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A NOTE ON ORDERINGS ON ALGEBRAIC VARIETIES

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It was proven in [A–G–R] that if $V \subset \mathbf{R}^n$ is a surface and α a total ordering in its coordinate polynomial ring, α can be described by a half branch (i.e., there exists $\gamma(0,\varepsilon) \to V$, analytic, such that for every $f \in \mathbf{R}[V]\operatorname{sgn}_\alpha f = \operatorname{sgn} f(\gamma(t))$ for t small enough). Here we prove (in any dimension) that the orderings with maximum rank valuation can be described in this way. Furthermore, if the ordering is centered at a regular point we show that the curve can be extended C^∞ to $t = 0$.

1. (1.0) Let V be an algebraic variety over $\mathbf{R}$ and α an ordering in $K = \mathbf{R}(V)$. If α is described by a half-branch γ: $(0,\varepsilon) \to V$, no non-zero polynomial vanishes over $\gamma(t)$ for t small enough. Consequently, if V' is birrationally equivalent to V (i.e., $\mathbf{R}(V') = \mathbf{R}(V)$), $\alpha \cap \mathbf{R}[V']$ is also described by a curve in V'.

(1.1) PROPOSITION. *Let V be an algebraic variety over $\mathbf{R}$ and $n = \dim V$. If $\mathbf{R}[V]$ is an integral extension of $\mathbf{R}[x_1,\dots,x_n] = \mathbf{R}[\underline{x}]$ and α an ordering on $\mathbf{R}[V]$ such that $\beta = \alpha \cap \mathbf{R}[\underline{x}]$ can be described by a half-branch, then the same holds true for α.*

Proof. By our previous remark (1.0) we can suppose V is a hypersurface. Thus $\mathbf{R}[V] = \mathbf{R}[\underline{x}, x_{n+1}](P)$ where $P \in \mathbf{R}[\underline{x}][x_{n+1}]$ is a monic polynomial in x_{n+1}. Let δ be the discriminant of P and π: $V \to \mathbf{R}^n$ the projection on the first n-coordinates. Then the restriction

$$\pi_|: V \setminus \pi^{-1}(\delta = 0) \to \mathbf{R}^n \setminus \{\delta = 0\}$$

has finite fibers with constant cardinal over every connected component. Moreover, by the implicit function theorem, $\pi_|$ is an analytic diffeomorphism from every connected component of $V \setminus \pi^{-1}(\delta = 0)$ onto someone of $\mathbf{R}^n \setminus \{\delta = 0\}$.

Let γ: $(0,\varepsilon) \to \mathbf{R}^n$ be the curve describing β. The connected components $C_1,\dots,C_p$ of $\mathbf{R}^n \setminus \{\delta = 0\}$ are open semi-algebraic sets, and we can write

$$C_i = \bigcup_{j=1}^{q} \{f_{ij1} > 0,\dots, f_{ijr} > 0\}, \qquad f_{ijl} \in \mathbf{R}[\underline{x}].$$

As γ describes the ordering in $\mathbf{R}[\underline{x}]$ and the C_i's are pairwise disjoint, for t small enough, $f_{ijl}(\gamma(t))$ does not change the sign and $\gamma(t)$ is contained in a unique C_{i_0}. We put $C = C_{i_0}$.

Let $D_1, \ldots, D_s$ (we shall see below that s is not zero) be the connected components of $V \setminus \pi^{-1}\{\delta = 0\}$ diffeomorphic to C via π. We claim that

$$s = \text{number of extensions of } \beta \text{ to } R(V).$$

By construction s is the number of roots of $P(\underline{x}, x_{n+1})$ for every $\underline{x} \in C$. On the other hand, the number of extensions of β to $\mathbf{R}(V)$ coincides with the number of roots of $P \in \mathbf{R}(\underline{x})[x_{n+1}]$ in a real closure of $(\mathbf{R}(\underline{x}), \beta)$ (see **[Pr]** 3.12). We shall prove now the latter is also the number of real roots of $P(\underline{x}, x_{n+1})$ for $\underline{x} \in C$.

