PACIFIC JOURNAL OF MATHEMATICS
Vol. 123, No. 2, 1986

ON THE RANGE OF AN ANALYTIC MULTIVALUED FUNCTION

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Proofs are given of set-valued analogues of Rouché's theorem, the argument principle, and the Picard theorems. This is achieved by developing a rudimentary theory of upper semicontinuous multivalued lifts.

Introduction. Analytic multivalued (a.m.v) functions first appeared in 1934 in a paper by K. Oka [**14**], where they were used to generalize a theorem of Hartogs. After being ignored for half a century, the theory of these functions has been rejuvenated by recent successful applications to functional analysis, beginning with the work of Z. Słodkowski [**21**], and continuing in [**5, 6, 8, 9, 10, 11, 15, 16, 19, 22, 24, 25, 26, 27, 29, 30**]. This has motivated a study of their properties, notably in [**1, 2, 6, 7, 8, 9, 11, 15, 17, 18, 21, 23, 29**], and the present paper continues this development.

Let G be a bounded open subset of $\mathbf{C}$, and suppose K is an upper semicontinuous set-valued map on $\overline{G}$ which is a.m.v. on G (these terms are explained in §1). The behaviour of K within G is *not* uniquely determined by its values on the boundary of G, as would be the case for a single-valued analytic function. As a simple example, consider $G = \{\lambda: |\lambda| < 1\}$ with

$$K_1(\lambda) = \{z: |z| \leq |\lambda|\},$$

$$K_2(\lambda) = \{z: |z| \leq 1\}.$$

However, in [**18**] it is shown that the boundary values of K do place certain constraints on the size of K at each point in G, and the underlying theme of the present paper is an investigation into what extent they determine its range, $K(\overline{G})$. This is motivated by the fact (Proposition 2.1) that $K(\overline{G})$ is the union of $K(\partial G)$ with a subcollection of the bounded components of $\mathbf{C} \setminus K(\partial G)$, so that one might expect that as in the classical theory, the range of K is computable via some form of argument principle. This hope is strengthened by a multivalued form of Rouché's theorem, proved in §2. To define winding numbers for multivalued functions, it is convenient first to set up a simple theory of covering spaces and upper semicontinuous lifts. This is carried out in §3, and then used in §4 to prove a simple topological argument principle. The main result appears in §5: it is a strong form of argument principle which is essentially analytic in nature. Finally, in §6 we give a second application of the theory of u.s.c. lifts to prove two Picard theorems for those a.m.v. functions whose values are connected and polynomially convex.

Acknowledgments. This paper was written partly while the author was a research fellow at Trinity College, Cambridge, and partly while he was an E. R. Hedrick assistant professor at the University of California, Los Angeles. He would like to express his gratitude to both bodies.

1. Preliminaries. The purpose of this section is to summarize some basic facts about analytic multivalued functions. We begin with some notation.

Let X and Y be Hausdorff topological spaces. Denote by $\kappa(Y)$ the collection of all non-empty compact subsets of Y. Given a map S: $X \to \kappa(Y)$ and subsets $A \subset X$ and $B \subset Y$, we write

$$S^{-1}(B) = \{x \in X: S(x) \subset B\},$$
$$S(A) = \bigcup\{S(x): x \in A\},$$
$$S \mid A = \text{the restriction of } S \text{ to } A.$$

The map S: $X \to \kappa(Y)$ is *upper semicontinuous* (*u.s.c*) if $S^{-1}(U)$ is open in X whenever U is open in Y. It is *continuous* if in addition $S^{-1}(F)$ is closed in X whenever F is closed in Y. A basic fact about u.s.c. maps S which we shall often use is that if $A \in \kappa(X)$ then $S(A) \in \kappa(Y)$.

Let G be an open subset of $\mathbf{C}$, and let K: $G \to \kappa(\mathbf{C})$ be u.s.c. Then K is an *analytic multivalued* (or *a.m.v*) *function* if whenever G_1 is open in G, and whenever ψ is a function plurisubharmonic on a neighborhood of $\{(\lambda, z): \lambda \in G_1,\ z \in K(\lambda)\}$, then the function

$$\phi(\lambda) = \sup\{\psi(\lambda, z): z \in K(\lambda)\} \qquad (\lambda \in G_1)$$

is subharmonic on G_1. This definition was introduced by Z. Słodkowski in [**21**], where he also gave a number of equivalent characterizations of a.m.v. functions. Further information regarding these functions and their applications may be found in the references cited in the Introduction. We shall only summarize those properties which will be needed in the sequel.

For the rest of this section, G denotes a non-empty open subset of $\mathbf{C}$. Firstly, here are some basic examples of a.m.v functions. (Part (a) is obvious, part (b) is a consequence of [**6**, Lemma 2.6], and part (c) is just [**17**, Proposition 4.5].)

PROPOSITION 1.1. (a) *If K: $G \to \kappa(\mathbf{C})$ is constant, then it is a.m.v.*

(b) *If f: $G \to \mathbf{C}$ is any function, then $K(\lambda) = \{f(\lambda)\}$ is a.m.v. if and only if f is analytic (in the usual sense) on G.*

(c) *If r: $G \to [0, \infty)$ is any function, then $K(\lambda) = \{z\colon |z| \le r(\lambda)\}$ is a.m.v. if and only if* $(\log r)$ *is subharmonic on G.* □

The next result provides several ways of building new a.m.v. functions from old. It is a special case of [**21**, Proposition 5.1].

PROPOSITION 1.2. *Let K: $G \to \kappa(\mathbf{C})$ be an a.m.v. function.*

(a) *If f: $G \to \mathbf{C}$ is analytic, then $L(\lambda) = f(\lambda) \cdot K(\lambda)$ is a.m.v. on G.*

(b) *If U is an open neighborhood of $K(G)$, and g: $U \to \mathbf{C}$ is analytic, then $(g \circ K)$ is a.m.v. on G, where $(g \circ K)(\lambda) = \{g(z)\colon z \in K(\lambda)\}$.*

(c) *If H is open in $\mathbf{C}$, and h: $H \to G$ is analytic, then $(K \circ h)$ is a.m.v. on H.*

(d) *If L: $G \to \kappa(\mathbf{C})$ is a.m.v., then $(K + L)$ is a.m.v. on G, where $(K + L)(\lambda) = \{z + w\colon z \in K(\lambda), w \in L(\lambda)\}$.* □

Given $T \in \kappa(\mathbf{C})$, its *polynomial hull* $\hat{T}$ is the set of $w \in \mathbf{C}$ such that $|p(w)| \le \|p\|_T$ for every polynomial p. The set $\hat{T}$ is just the union of T with all the bounded components of $\mathbf{C} \setminus T$. We say T is *polynomially convex* if $\hat{T} = T$. We shall be very much concerned with multivalued functions Q whose values $Q(\lambda)$ are all connected, polynomially convex sets; such an hypothesis will normally be abbreviated simply to "Q is connected and polynomially convex". This involves less loss of generality than might be supposed, since the polynomial hull $\hat{Q}$ of an u.s.c. (respectively a.m.v.) function Q, taken pointwise, is also u.s.c. (respectively a.m.v.); the proof is easy.

