_{ij} + d\left(e_1x^*c_2\right)e_{ji}\right) \leq 2,$$

hence $d(xe_{ij} + e_1x^*c_2e_{ji}) = 0$, and, multiplying by $e_{ji}$ from the right we get $d(x)e_{ii} = 0$, for all $i = 1,\dots,n$. Thus $d(x) = d(xI) = \Sigma_i d(x)e_{ii} = 0$, i.e. $d(D) = 0$. In short $d = 0$ in $D_n$.

Now suppose that * is symplectic. In this case $D = F$ is a field and suppose $n > 4$. Let $I_1 = e_{11} + e_{22}$; $I_1^2 = I_1 \in S$, so $\operatorname{rank} d(I_1) = \operatorname{rank}(I_1d(I_1) + d(I_1)I_1) \leq 4$ implies $d(I_1) = 0$. Now, for $i$ odd, $a = e_{1i} + e_{i+1,2} \in S$; hence $d(a) = d(I_1a + aI_1) = I_1d(a) + d(a)I_1$ has rank $\leq 4$. It follows $d(a) = 0$ and, so, for $i \neq 1$, $0 = d(I_1a) = d(e_{1i})$. On the other hand, if $i$ is even, $e_{1i} - e_{i-1,2} \in S$ and by the same argument we get $d(e_{1i}) = 0$ for $i \neq 2$. Moreover by looking at $e_{1i} + e_{i1}^*$ as above, we obtain $d(e_{i1}) = 0$ for $i \neq 1, 2$. At this stage it easily follows $d(e_{ij}) = 0$ for all $i$, $j = 1,\dots,n$. Since $d(I_1) = 0$ implies $d(F) = 0$, then $d = 0$ in $F_n$ and we are done.

We are left with the case $R = F_4$ and * symplectic. We will prove that in this case $d$ must be inner. By a well known result on finite dimensional simple algebras it is enough to prove that $d(F) = 0$. So, suppose by contradiction that there exists $\alpha \in F$ such that $d(\alpha) \neq 0$ and let $s \in S$, $s \neq 0$, be such that $d(s) = 0$. Then, since $d(\alpha) \in F$, $d(\alpha s) = d(\alpha)s \neq 0$ implying $s$ invertible. Therefore, for every $s \in S$, $s \neq 0$, $d(s) = 0$ implies $s$ invertible.

Now, if $I$ is the identity matrix in $F_2$, $t = \left(\begin{smallmatrix}0&0\\0&I\end{smallmatrix}\right) \in S$ and, since $t$ is not invertible, $d(t) \neq 0$. Moreover it is easy to prove that $d(t) = \left(\begin{smallmatrix}0&A\\B&0\end{smallmatrix}\right)$ where $A$, $B \in F_2$. Now let $V$ be a 4-dimensional vector space over $F$ and let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for $V$. Then since $d(t)$ is invertible, $e_1d(t)$, $e_2d(t)$ are linearly independent over $F$; moreover $e_1d(t)$, $e_2d(t) \in \operatorname{Span}_F\{e_3, e_4\}$.

Clearly, there exists an element $x \in F_4$ such that $e_1d(t)x = e_2d(t)x = 0$ and $\operatorname{span}_F\{e_1x, e_2x\} = \operatorname{span}_F\{e_3, e_4\}$.Now writing

$$x = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

where $X_{ij} \in F_2$, we have that $X_{21}$ is a unit and that $(txx^*t)_{2,2} = X_{21}X_{21}^* \neq 0$, a contradiction. □

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