Let $S = \{P_0, \ldots, P_l\}$ $\mathbf{R}(\underline{x})[x_{n+1}]$ be the standard Sturm sequence of

$$P(\underline{x}, x_{n+1}) = x_{n+1}^m + a_1 x_{n+1}^{m-1} + \cdots + a_m, \qquad M = 1 + m + \sum_{i=1}^m a_i^2$$

and Δ the product of all numerators and denominators of the non-zero coefficients of the polynomials in x_{n+1} used in the construction of S. In this situation, by Artin's specialization theorem there exists $\underline{x}_0 \in \mathbf{R}^n$ such that

(a) $f_{i_0jh}(\underline{x}_0) > 0$, $\Delta(\underline{x}_0) \neq 0$, some $j = 1, \ldots, q$, all $h = 1, \ldots, r$

(b) $\operatorname{sgn}_\beta P_k(\pm M) = \operatorname{sgn}_{\mathbf{R}} P_k(\underline{x}_0, \pm M(\underline{x}_0))$, $k = 0, \ldots, l$.

By (a), $\underline{x}_0 \in C$ and $S_{x_0} = \{P_1(\underline{x}_0), \ldots, P_l(\underline{x}_0)\}$ is the standard Sturm sequence of $P(\underline{x}_0, x_{n+1})$. By (b) the number of sign changes of $S_{\underline{x}_0}$ and S coincides. Then the claim is proven.

Now, let us denote by $\gamma_k = (\pi_{|D_k})^{-1} \circ \gamma$, $k = 1, \ldots, l$ the liftings of γ. Then it is easy to prove:

(a′) If $f \in \mathbf{R}[V] \setminus \{0\}$, $f(\gamma_k(t)) \neq 0$ and its sign does not change for t small enough. Consequently every γ_k defines an ordering that we call α_k.

(b′) If $k \neq k'$, $\alpha_k \neq \alpha_{k'}$.

From the remarks above, α must be equal to some α_k, hence it is described by the corresponding α_k.

2. (2.0) Let K and Δ be ordered fields and $p\colon K \to \Delta, \infty$ a place such that for x positive, $p(x)$ is not negative. Then we define a signed place $\hat{p}\colon K \to \Delta \cup \{+\infty, \infty\} = \Delta, \pm\infty$ in the following way:

$$\hat{p}(x) = p(x) \quad \text{if } p(x) \neq \infty; \qquad \hat{p}(x) = \operatorname{sign}(x) \cdot \infty \quad \text{if } p(x) = \infty.$$

Now assume K is the function field of a real algebraic variety V, and α an ordering in K. A point $O \in V$ is the center of α in V if the real valued canonical place p_α associated to α (see **[B]** Chap. VII) is finite over

$\mathbf{R}[V]$ and the ideal of O is the center of p_α in $\mathbf{R}[V]$. In that case, every function positive at O is positive in α, and if α is described by γ, then $\lim_{t\to 0}\gamma(t) = O$.

We are interested in the case when the rank of p_α is maximum (i.e., it coincides with the dimension of V). In this situation the decomposition of p_α in rank 1 places is

$$K = K \overset{\theta_{n-1}}{\to} K_{n-1}, \infty \to \cdots \to \mathbf{R}, \infty, \tag{2.0.1}$$

where K_j is a function field over $\mathbf{R}$ of dimension j. Then it is possible to define uniquely orderings in K_j ($j = 1, \dots, r$) such that, considering α in K, all places verify the compatibility conditions. Thus we consider the associated signed places $\hat{\theta}_j$: $K_j \to K_{j-1}, \pm\infty$ (see [**B**] Chap. VIII), to get a decomposition of $\hat{p}_\alpha$ in rank 1 signed places.

(2.1) Proposition. *If p_α has a maximum rank, α can be described by a half-branch.*

Proof. The proof goes by induction. If $n = 1$, by 1.1 and 1.0 we can suppose $K = \mathbf{R}(x)$, α centered at $x = 0$, and $x >_\alpha 0$. Then, there is a unique ordering with this property (i.e., making x infinitesimal with respect to $\mathbf{R}$ and positive), and it is described by the curve $\gamma(t) = t$.

In the general situation we can choose $\zeta_1, \dots, \zeta_{n-1}, \zeta_n$ in K such that $\theta_{n-1}(\zeta_1), \dots, \theta_{n-1}(\zeta_n) \in K_{n-1}$ and:

(i) $\theta_{n-1}(\zeta_1), \dots, \theta_{n-1}(\zeta_{n-1})$ are algebraically independent.