The next result is a form of Liouville's Theorem for a.m.v. functions (see [**6**]).

PROPOSITION 1.3. *Let K: $\mathbf{C} \to \kappa(\mathbf{C})$ be a.m.v. and polynomially convex. If K is non-constant, then $K(\mathbf{C})$ is dense in $\mathbf{C}$.* □

Lastly, we shall require a form of removable singularity theorem.

PROPOSITION 1.4. *Let $0 < r_1 < r_2$, and suppose that V is either the disc $\{|z| < r_2\}$ or the annulus $\{r_1 < |z| < r_2\}$. If K: $\{0 < |\lambda| < R\} \to \kappa(V)$ is an a.m.v. function, then it can be extended to an a.m.v. function K_1: $\{|\lambda| < R\} \to \kappa(V)$.*

Proof. For $0 < |\lambda| < R$ set $K_1(\lambda) = K(\lambda)$, and at 0 define

$$K_1(0) = \bigcap_{s>0} \left(\overline{\bigcup_{0<|\lambda|<s} K(\lambda)} \right)$$

It is shown in [**18**, Proposition 6.1] that the resulting function K_1: $\{|\lambda| < R\} \to \kappa(\mathbf{C})$ is a.m.v. It remains to prove that $K_1(0) \subset V$. Define ϕ: $\{|\lambda| < R\} \to [0, \infty)$ by

$$\phi(\lambda) = \sup\{|z|: z \in K_1(\lambda)\}.$$

By definition of a.m.v. function, ϕ is subharmonic on $\{|\lambda| < R\}$. Also, since $K(\lambda) \subset V$ on $\{|\lambda| = R/2\}$, we have $\phi(\lambda) < r_2$ on $\{|\lambda| = R/2\}$. It follows that $\phi(0) < r_2$, and consequently $K_1(0) \subset V$ when V is the disc. In the other case, since clearly $K_1(0) \subset \overline{V}$, the definition of a.m.v. function implies that

$$\phi_1(\lambda) = \sup\{|1/z|: z \in K_1(\lambda)\}$$

is subharmonic on $\{|\lambda| < R\}$, and arguing as above we deduce that $\phi_1(0) < 1/r_1$. This proves $K_1(0) \subset V$ when V is an annulus. □

2. Rouché's Theorem. Throughout this section, G denotes a bounded non-empty open subset of $\mathbf{C}$. We begin with a simple result, which motivates much of what follows.

PROPOSITION 2.1. *Let* K: $\overline{G} \to \kappa(\mathbf{C})$ *be u.s.c. and* $K\,|\,G$ *be a.m.v. Then* $K(\overline{G})$ *is the union of* $K(\partial G)$ *with a subcollection of the bounded components of* $\mathbf{C} \setminus K(\partial G)$.

Proof. Let D be a component of $\mathbf{C} \setminus K(\partial G)$, and for a contradiction suppose that there exist z_0, $z_1 \in D$ with $z_0 \in K(\overline{G})$ and $z_1 \notin K(\overline{G})$. As D is connected, we can further suppose that

$$|z_0 - z_1| < \operatorname{dist}(z_1, \partial D).$$

Define ψ: $\overline{G} \times (\mathbf{C} \setminus \{z_1\}) \to [0, \infty)$ by $\psi\,(\lambda, z) = 1/|z - z_1|$. As ψ is continuous on $\overline{G} \times (\mathbf{C} \setminus \{z_1\})$, and plurisubharmonic on $G \times (\mathbf{C} \setminus \{z_1\})$, it follows from the definition of a.m.v. function that ϕ: $\overline{G} \to [0, \infty)$, given by

$$\phi(\lambda) = \sup\{\psi(\lambda, z): z \in K(\lambda)\},$$

is u.s.c. on $\overline{G}$ and subharmonic on G. Now if $\zeta \in \partial G$ and $z \in K(\zeta)$, then $|z - z_1| \geq \operatorname{dist}(z_1, \partial D)$; hence

$$\phi(\zeta) < 1/|z_0 - z_1| \qquad (\zeta \in \partial G).$$

But on the other hand, as z_0 belongs to $K(\overline{G})$ and not to $K(\partial G)$, there exists $\lambda_0 \in G$ with $z_0 \in K(\lambda_0)$, and hence

$$\phi(\lambda_0) \geq \psi(\lambda_0, z_0) = 1/|z_0 - z_1|,$$

which contradicts the maximum principle for ϕ.

Thus we have shown that $K(\overline{G})$ is the union of $K(\partial G)$ with a number of complete components of $\mathbf{C} \setminus K(\partial G)$, and as $K(\overline{G})$ is compact, all these components must be bounded. □

As a simple consequence we obtain a multivalued form of Rouché's theorem (to recover the classical theorem, though without multiplicities, take $K_t(\lambda) = \{f(\lambda) + t \cdot g(\lambda)\}$).

THEOREM 2.2. *Let* $(t, \lambda) \mapsto K_t(\lambda)$: $[0,1] \times \overline{G} \to \kappa(\mathbf{C})$ *be u.s.c., and suppose further that*

(i) *for each* $t \in [0,1]$, *the function* $\lambda \mapsto K_t(\lambda)$ *is a.m.v. on* G, *and*

(ii) *for each* $\lambda \in G$, *the function* $t \mapsto K_t(\lambda)$ *is continuous on* $[0,1]$. *Then*

$$[K_0(G) \Delta K_1(G)] \subset \bigcup\{K_t(\zeta)\colon t \in [0,1], \zeta \in \partial G\}. \tag{1}$$

Proof. Let B be the union on the right hand side of (1), and let $z_0 \notin B$. Define $I = \{t \in [0,1]\colon z_0 \in K_t(G)\}$. We shall show that I is both open and closed in $[0,1]$, whence it follows that $z_0 \notin [K_0(G) \Delta K_1(G)]$, proving (1).

