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ON RESTRICTION THEOREMS OF MAXIMAL-TYPE

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In this paper we prove a vector-valued restriction theorem for the Fourier transform. This result allows us to establish certain sharp "maximal-type" restriction theorems and a partial generalization of the Riemann-Lebesgue lemma for certain $p > 1$.

1. Introduction. Vector-valued inequalities for Bochner-Riesz multipliers have been known for some time in two dimensions and for certain exponents in higher dimensions (cf. [**2**], [**3**], [**4**]). In this note our main result is the corresponding vector-valued restriction theorem for the Fourier transform which is valid everywhere the restriction problem is now known to be true. In two dimensions the result is proved by using the techniques of oscillatory integrals while in higher dimensions our inequalities follow trivially from known ones. Finally, using these results we then obtain a lacunary restriction theorem of maximal-type for the Fourier transform.

2. Vector-valued inequalities. Our main results for this section will be the following sharp vector-valued restriction theorems. By $d\sigma$ we will mean Lebesgue measure on the unit sphere $S^{n-1} \subset \mathbf{R}^n$.

THEOREM 1. *If* $n = 2, 1 \leq p < 4/3, q = p'/3, R_j > 0, j = 1, 2, 3, \dots$

$$\left\| \left(\sum |\hat{f}_j(R_j \cdot) R_j^{2/p'}|^2 \right)^{1/2} \right\|_{L^q(S^1)} \leq C_p \left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_{L^p(R^2)},$$

whenever $f_j \in \mathscr{S}$.

THEOREM 2. *If*

$$n \geq 3, \quad 1 \leq p \leq 2(n+1)/(n+3),$$

$$q = (n-1)p'/(n-1), \quad R_j > 0,$$

$$\left\| \left(\sum |\hat{f}_j(R_j \cdot) R_j^{n/p'}|^2 \right)^{1/2} \right\|_{L^q(S^{n-1})} \leq C_p \left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_{L^p(R^n)},$$

whenever $f_j \in \mathscr{S}$.

We start out by proving Theorem 1. For this purpose we define for $x \in \mathbf{R}^2$, $R_j > 0$,

$$T_j g(x) = \int_{S^1} \exp\left[iR_j(x \cdot y)\right] g(y)\, d\sigma(y).$$

It then follows from duality that Theorem 1 is equivalent to the following:

THEOREM 1′. *If* $1 \le p < 4$, $q = 3p'$

$$\left\| \left(\sum |T_j g_j|^2 \right)^{1/2} \right\|_{L^q(R^2)} \le C_p \left\| \left(\sum |g_j(\cdot) R_j^{-2/q}|^2 \right)^{1/2} \right\|_{L^p(S^1)}.$$

To prove this we will make use of the following lemma which follows trivially from Minkowski's integral inequality (cf. **[8]**, p. 19).

LEMMA. *If* (X, dx) *is a measure space and* $1 \le r \le 2$ *then*

$$\left(\sum \|f_j\|_r^2 \right)^{1/2} \le \left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_r \tag{1}$$

and,

$$\left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_{r'} \le \left(\sum \|f_j\|_{r'}^2 \right)^{1/2}. \tag{2}$$

We now come to the proof of Theorem 1′. It is based on a modification of the arguments of Zygmund **[9]** using inequalities (1) and (2).

Proof of Theorem 1′. Fix p and q as in the statement and put $r = q/2$. Then $r > 2$ and so by (2) we have

$$\left\| \left(\sum |T_j g_j|^2 \right)^{1/2} \right\|_q^2 \le \left(\sum \sum \|T_j g_j T_k g_k\|_r^2 \right)^{1/2}. \tag{3}$$

However,

$$\begin{aligned} &\|T_j g_j T_k g_k\|_r \\ &= \left(\int_{R^2} \left| \int_0^{2\pi} \int_0^{2\pi} \exp\left[ix \cdot (R_j \cos s - R_k \cos t, R_j \sin s - R_k \sin t)\right] \right.\right. \\ &\qquad\qquad \left.\left. \times g_j(s) g_k(t)\, ds\, dt \right|^r dx \right)^{1/r}. \end{aligned}$$

Therefore, we put for a fixed pair j, k

$$u = \left(R_j \cos s - R_k \cos t, R_j \sin s - R_k \sin t\right)$$

and since $|\partial u/\partial(s,t)| = R_j R_k |\sin(s-t)|$ we see that the last integral equals

$$\begin{aligned}
&\left(\int_{R^2}\left|\int_{R^2} e^{iu\cdot x} g_j(s) g_k(t) \left|R_j R_k \sin(s-t)\right|^{-1} du\right|^r dx\right)^{1/r} \\
(4)\qquad &\leq \left(\int \left|g_j(s) g_k(t)\left(R_j R_k \sin(s-t)\right)^{-1}\right|^{r'} du\right)^{1/r'} \\
&= \left(\iint \left|R_j^{-2/q} g_j(s)\right|^{r'} \left|R_k^{-2/q} g_k(t)\right|^{r'} \left|\sin(s-t)\right|^{-r'+1} ds\, dt\right)^{1/r'}.
\end{aligned}$$

The last inequality comes from the Hausdorff-Young Theorem.