(ii) $\zeta_1, \dots, \zeta_n$ are algebraically independent

(iii) $p_\alpha(\zeta_i) = 0$ ($i = 1, \dots, n$).

Since K is the quotient field of the integral closure of $B = \mathbf{R}[\zeta_1, \dots, \zeta_{n-1}, \zeta_n]$ we can suppose $K = q \cdot f(B)$ by 1.1. Then the kernel of $\theta_{n-1|}$: $B \to K_{n-1}$ is an height one prime ideal and hence it is generated by some $F \in B$. The field K_{n-1} is the function field of the hypersurface $\{F = 0\}$. Moreover we may assume $F >_\alpha 0$.

Let us consider, according to 2.0, the ordering β associated to $r = \theta_0 \circ \cdots \circ \theta_{n-2}$ in K_{n-1}. Then $p_\beta = r$ and β is centered at $\underline{0} = (0, \dots, 0)$ which belongs to the hypersurface. Consequently, for every $f \in B$ we have:

(2.1.1) if $f(\underline{0}) = p_\alpha(f) \neq 0$, then $\operatorname{sgn}_\alpha f = \operatorname{sgn} f(\underline{0})$
if $\theta_{n-1}(f) \neq 0$, $\operatorname{sgn}_\alpha f = \operatorname{sgn}_\beta \bar{f}$, where $\bar{f}$ is $f + (F)$
if $\theta_{n-1}(f) = 0$ and $f = u \cdot F^r$ with g.c.d $(u, F) = 1$,
then $\operatorname{sgn}_\alpha f = \operatorname{sgn}_\alpha u = \operatorname{sgn}_\beta \bar{u}$.

Now we need a lemma:

(2.2) LEMMA. *Let $H = \{F(\underline{x}) = 0\}$ be a real irreducible hypersurface in $\mathbf{R}^n$ and β a rank $(n-1)$ ordering in H (i.e., in $\mathbf{R}[\underline{x}]/(F)$) centered at the point $\underline{0}$ and described by $\gamma\colon (0, \varepsilon) \to H$. Then, there is not more than one ordering α in $\mathbf{R}[\underline{x}]$ making F infinitesimal and positive, and inducing β in $\mathbf{R}[\underline{x}]/(F)$. Moreover α can be described by a half-branch.*

Proof. The first claim is an easy consequence of 2.1.1.

Next, as p_β has rank $n-1$, p_β is discrete and its value group is isomorphic to $Z \oplus \overset{n-1}{\cdots} \oplus Z$, lexicographically ordered. Let $\bar{h} \in \mathbf{R}[\underline{x}]/(F)$ have value $(a_1, \ldots, a_{n-1})$ with $a_1 \geq 1$ (notice that this is possible because the valuation ring of p_β contains $\mathbf{R}[\underline{x}]/(F)$), and put $\psi(t) = h(\gamma(t))$. Since $p_\beta(\bar{h}) = 0$, $h(\underline{0}) = 0$ *and* $\mathit{lim}_{t\to 0}\psi(t) = 0$, ψ is analytic in $(0, \varepsilon)$. Now we define the analytic curve:

$$\gamma^*\colon (0, \varepsilon) \to \mathbf{R}^n\colon t \mapsto \left(\gamma_i(t) + c_i e^{-1/\psi(t)^2}\right) \qquad i = 1, \ldots, n$$

where the c_i's will be determined later.

Thus, the result follows from the statements (a) and (b) below.

(a) For any c_i's, if $G \in \mathbf{R}[x]$ is positive along γ, so is along γ^*.

(b) There is $(c_1, \ldots, c_n) \in \mathbf{R}^n$ such that $F(\gamma^*(t)) > 0$ for t small enough.