Closed: For each $t \in [0,1]$, it follows from $z_0 \notin B$ that $z_0 \in K_t(G)$ if and only if $z_0 \in K_t(\overline{G})$. But $t \mapsto K_t(\overline{G})$ is u.s.c. on $[0,1]$, and so I is closed.

Open: Let $t_0 \in I$. Then there exists $\lambda_0 \in G$ such that $z_0 \in K_{t_0}(\lambda_0)$. Let D be the component of $\mathbf{C} \setminus B$ which contains z_0, so D is open. By hypothesis (ii), the set $\{t \in [0,1]\colon K_t(\lambda_0) \subset \mathbf{C} \setminus D\}$ is closed, so there exists a neighborhood N of t_0 such that

$$t \in N \Rightarrow K_t(\lambda_0) \cap D \neq \varnothing.$$

But for any t, the set D is contained in a single component of $\mathbf{C} \setminus K_t(\partial G)$, so by hypothesis (i) and Proposition 2.1

$$K_t(G) \cap D \neq \varnothing \Rightarrow D \subset K_t(G).$$

Consequently

$$t \in N \Rightarrow D \subset K_t(G) \Rightarrow z_0 \in K_t(G),$$

and so I is open. □

3. Lifts of U.S.C. multivalued functions. In this section we develop the topological theory used for defining winding numbers in §4 and proving the Picard theorems in §6. For the time being we forget about analyticity, and concentrate on upper semicontinuity.

Fix the following notation.

(i) Let X, Y, Z be connected Hausdorff spaces possessing a base of path-connected, simply connected open sets. Thus they are necessarily path-connected, and components of open sets are open.

(ii) Let $p\colon Z \to Y$ be a *covering map*, that is, a map satisfying the following property: given $y \in Y$, there exists an open neighborhood U of y with $p^{-1}(U)$ equal to a non-empty union of disjoint open subsets V_γ of Z, such that $p|V_\gamma\colon V_\gamma \to U$ is a homeomorphism for each γ. We call any such U *fundamental*. A covering map is necessarily continuous, open and surjective.

(iii) Let A be the *cover group* of $p\colon Z \to Y$, that is, the group of homeomorphisms $h\colon Z \to Z$ such that $ph = p$. Call A *transitive* if $p(z_1) = p(z_2)$ implies that $z_2 = h(z_1)$ for some $h \in A$; unless specifically stated, we do not assume this is the case.

We recall two basic facts about covering maps, together with a few of their elementary consequences (details may be found for example in **[13]**).

Uniqueness of lifts (UL). *If g_1, g_2: $X \to Z$ are continuous maps such that $pg_1 = pg_2$, then the set $\{x \in X\colon g_1(x) = g_2(x)\}$ is either empty or the whole of X. Hence*:

(a) *the group A is a fixed-point-free* (*i.e. if $h \in A$ is not the identity, then $h(z) \neq z$ for all $z \in Z$*);

(b) *if g_1, g_2 are as above, and if also A is transitive, then there exists a unique $h \in A$ with $g_2 = hg_1$.*

Existence of Lifts (EL). *Assume that X is simply connected. Let f: $X \to Y$ be continuous, and suppose $x_0 \in X$ and $z_0 \in Z$ satisfy $f(x_0) = p(z_0)$. Then there exists a continuous map $g\colon X \to Z$ with $g(x_0) = z_0$ such that $pg = f$. Hence*:

(a) *if U is a connected, simply connected, open subset of Y, then U is fundamental*;

(b) *if Z is simply connected, then A is automatically transitive.*

We now seek to extend these results to multivalued mappings. The first task is to define 'lift'.

Let $Q_0 \in \kappa(Y)$. We say $L_0 \in \kappa(Z)$ is a *lift* of Q_0 if there are open sets U in Y and V in Z with $p\colon V \to U$ a homeomorphism, such that

$Q_0 \subset U$ and $L_0 = (p|V)^{-1}(Q_0)$. Note that if $z_1, z_2 \in L_0$, then $p(z_1) = p(z_2)$ implies $z_1 = z_2$.

Let Q: $X \to \kappa(Y)$ be u.s.c. We say L: $X \to \kappa(Z)$ is an *u.s.c. lift* of Q if L is u.s.c., and if for each $x \in X$ the set $L(x)$ is a lift of $Q(x)$. Clearly, if L is an u.s.c. lift of Q, then so is $h \circ L$ for any $h \in A$.

The following result extends (UL) to connected u.s.c. Q: $X \to \kappa(Z)$ (i.e. those u.s.c. Q such that $Q(x)$ is connected for each $x \in X$).

PROPOSITION 3.1. *Let Q: $X \to \kappa(Y)$ be u.s.c. and connected, and let L_1, L_2: $X \to \kappa(Z)$ be two u.s.c. lifts of Q.*

(a) *Either $L_1(x) = L_2(x)$ for all $x \in X$, or $L_1(x) \cap L_2(x) = \varnothing$ for all $x \in X$.*

(b) *If also A is transitive, then there exists a unique $h \in A$ such that $L_2 = h \circ L_1$.*

Proof. (a) Define

$$X_0 = \{x \in X: L_1(x) \cap L_2(x) = \varnothing\},$$
$$X_1 = \{x \in X: L_1(x) \cap L_2(x) \neq \varnothing\},$$
$$X_2 = \{x \in X: L_1(x) = L_2(x)\}.$$

Since L_1 and L_2 are u.s.c. and Z is Hausdorff, the set X_0 is open. By the connectedness of X, it therefore suffices to prove that X_1 is contained in the interior of X_2. Let $x_0 \in X_1$, so that $L_1(x_0) \cap L_2(x_0)$ contains a point z_0, say, and set $y_0 = p(z_0)$, so that $y_0 \in Q(x_0)$. As $L_1(x_0)$ and $L_2(x_0)$ are both lifts of $Q(x_0)$, these are open sets U_j in Y and V_j in Z with $p|V_j$: $V_j \to U_j$ a homeomorphism ($j = 1, 2$), such that

$$Q(x_0) \subset U_j \quad \text{and} \quad L_j(x_0) = (p|V_j)^{-1}(Q(x_0)) \qquad (j = 1, 2).$$