Since $r' < 2$ we now see from inequalities (1), (3), (4) that

$$\begin{aligned}
&\left\|\left(\sum |T_j g_j|^2\right)^{1/2}\right\|_q^2 \\
&\leq \left[\sum\sum\left(\iint \left|R_j^{-2/q} g_j(s)\right|^{r'} \left|R_k^{-2/q} g_k(t)\right|^{r'} \times \left|\sin(s-t)\right|^{-r'+1} ds\, dt\right)^{2/r'}\right]^{1/2} \\
&\leq \left[\iint \left(\sum \left|R_j^{-2/q} g_j(s)\right|^2\right)^{r'/2} \left(\sum \left|R_k^{-2/q} g_k(T)\right|^2\right)^{r'/2} \times \left|\sin(s-t)\right|^{-r'+1} ds\, dt\right]^{1/r'}.
\end{aligned}$$

If now as in [9] we use the classical inequality or Hardy-Littlewood for fractional integrals we find that this last expression is dominated by

$$\left(\int \left(\sum \left|R_j^{-2/q} g_j(s)\right|^2\right)^{p/2} ds\right)^{2/p} = \left\|\left(\sum \left|R_j^{-2/q} g_j\right|^2\right)^{1/2}\right\|_p^2.$$

This completes the proof of Theorem 1′.

Theorem 2 follows easily from the L^2-restriction theorem of Stein-Tomas [7] and our lemma. In fact if, as before, we define

$$T_j g(x) = \int_{S^{n-1}} \exp\left[iR_j(x\cdot y)\right] g(y)\, d\sigma(y)$$

then the restriction theorem of Stein-Tomas implies that for each $1 \leq p \leq 2$, $q = p'(n+1)/(n-1)$,

$$\|T_j g\|_{L^q(R^n)} \leq C\left\|R_j^{-n/q} g_j\right\|_{L^p(S^{n-1})}.$$

Therefore, since $q > 2$ and $p < 2$ this inequality and the lemma show that

$$\left\| \left(\sum |T_j g_j|^2 \right)^{1/2} \right\|_q \leq \left(\sum \|T_j g_j\|_q^2 \right)^{1/2} \leq C \left(\sum \|R_j^{-n/q} g_j\|_p^2 \right)^{1/2}$$
$$\leq C \left\| \left(\sum |R_j^{-n/q} g_j|^2 \right)^{1/2} \right\|_p .$$

As before, Theorem 2 follows by duality from this last inequality.

3. Restriction theorems of maximal-type. In this section we indicate how the above results imply the following sharp lacunary restriction theorems of maximal-type.

We define the lacunary maximal restriction operators:

$$M_p \hat{f}(y) = \sup_{j \in Z} |\hat{f}(2^j y) 2^{nj/p'}|, \qquad y \in S^{n-1},\ f \in \mathcal{S}.$$

THEOREM 3. *Let* $n = 2$, $1 \leq p < 4/3$, $q = p'/3$. *Then*

$$\|M_p \hat{f}\|_{L^q(S^{n-1})} \leq C_p \|f\|_{L^p(R^n)}, \qquad f \in \mathcal{S}.$$

THEOREM 4. *Let* $n \geq 3$,

$$1 \leq p \leq 2(n+1)/(n+3), \quad q = p'(n-1)/(n+1).$$

Then

$$\|M_p \hat{f}\|_{L^p}(S^{n-1}) \leq C_p \|f\|_{L^p(R^n)}, \qquad f \in \mathcal{S}.$$

Proofs. Fix $n \geq 2$ and p and q as above. Clearly, to prove these results it suffices to show that whenever $\{E_j\}$, $j \in \mathbf{Z}$, is a collection of disjoint measureable subsets of S^{n-1} there is an absolute constant C for which

$$\left\| \sum \chi_{E_j}(\cdot) \hat{f}(2^j \cdot) 2^{nj/p'} \right\|_{L^q(S^{n-1})} \leq C \|f\|_p . \tag{5}$$

Now let $b \in C_0^\infty$ be a bump function with the properties that $b(1) = 1$, $\operatorname{supp}(b) \subset (1/4, 2)$ and $\Sigma b(2^j t) = 1$, $t \in \mathbf{R}_+$. If we define f_j by $\hat{f}_j = b(2^{-j}|y|)\hat{f}(y)$ then the left hand side of (5) is majorized by

$$\left\| \left(\sum |\hat{f}_j(2^j \cdot) 2^{nj/p'}|^2 \right)^{1/2} \right\|_{L^q(S^{n-1})} \leq C \left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_{L^p(R^n)} .$$

The inequality follows from Theorems 1 and 2. Our desired result now follows via Littlewood-Paley theory from the inequality $\|(\Sigma |f_j|^2)^{1/2}\|_p \leq C\|f\|_p$ if $p > 1$. The case $p = 1$ is the Riemann-Lebesgue lemma.