To prove (a) we first write:

$$G(\gamma^*(t)) = G(\gamma(t)) + m(t)e^{-1/\psi(t)^2} \tag{2.2.1}$$

where $m(t)$ is a polynomial in $\gamma_1(t), \ldots, \gamma_n(t)$ and $e^{-1/\psi(t)^2}$. On the other hand, looking at the value of $\bar{h}$, for large $m \in N$ we know that $\bar{h}^m/\bar{G}$ ($\bar{G} = G + (F) \in \mathbf{R}[\underline{x}]/(F)$) is infinitesimal in β w.r.t. $\mathbf{R}$ and so, $1 - \bar{h}^m/\bar{G} >_\beta 0$. Since $\bar{G}$ is positive in β, taking an even m we have $\bar{G} >_\beta \bar{h}^m >_\beta 0$. Hence $G(\gamma(t)) >_\beta \psi(t)^m >_\beta 0$ for small t enough, what implies $\lim_{t\to 0} e^{-1/\psi(t)^2}/G(\gamma(t)) = 0$. Thus, we get (a) after dividing in 2.2.1 by $G(\gamma(t))$ and taking the limit when $t \to 0$.

For (b), we take the Taylor expansion of F at $\gamma(t)$ and compute it at $\gamma^*(t)$:

$$F(\gamma^*(t)) = \sum_{i=1}^{n} \frac{\partial F(\gamma(t))}{\partial x_i} c_i e^{-1/\psi(t)^2} + \sum_{i,j} \frac{\partial^2 F(\gamma(t))}{\partial x_i \partial x_j} c_i c_j e^{-2/\psi(t)^2} + \cdots . \tag{2.2.2}$$

As $\partial F/\partial x_i \notin (F)$ for some i, we have $c_i = \operatorname{sgn}_\beta(\partial F/\partial x_i)$ $(= \pm 1 \neq 0)$ and we take $c_j = 0$ for $j \neq i$. Then, β being described by γ:

$$(2.2.3) \qquad H(t) = \sum_{i=1}^{n} \frac{\partial F(\gamma(t))}{\partial x_i} c_i > 0, \quad \text{for small } t.$$

Again we have $\lim_{t\to 0} e^{-1/\psi(t)^2}/H(t) = 0$. Then, dividing in 2.2.2 by $H(t)$, we find $F(\gamma^*(t))/H(t) > 0$, hence $F(\gamma^*(t)) > 0$, for small t.

(2.3) Remark. Looking at the class of the curve γ at 0, we see that if $O \in \operatorname{Reg} H$, and γ can be extended C^∞ to $t = 0$, the same holds true for γ^*.

(2.4) Remark. Notice that 2.2 and 2.3 hold also true if we replace $\mathbf{R}^n$ by an algebraic variety V with $O \in \operatorname{Reg} V$. In fact the same proof applies, by taking a regular system of parameters at O in the place of $x_1, \ldots, x_n$.

(2.5) *Application*. As an example of the constructibility of the proof of 2.1 we determine the curves describing the rank 2 orderings in $\mathbf{R}^2$ (see [**A-G-R**]).

Firstly, after changes $x \to \pm(x \pm a)^{\pm 1}$, $y \to \pm(y \pm b)^{\pm 1}$, we can suppose $(0,0)$ is the center of the ordering α and $x >_\alpha 0$, $y >_\alpha 0$. Assume the divisor w which specializes p_α is centered in $\mathbf{R}[x, y]$ at $F(x, y) = 0$, and $x = t^n$, $y = a_1 t^{n_1} + \cdots$ $(n \leq n_1)$, $t > 0$, is a primitive parametrization of the half-branch describing the corresponding ordering in $\mathbf{R}[x, y]/(F)$. According to the above parametrization and looking at the proof of 2.2, we may choose $h(x) = x$, $c_1 = 0$ and $c_2 = \pm 1$ in the proof of 2.2, and we get a half-branch describing α of the form:

$$\gamma(t) = \left(t^n, \pm e^{-1/t^{2n}} + a_1 t^{n_1} + \cdots\right)$$

Now assume that the prime divisor w is centered at the maximal ideal, (x, y). Let us call v the valuation corresponding to p_α. Following Abhyankar [**A**], after a finite number of quadratic transforms along w we get the previous situation. In fact, we call $A_0 = \mathbf{R}[x, y]$ and, if $v(x) \leq v(y)$ (so $w(x) \leq w(y)$) we put: $r_0 = p_\alpha(y/x)$, $y_1 = (y - r_0 x)/x$, $x_1 = x$ and $A_1 = A_1[x_1, y_1]$. Repeating this procedure we end at $A_s = A_{s-1}[x_s, y_s] = \mathbf{R}[x_s, y_s]$ such that, the center of w in A_s is 1-dimensional, and w is centered at (x_{s-1}, y_{s-1}) in A_{s-1}. We have, say,