Now $Q(x_0)$ is connected, so is contained within a single component U of $U_1 \cap U_2$; thus without loss of generality we may suppose that $U_1 = U_2 = U$. Also $y_0 \in Q(x_0) \subset U$ and $(p|V_j)^{-1}(y_0) = z_0 (j = 1, 2)$, so by (UL) it follows that $(p|V_1)^{-1} = (p|V_2)^{-1}$ on U, and in particular that $V_1 = V_2 = V$, say. As L_1 and L_2 are u.s.c., there exists a neighborhood N of x_0 such that $x \in N$ implies $L_j(x) \subset V$ ($j = 1, 2$). But then if $x \in N$

$$p(L_1(x)) = Q(x) = p(L_2(x)),$$

and since $p|V$: $V \to U$ is a homeomorphism, $L_1(x) = L_2(x)$. Thus we have shown that any $x_0 \in X_1$ possesses a neighborhood $N \subset X_2$, as was required.

(b) Fix $x_0 \in X$, and $y_0 \in Q(x_0)$. Choose $z_j \in L_j(x_0)$ with $p(z_j) = y_0$ ($j = 1, 2$). If A is transitive, there exists $h \in A$ with $h(z_1) = z_2$. Then

$h \circ L_1$ is a u.s.c. lift of Q, and $(h \circ L_1)(x_0) \cap L_2(x_0)$ is non-empty (it contains z_2), so from part (a) we deduce that $L_2 = h \circ L_1$.

For uniqueness, note that if $k \in A$ also satisfies $L_2 = k \circ L_1$, then $k(z_1) \in L_2(x_0)$ and $p(k(z_1)) = p(z_1) = y_0$. This implies $k(z_1) = z_2 = h(z_1)$, and as A is fixed-point-free it follows that $k = h$. □

Now we turn to the question of existence of u.s.c. lifts of Q: $X \to \kappa(Y)$. Plainly, for such a lift to exist at all, the set $Q(x)$ must possess a lift at each point $x \in X$. A condition guaranteeing this is that for each $x \in X$ the set $Q(x)$ lies inside some fundamental open set U_x. We take this as our hypothesis.

PROPOSITION 3.2. *Assume that X is simply connected and let $x_0 \in X$. Let Q: $X \to \kappa(Y)$ be u.s.c. and connected, such that for each $x \in X$ the set $Q(x)$ is contained in some fundamental open set.*

(a) *There exists a lift L_0 of $Q(x_0)$.*

(b) *Given any lift L_0 of $Q(x_0)$, there exists an u.s.c. lift L: $X \to \kappa(Z)$ of Q such that $L(x_0) = L_0$.*

Proof. Part (a) is clear. The proof of (b) is very similar to that for the classical case (EL), so we give only a sketch.

Stage 1. The special case $X = [0,1] \times [0,1]$ and $x_0 = (0,0)$.

As Q is u.s.c., the Lebesgue covering theorem may be used to decompose this X as a finite union of compact squares I_j, such that for each j the set $Q(I_j)$ is contained within some fundamental open set. Moreover we can suppose that $x_0 \in I_0$, and that $J_n = (I_0 \cup \cdots \cup I_{n-1}) \cap I_n$ is always non-empty and connected. We construct an u.s.c. lift L of Q inductively, starting on I_0 with $L(x_0) = L_0$. At the nth stage, if L is already constructed on $(I_0 \cup \cdots \cup I_{n-1})$, use p^{-1} to obtain a lift L_1 of Q on I_n with the property that $L_1(x_n) = L(x_n)$ for some point $x_n \in J_n$. As J_n is connected, it follows from Proposition 3.1 that $L_1 = L$ on *all* of J_n, whence setting $L = L_1$ on I_n gives us an u.s.c. lift L of Q on $(I_0 \cup \cdots \cup I_n)$. After a finite number of such extensions, this lift L is defined on all of X.

Note that in particular, Stage 1 also implies the result holds when $X = [0,1]$ and $x_0 = 0$ (i.e. u.s.c. lifts exist along paths).

Stage 2. The general case.

Define L: $X \to \kappa(Z)$ as follows. Given $x \in X$, join x_0 to x by a path α: $[0,1] \to X$, let L_α: $[0,1] \to \kappa(Z)$ be an u.s.c. lift of $(Q \circ \alpha)$ with

$L_\alpha(0) = L_0$ (its existence being guaranteed by the remark above), and set $L(x) = L_\alpha(1)$.

The crucial fact is that this should not depend upon the path α chosen. To check this, let β be another path joining x_0 to x, with L_β the corresponding u.s.c. lift. As X is simply connected, there is a homotopy H: $[0,1] \times [0,1] \to X$ between α and β, so that for s, $t \in [0,1]$ we have

$$H(0,s) = \alpha(s) \qquad H(t,0) = x_0$$
$$H(1,s) = \beta(s) \qquad H(t,1) = x.$$

By Stage 1 there is an u.s.c. lift M: $[0,1] \times [0,1] \to \kappa(Z)$ of $Q \circ H$, with $M(0,0) = L_0$. Since $Q \circ H(t,0)$ is constant, Proposition 3.1 implies $M(t,0) = L_0$ for all t. By Proposition 3.1 again, $M(0,s) = L_\alpha(s)$ and $M(1,s) = L_\beta(s)$. Finally, since $Q \circ H(t,1)$ is constant, Proposition 3.1 once again implies $M(t,1)$ is constant, whence $L_\alpha(1) = L_\beta(1)$ as desired.

As $L(x)$ is clearly a lift of $Q(x)$ for each x, it remains to show that L is u.s.c. This follows easily from the fact just proved, that $L(x) = L_\alpha(1)$ for *any* path α joining x_0 to x, and from the uniqueness of lifts Proposition 3.1; the details are omitted. □

Proposition 3.2 begs the following question: how can we tell if each set $Q(x)$ lies in a fundamental open set? Using (EL) (b), a simple sufficient condition is that for every $x \in X$, the set $Q(x)$ should possess a connected, simply connected open neighborhood within Y. When Y is a subdomain of $\mathbf{C}$, this is equivalent to demanding that $Q(x)^\wedge \subset Y$ for each x. We have therefore justified the following corollary.