This finishes the proofs.

If now $n \geq 2$ and p is as in Theorem 1 or 2 it follows that $\hat{f}(2^j\xi)$ can be defined as an element of $L^q(S^{n-1})$, $q = p'(n-1)/(n+1)$. Elements of $L^q(S^{n-1})$ of course are only defined almost everywhere so in what follows to avoid ambiguity we assume that for each $f \in L^p(\mathbf{R}^n)$ and $j \in \mathbf{Z}$ we have fixed a function $\hat{f}(2^j\xi)$ which represents this element. If we now define $M_p\hat{f}$ as before, it follows from a simple limiting argument that the a priori inequalities of Theorems 3 and 4 now hold for general $f \in L^p(\mathbf{R}^n)$. With these things in mind we now come to the following partial generalizations of the classical Riemann-Lebesuge lemma.

COROLLARY 1. *If* $1 < p < 4/3$, $f \in L^p(\mathbf{R}^2)$ *then for almost every direction* $\xi \in S^1$ *we have*:

$$2^{2j/p'}\hat{f}(2^j\xi) \to 0 \quad as\ |j| \to \infty.$$

COROLLARY 2. *If* $n \geq 3$, $1 < p \leq (2n+2)/(n+3)$ *and* $f \in L^p(\mathbf{R}^n)$ *then for almost every every direction* $\xi \in S^{n-1}$ *we have*:

$$2^{nj/p'}\hat{f}(2^j\xi) \to 0 \quad as\ |j| \to \infty.$$

Proofs. Fix $n \geq 2$ and p and q as above. Define

$$\Omega\hat{f}(\xi) = \limsup_{j\to\infty}\left[\left|2^{nj/p'}\hat{f}(2^j\xi)\right| + \left|2^{-nj/p'}\hat{f}(2^{-j}\xi)\right|\right].$$

We must show that $\Omega\hat{f} = 0$ for almost every direction $\xi \in S^{n-1}$. Clearly, $\Omega\hat{f}(\xi) \leq 2M_p\hat{f}(\xi)$ and so by Theorems 3 and 4

$$m\{\xi \in S^{n-1}: \Omega\hat{f}(\xi) > \varepsilon\} \leq C/\varepsilon^q \|f\|_p^q.$$

It is also clear that if $h \in \mathcal{S}$ then $\Omega\hat{h} = 0$ almost everyone. Furthermore, $f \in L^p(\mathbf{R}^n)$ can be written as $f = g + h$ with $h \in \mathcal{S}$ and $\|g\|_p$ arbitrarily small. Consequently, as $\Omega\hat{f} \leq \Omega\hat{g} + \Omega\hat{h}$ we get that

$$m\{\xi \in S^{n-1}: \Omega\hat{f}(\xi) > \varepsilon\} \leq C/\varepsilon^q \|g\|_p^q.$$

Finally, since the size of $\|g\|_p$ is at our disposal, we see that $\Omega\hat{f} = 0$ almost everywhere as desired.

REMARKS. We now show that how the results of this section are of the best possible nature in the sense that in $\mathbf{R}^n$, $n \geq 2$, no inequality of the kind

$$\int_{S^{n-1}} \sup_{1<r<2} |\hat{f}(r\xi)|\, d\sigma \leq A_p \|f\|_{L^p(R^n)}$$

can hold for any $p > 1$.

In fact consider $f = f_1(x')f_2(x_n)$ where $f_1(x') = e^{-|x'|^2}$, and

$$\hat{f}_2(\xi_n) = \begin{cases} |\xi_n - 1|^{-1/2p'}, & |\xi_n| < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\hat{f}(\xi', \xi_n) = c_n|\xi_n - 1|^{-1/2p'}e^{-|\pi\xi'|^2}$ for (ξ', ξ_n) near $(0, \ldots, 0, 1)$, and hence $\sup_{1<r<2}|\hat{f}(r\xi)| = \infty$ $\xi \in S^{n-1}$ in a neighborhood of $(0, \ldots, 0, 1)$ even though $f \in L^p(\mathbf{R}^n)$.

This counterexample was kindly pointed out to the author by E. M. Stein.

Finally, we also remark that one would expect that Theorems 2 and 4 should also hold for $(2n + 2)/(n + 3) < p < 2n/(n + 1)$, since this is the conjectured range for the restriction problem for the Fourier transform (cf. **[5]**).

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