$$y_s = (y_{s-1} - r_{s-1} x_{s-1})/x_{s-1}$$

and $x_s = x_{s-1}$. Hence $w(x_s) = w(x_{s-1}) > 0$ and $M_w \cap A_s = (x_s)$. Thus, according to the proof of 2.2, the half-branch $x_s = \pm e^{-1/t^2}$, $y_s = t$ describes the ordering in A_s. Hence, going backwards in the quadratic transformations, it follows easily that the ordering α can be described by a curve

$$\left(P(t, e^{-1/t^2}), Q(t, e^{-1/t^2})\right)$$

for some polynomials P and Q.

3. (3.0) We finish this note with some considerations about the class at $t = 0$ of the γ's describing orderings (see also **[R]** §3). To start with notice that any algebraically independent power series $x_1(t), \dots, x_n(t)$, describe an ordering in $\mathbf{R}[x]$. Then by **[An]** the set of such orderings is dense in the space of all orderings endowed with the Harrison Topology **[H]**. Moreover, the valuations associated to these orderings are discrete of rank one. Hence the orderings with maximum rank valuation, cannot be described by curves which are analytic at $t = 0$ unless the variety is a curve. So, the best result we can expect is the following:

(3.1) PROPOSITION. *If $V \subset \mathbf{R}^n$ is an algebraic variety an α an ordering centered at $0 = (0, \dots, 0)\operatorname{Reg} V$, with associated valuation of maximum rank, there is a half-branch describing α which can be extended C^∞ (but not analytically) to $t = 0$. Furthermore the set of orderings of $\mathbf{R}[V]$ described by half-branches C^∞ at $t = 0$ but not by analytic ones, is dense in the space of orderings.*

Proof. The proof goes by induction on $d = \dim V$. If $d = 1$, the valuation associated to the ordering α is discrete, has rank one, and the ordering is described by the unique branch of V through 0:

$$(t, u_2(t), \dots, u_n(t))$$

where each $u_i(t)$ is analytic and the choice $t > 0$ or $t < 0$.

In the general case, set $\hat{p}_\alpha = p$ and consider again

$$K = \mathbf{R}(V) \xrightarrow{q} K_{n-1}, \pm\infty \xrightarrow{r} \mathbf{R}, \pm\infty, \qquad p = r \circ q,$$

the decomposition of p in signed places of rank one.

As we did in 2.1 we can find an (affine) algebraic variety V_1 and π: $V_1 \to V$ birational morphism such that the center of q in V_1, say H_1, has dimension $d - 1$. By means of Hironaka's desingularization I **[Hi]** we may assume V_1 is smooth. Then by Hironaka's desingularization II (loc. cit), we find $\tilde{V}$ and $\tilde{\pi}$: $\tilde{V} \to V_1$, a proper birrational map such that $\tilde{\pi}^{-1}(H_1)$ is a normal crossing divisor. Let $\tilde{0}$ be the center of p in $\tilde{V}$ and $\tilde{H}$ the center of q. Since the valuation ring of q, $\mathbf{R}[V_1]_{\mathscr{I}(H_1)}$, dominates $\mathbf{R}[\tilde{V}]$ and $\tilde{H}$ lies over H_1, we have $K_{n-1} = qf \cdot \tilde{H}$ and the center of r in $\tilde{H}$ is $\tilde{0}$.

We call β the ordering in K_{n-1} corresponding to the precedent decomposition (i.e. $\hat{p}_\beta = r$). Since r has maximum rank, by our inductive hypothesis the ordering $\beta \cap \mathbf{R}[\tilde{H}]$ can be described by $\gamma\colon (0, \varepsilon) \to \tilde{H}$, with $\lim_{t \to 0} \gamma(t) = 0$, and γ can be extended C^∞ to $t = 0$. Then, considering a modification γ^* of γ as we did in 2.2 and using Remarks 2.3 and 2.4, α is described in $\tilde{V}$ by γ^* and it can be extended C^∞ to $t = 0$. Finally $\pi_1 \circ \tilde{\pi} \circ \gamma^*$ is a curve which defines the ordering α and can be extended C^∞ to $t = 0$.

The second part comes from the first one, the above remark 3.0, and the fact that the set of orderings with maximum rank are dense (see **[B]**, 8.4.9).

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