COROLLARY 3.3. *Assume that X is simply connected and let $x_0 \in X$. Assume also that Y is a connected open subset of $\mathbf{C}$. Let Q: $X \to \kappa(Y)$ be u.s.c., connected and polynomially convex.*

(a) *There exists a lift L_0 of $Q(x_0)$.*

(b) *Given any lift L_0 of $Q(x_0)$, there exists an u.s.c. lift L: $X \to \kappa(Z)$ of Q such that $L(x_0) = L_0$.* □

4. Winding numbers and a topological argument principle. We shall now apply the theory of §3 to the spaces $Z = \mathbf{C}$ and $Y = \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ (but X still arbitrary), with p: $Z \to Y$ defined by $p(z) = \exp(z)$. In this case the cover group A consists of those maps of the form $z \mapsto z + 2\pi i n$ (n an integer), and is transitive. If $Q_0 \in \kappa(\mathbf{C}^*)$ and L_0 is a lift of Q_0, we shall call L_0 a choice of log Q_0. For convenience, we restate the results of §3 in the context of this particular set-up.

PROPOSITION 4.1. *Let Q: $X \to \kappa(\mathbf{C}^*)$ be u.s.c. and connected.*

(a) *If L_1 and L_2 are u.s.c. choices of* $\log Q$, *then there is a unique integer n such that*

$$L_2(x) = L_1(x) + \{2\pi i n\} \qquad (x \in X),$$

and if m is any other integer then

$$L_2(x) \cap (L_1(x) + \{2\pi i m\}) = \varnothing \qquad (x \in X).$$

(b) *Suppose that X is simply connected, that $x_0 \in X$, and that Q is also polynomially convex. Then there exists a choice L_0 of* $\log Q(x_0)$; *and given any such choice L_0, there exists an u.s.c. choice L: $X \to \kappa(\mathbf{C})$ of* $\log Q$ *such that $L(x_0) = L_0$.* □

Let Q: $[0,1] \to \kappa(\mathbf{C})$ be u.s.c., connected and polynomially convex, with $Q(0) = Q(1)$. Suppose that $0 \notin Q([0,1])$. Then by Proposition 4.1(b) there exists an u.s.c. choice L: $[0,1] \to \kappa(\mathbf{C})$ of $\log Q$. Now $L(0)$ and $L(1)$ are lifts of the same set, so Proposition 4.1(a) applied to constant functions shows that

$$L(1) = L(0) + \{2\pi i n\}$$

for some unique integer n. By Proposition 4.1(a) again, this integer is independent of the choice of L. We call n the *winding number* of Q about 0, written $n(Q,0)$. In general, the winding number of Q about $w \in \mathbf{C} \setminus Q([0,1])$ is defined as

$$n(Q,w) = n(Q + \{-w\},0).$$

If Q is in fact singleton-valued, say $Q(t) = \{q(t)\}$, then this definition of $n(Q,w)$ agrees with the classical definition of $n(q,w)$.

PROPOSITION 4.2. *Let Q: $[0,1] \to \kappa(\mathbf{C})$ be u.s.c., connected and polynomially convex, with $Q(0) = Q(1)$. Then $w \mapsto n(Q,w)$ is constant on each component of $\mathbf{C} \setminus Q([0,1])$, and equals zero on the unbounded component.*

Proof. To prove the first conclusion, it is enough to show that $w \mapsto n(Q,w)$ is constant on any open disc N contained in $\mathbf{C} \setminus Q([0,1])$. Given such an N, define an u.s.c. map P: $[0,1] \times N \to \kappa(\mathbf{C}^*)$ by

$$P(t,w) = Q(t) + \{-w\}.$$

By Proposition 4.1(b), there exists an u.s.c. choice L of $\log P(t,w)$. Now $P(0,w) = P(1,w)$ for all $w \in N$, so $L(0,w)$ and $L(1,w)$ are both u.s.c. choices of $\log P(0,w)$ on N. Hence by Proposition 4.1(a) there exists an integer n such that

$$(2) \qquad L(1,w) = L(0,w) + \{2\pi i n\} \qquad (w \in N).$$

For each $w \in N$, the map $t \mapsto L(t,w)$ is an u.s.c. choice of $\log(Q(t) + \{-w\})$, so by (2) and the definition of winding number, $n(Q,w) = n$ for all $w \in N$, and hence $n(Q,w)$ is indeed constant on N.

From what has just been proved, to show that $n(Q,w) = 0$ on the unbounded component of $\mathbf{C} \setminus Q([0,1])$ it is sufficient to prove it for just one w in this component. Choose w_0 so that

$$\operatorname{Re} w_0 > \sup\{\operatorname{Re} z\colon z \in Q([0,1])\}.$$

Then $Q(t) + \{-w_0\}$ always lies in $D = \{z\colon \operatorname{Re} z < 0\}$, so if f is a branch of log on D and $L(t) = f(Q(t) + \{-w_0\})$, then L is an u.s.c. choice of $\log(Q + \{-w_0\})$ with $L(0) = L(1)$, whence $n(Q, w_0) = 0$. □

We now prove a topological argument principle. First recall that if α: $[0,1] \to \mathbf{C}$ is a (closed) Jordan curve and if G is its interior, then $\partial G = [\alpha]$ (which is by definition $\alpha([0,1])$), and $\overline{G}$ is homeomorphic to the closed unit disc.

THEOREM 4.3. *Let α be a Jordan curve and G be its interior. Let K: $\overline{G} \to \kappa(\mathbf{C})$ be u.s.c., connected and polynomially convex. If $z_0 \notin K(\partial G)$ and $n(K \circ \alpha, z_0) \neq 0$, then $z_0 \in K(G)$.*

Proof. If the conclusion fails, then by Proposition 4.1(b) there exists an u.s.c. choice L of $\log(K + \{-z_0\})$ on $\overline{G}$. In particular, $L \circ \alpha$ is an u.s.c. choice of $\log(K + \{-z_0\}) \circ \alpha$ on $[0,1]$, and since $L(\alpha(0)) = L(\alpha(1))$ it follows that $n(K \circ \alpha, z_0) = 0$. □

REMARK. In Theorem 4.3, polynomial convexity is used for more than just defining winding number; for to do this, one only needs K to be polynomial convex on ∂G, but in fact this is not enough to ensure that $n(K \circ \alpha, z_0) \neq 0$ implies $z_0 \in K(G)$. For example, let $z_0 = 0$ and consider K: $\{|\lambda| \leq 2\} \to \kappa(\mathbf{C})$ given by

$$K(\lambda) = \begin{cases} \{\lambda\}, & \text{if } 1 < |\lambda| \leq 2 \\ \{z\colon |z| = 1\}, & \text{if } |\lambda| \leq 1. \end{cases}$$

5. An analytic argument principle. For $z \in \mathbf{C}$ and $\theta \in \mathbf{R}$, write

$$H(z,\theta) = \{z + re^{i\theta}\colon r \geq 0\},$$

the half-line with endpoint z and angle of inclination θ. With this notation established, we can state our analytic argument principle.

THEOREM 5.1. *Let α be a Jordan curve and G be its interior, and suppose α is positively oriented (i.e. $n(\alpha,\lambda)=1$ for $\lambda\in G$). Let K: $\overline{G}\to\kappa(\mathbf{C})$ be u.s.c., such that $K\,|\,G$ is a.m.v. and $K\,|\,\partial G$ is connected. Let $z_0\notin K(\partial G)$ and suppose that for each $\zeta\in\partial G$ there exists $\theta\in\mathbf{R}$ such that*

$$K(\zeta)\cap H(z_0,\theta)=\varnothing. \tag{3}$$

Then $n(\hat{K}\circ\alpha,z_0)\geq 0$, and $z_0\in K(G)$ if and only if $n(\hat{K}\circ\alpha,z_0)>0$.

REMARKS. (i) The hypothesis (3) guarantees that

$$z_0\notin K(\zeta)^{\wedge}\qquad(\text{all }\zeta\in\partial G), \tag{3$'$}$$

and so the winding number $n(\hat{K}\circ\alpha,z_0)$ is well-defined. Key use is made of the hypothesis in the proof below, but it seems somehow unnatural and we conjecture that the theorem remains true if (3) is replaced by (3′). This conjecture is closely related to the work in **[3, 4, 28]**.

(ii) The added assumption of analyticity allows a much stronger conclusion to be drawn than in Theorem 4.3, namely a condition for $z_0\in K(G)$ which is both necessary and sufficient. Note also that in contrast with Theorem 4.3, we no longer require that K be connected or polynomially convex on G.

Proof. We begin by making a number of simplifications. Firstly, by considering $K+\{-z_0\}$ instead of K, we may as well suppose that $z_0=0$. Secondly, replacing $K(\zeta)$ by $K(\zeta)^{\wedge}$ for all $\zeta\in\partial G$, we can assume that $K\,|\,\partial G$ is polynomially convex as well as connected. Thirdly, there is no loss of generality in taking α to be the curve $\gamma(t)=e^{2\pi it}$, and G to be Δ, the open unit disc. For by the Riemann mapping theorem there exists a homeomorphism f: $\overline{\Delta}\to\overline{G}$ which is analytic on Δ and satisfies $f(1)=\alpha(0)$. This implies that $f\circ\gamma$ is just a re-parameterisation of α, because α is positively oriented. So if K_1: $\overline{\Delta}\to\kappa(\mathbf{C})$ is defined by $K_1=K\circ f$, then K_1 satisfies all the hypotheses of the theorem, with α and G replaced by γ and Δ respectively, and also $n(K_1\circ\gamma,0)=n(K\circ\alpha,0)$. Thus once the theorem is proved for K_1, it is also proved for K.

Set $n=n(K\circ\gamma,0)$. By Proposition 4.1(b), there exists an u.s.c. choice L_1 of $\log K\circ\gamma$ on $[0,1]$, and by definition of winding number we have

$$L_1(1)=L_1(0)+\{2\pi in\}.$$

Therefore the function L: $\partial\Delta\to\kappa(\mathbf{C})$ defined by

$$L(e^{2\pi it})=L_1(t)+\{-2\pi int\}$$

is u.s.c. on $\partial\Delta$. Now the hypothesis (3) implies that for each $\zeta\in\partial\Delta$, there exists some choice L_ζ of $\log K(\zeta)$ such that

$$\sup\{\operatorname{Im}(z_1-z_2)\colon z_1,z_2\in L_\zeta\}<2\pi.$$

By Proposition 4.1(a) applied to constant functions, the set $L(\zeta)$ is a translate of L_ζ, so

$$\sup\{\mathrm{Im}(z_1 - z_2)\colon z_1, z_2 \in L(\zeta)\} < 2\pi.$$

Therefore if we set

$$\left.\begin{aligned} u(\zeta) &= \sup\{\mathrm{Im}(z)\colon z \in L(\zeta)\} \\ v(\zeta) &= \inf\{\mathrm{Im}(z)\colon z \in L(\zeta)\} \end{aligned}\right\} \qquad (\zeta \in \partial\Delta)$$

then $u < (v + 2\pi)$ on $\partial\Delta$. Also since L is u.s.c., it follows that u is u.s.c. and v is l.s.c. on $\partial\Delta$. By a standard partition-of-unity argument, there exists a continuous function $g\colon \partial\Delta \to \mathbf{R}$ such that $u < g < (v + 2\pi)$ on $\partial\Delta$. As $(g - u)$ and $(v + 2\pi - g)$ are strictly positive l.s.c. functions on the compact set $\partial\Delta$, they are bounded away from zero, their infima being at least $\delta > 0$, say. Using the Stone-Weierstrass theorem we may find a polynomial $p(\lambda)$ such that $|g - \mathrm{Im}(p)| < \delta$ on $\partial\Delta$, and consequently $u < \mathrm{Im}(p) < (v + 2\pi)$ on $\partial\Delta$. Hence if $\zeta \in \partial\Delta$ and $z \in (L(\zeta) + \{-p(\zeta)\})$, then $-2\pi < \mathrm{Im}(z) < 0$. In other words,

$$e^{-p(\zeta)} \cdot \zeta^{-n} \cdot K(\zeta) \cap H(0,0) = \varnothing \qquad (\zeta \in \partial\Delta). \tag{4}$$

From this we can deduce the theorem. First suppose that $n < 0$. Then $M(\lambda) = e^{-p(\lambda)} \cdot \lambda^{-n} \cdot K(\lambda)$ is u.s.c. on $\overline{\Delta}$ and a.m.v. on Δ, and (4) implies that 0 lies in the unbounded component of $\mathbf{C} \setminus M(\partial\Delta)$. Hence by Proposition 2.1, $0 \notin M(\Delta)$, which is patently absurd since $M(0) = \{0\}$. We conclude that necessarily $n \geq 0$. Now for $t \in [0,1]$ and $\lambda \in \overline{\Delta}$ define

$$K_t(\lambda) = t \cdot K(\lambda) + (t - 1) \cdot \{\lambda^n e^{p(\lambda)}\}.$$

Then $K_t(\lambda)$ satisfies the hypotheses of Theorem 2.2, and moreover (4) implies that

$$0 \notin \bigcup\{K_t(\zeta)\colon t \in [0,1], \zeta \in \partial\Delta\}.$$

Hence by Theorem 2.2, $0 \in K(\Delta)$ if and only if 0 lies in the range of $\lambda^n \cdot e^{p(\lambda)}$ on Δ, that is, if and only if $n > 0$. □

The fact that $n(\hat{K} \circ \alpha, z_0) \geq 0$ demonstrates that in some sense a.m.v. functions preserve orientation. Classically (i.e. when $K(\lambda) = \{f(\lambda)\}$) the winding number is obviously non-negative because it registers exactly the number of zeros of the analytic function $(f(\lambda) - z_0)$ within α. No such precise interpretation seems possible in the multivalued case, though there is something that can be said.

First we need one more definition. Let us call $S \subset \mathbf{C}$ *costarshaped* if its complement is "starshaped about ∞", i.e. for each $z \notin S$ there exists

$\theta \in \mathbf{R}$ with $S \cap H(z, \theta) = \varnothing$. Any convex or starshaped set is costarshaped. Also a compact costarshaped set is automatically polynomially convex.

THEOREM 5.2. *Let α be a positively oriented Jordan curve and G be its interior. Let K: $\overline{G} \to \kappa(\mathbf{C})$ be u.s.c., connected and costarshaped, and $K \mid G$ be a.m.v. If $z_0 \notin K(\partial G)$, then $n(K \circ \alpha, z_0)$ is an upper bound for the number of components of $\hat{E}$, where E is the compact set $\{\lambda \in G\colon z_0 \in K(\lambda)\}$.*

REMARKS. (i) Since no account is taken of "multiplicity of zeros", we cannot hope for exact equality.

(ii) If the conjecture made in remark (i) after Theorem 5.1 were correct, then "costarshaped" could be replaced by "polynomially convex" in Theorem 5.2.

Proof. It is enough to show that whenever $\hat{E}$ can be written as the disjoint union

$$\hat{E} = F_1 \cup F_2 \cup \cdots \cup F_m,$$

where $F_1, \ldots, F_m$ are non-empty open-and-closed subsets of $\hat{E}$, then necessarily

$$n(K \circ \alpha, z_0) \geq m.$$

The proof is by induction on m. The case $m = 0$ follows immediately from Theorem 5.1. Let $m > 0$, and suppose that the statement above holds for any α, K, z_0 satisfying the hypotheses of the theorem, and any decomposition of the corresponding $\hat{E}$ into $m - 1$ or fewer disjoint non-empty open-and-closed subsets. With the particular decomposition $\hat{E} = F_1 \cup \cdots \cup F_m$ under consideration, since F_m is polynomially convex, we can choose a negatively oriented Jordan curve β in G (i.e. $n(\beta, \lambda) = -1$ if λ lies inside β) such that

$$\begin{aligned} &F_j \subset \text{exterior of } \beta \qquad (j = 1, 2, \ldots, m-1) \\ &F_m \subset \text{interior of } \beta. \end{aligned}$$

Select distinct points a_1, a_2 in $[\alpha]$ and b_1, b_2 in $[\beta]$. Since $F_1, \ldots, F_{m-1}$ are polynomially convex, we can find disjoint simple paths β_1, β_2 joining a_1 to b_1 and a_2 to b_2 respectively, such that

$$\beta_j(t) \in G \backslash (F_1 \cup \cdots \cup F_m) \qquad (0 < t < 1,\ j = 1, 2).$$

Denote by α' the part of the curve α running from a_2 to a_1, and by α'' the part running from a_1 to a_2. Similarly, let β' be the part of β going from b_1 to b_2, and β'' denote the remainder (the reader is urged to draw a picture). Finally, write δ' for the path formed by joining together α', β_1, β' and the reverse of β_2; likewise, form δ'' by joining up respectively α'', β_2, β'' and the reverse of β_1. Then δ' and δ'' are positively oriented Jordan curves, and

$$n(K \circ \delta', z_0) + n(K \circ \delta'', z_0) = n(K \circ \alpha, z_0) + n(K \circ \beta, z_0). \tag{5}$$

Let G' and G'' be the interiors of δ' and δ'' respectively. We are going to apply the inductive hypothesis separately to K on G' and on G''. The corresponding sets E' and E'' are just $(E \cap G')$ and $(E \cap G'')$; therefore we have the decompositions

$$(E')\hat{} = F_1' \cup \cdots \cup F_p',$$
$$(E'')\hat{} = F_1'' \cup \cdots \cup F_q'',$$

where the F_j' and F_j'' are disjoint non-empty open-and-closed subsets of $(E')\hat{}$ and $(E'')\hat{}$ respectively, and where p and q satisfy

$$p \leq m-1, \quad q \leq m-1 \quad \text{and} \quad p+q \geq m-1. \tag{6}$$

Since (6) holds, the inductive hypothesis implies that

$$n(K \circ \delta', z_0) \geq p \quad \text{and} \quad n(K \circ \delta'', z_0) \geq q. \tag{7}$$

Also, since F_m is contained in the interior of β, applying Theorem 5.1 to the reverse of β gives

$$n(K \circ \beta, z_0) \leq -1. \tag{8}$$

Combining (5), (6), (7) and (8), we deduce

$$n(K \circ \alpha, z_0) \geq m,$$

and the induction is complete. □

6. Two Picard theorems. Let K: $\mathbf{C} \to \kappa(\mathbf{C})$ be a non-constant (entire) a.m.v. function whose values are polynomially convex. By Proposition 1.3, $K(\mathbf{C})$ must be dense in $\mathbf{C}$. In fact the "little Picard theorem" of B. Aupetit [**11**] says that $\mathbf{C} \setminus K(\mathbf{C})$ is necessarily a polar set (alternative proofs appear in [**7, 17**]). Also A. Zraibi [**29**] has shown that if $K(\lambda)$ is always finite, with say $\#K(\lambda) \leq n$ (all $\lambda \in \mathbf{C}$), then in fact $\mathbf{C} \setminus K(\mathbf{C})$ contains at most $2n-1$ points, and this bound is best possible. At the other extreme, we shall now prove that if $K(\lambda)$ is always connected, then $\mathbf{C} \setminus K(\mathbf{C})$ can contain at most one point. This will be done applying the theory of §3 to a different covering map; namely we shall take $Y = \mathbf{C} \setminus \{0,1\}$ and $Z = H^+ = \{z \in \mathbf{C}\colon \operatorname{Im} z > 0\}$, and $p\colon Z \to Y$ will be the modular function $\Lambda\colon H^+ \to \mathbf{C} \setminus \{0,1\}$. This function is analytic, and is a covering map between the spaces concerned; its construction and other details may be found for example in [**20**].

THEOREM 6.1. *Let $K\colon \mathbf{C} \to \kappa(\mathbf{C})$ be a.m.v., connected and polynomially convex. If K is non-constant, then $\mathbf{C} \setminus K(\mathbf{C})$ contains at most one point.*

Proof. Suppose $K(\mathbf{C})$ omits more than one value. Rescaling if necessary, we may assume without loss of generality that $K(\mathbf{C}) \subset \mathbf{C} \setminus \{0,1\} = Y$. Applying Proposition 3.3 with $X = \mathbf{C}$ shows that there exists an u.s.c. lift $L\colon \mathbf{C} \to \kappa(H^+)$ of K (via Λ). Because Λ is analytic, it follows from Proposition 1.2(b) and the definition of u.s.c. lift that L must be a.m.v. Moreover, for each $\lambda \in \mathbf{C}$ the set $L(\lambda)$ is homeomorphic to $K(\lambda)$, so must also be connected and polynomially convex (polynomial convexity is a topological invariant of compact subsets of $\mathbf{C}$). Since $L(\mathbf{C}) \subset H^+$, Proposition 1.3 implies that L must be constant on $\mathbf{C}$, whence $K = \Lambda \circ L$ is also constant. □

Theorem 6.1 and the results mentioned in the paragraph preceding it may be thought of as analogues of the little Picard theorem. There is also a multivalued version of the big Picard theorem, proved by the author in [**18**]. In a slightly weakened form, it states that if $K\colon \{0 < |\lambda| < R\} \to \kappa(\mathbf{C})$ is an a.m.v. function, and if

$$E = \bigcap_{0<r<R} K(\{0 < |\lambda| < r\}), \tag{9}$$

then either $\mathbf{C} \setminus E$ is polar, or there exists a Möbius transformation g such that $g \circ K$ has an a.m.v. extension to a neighborhood of 0. If K is connected and polynomially convex then this conclusion can be strengthened by using the same general idea as in Theorem 6.1. The proof which follows is based on an exposition of the classical big Picard theorem given in [**12**].

THEOREM 6.2. *Let $K\colon \{0 < |\lambda| < R\} \to \kappa(\mathbf{C})$ be a.m.v., connected and polynomially convex, and let E be defined as in* (9). *Then either $\#(\mathbf{C} \setminus E) \leq 1$, or there exists a Möbius transformation g such that $g \circ K$ has an a.m.v. extension to a neighborhood of* 0.

Proof. Suppose that $\#(\mathbf{C} \setminus E) \geq 2$. Rescaling if necessary, we may assume that

$$K(\{0 < |\lambda| < 2\}) \subset \mathbf{C} \setminus \{0,1\}.$$

Let L_0 be a lift (via Λ) of the set $K(1)$. Define

$$D_1 = \{re^{i\theta}: 0 < r < 2, -3\pi/2 < \theta < \pi/2\},$$
$$D_2 = \{re^{i\theta}: 0 < r < 2, -\pi/2 < \theta < 3\pi/2\},$$

so that D_1 and D_2 are both simply connected domains containing the point 1. By Proposition 3.3 there exists u.s.c. lifts L_j: $D_j \to \kappa(H^+)$ of K (via Λ) such that $L_j(1) = L_0$ ($j = 1,2$). As Λ is analytic, both L_1 and L_2 are a.m.v. functions. By Proposition 3.1(a), $L_1(1) = L_2(1)$ implies that $L_1 = L_2$ on D^+, where

$$D^+\{\lambda \in \mathbf{C}: 0 < |\lambda| < 2, \operatorname{Re}\lambda > 0\}.$$

Also, if we set

$$D^- = \{\lambda \in \mathbf{C}: 0 < |\lambda| < 2, \operatorname{Re}\lambda < 0\},$$

then by Proposition 3.1(b), the transitivity of the cover group A of Λ: $H^+ \to \mathbf{C} \setminus \{0,1\}$ (note H^+ is simply connected) implies that there is an element $h \in A$ such that $L_2 = h \circ L_1$ on D^-. Now it is shown in **[12]** that given any $h \in A$, there exist a domain V of the form $\{z: s < |z| < 1\}$ for some $0 \leq s < 1$, and analytic maps q: $H^+ \to V$ and l: $V \to \mathbf{C} \setminus \{0,1\}$, such that $q \circ h = q$ and $l \circ q = \Lambda$. Construct V, q and l for our particular h. Then $q \circ L_1 = q \circ L_2$ on both D^+ and D^-, and hence on $D_1 \cap D_2$, so that if M: $\{0 < |\lambda| < 2\} \to \kappa(V)$ is given by

$$M(\lambda) = q \circ L_j(\lambda), \qquad (\lambda \in D_j)$$

then M is a well defined a.m.v. function. Also on D_j we have

$$l \circ M = \Lambda \circ L_j = K, \qquad (j = 1,2),$$

so that in fact $l \circ M = K$ on the whole of $\{0 < |\lambda| < 2\}$. There are now two cases to consider.

Case 1. $V = \{0 < |z| < 1\}$.

Set $V_0 = V \cup \{0\} = \{|z| < 1\}$. By the classical big Picard theorem, l: $V \to \mathbf{C} \setminus \{0,1\}$ can be extended to a meromorphic function l_0: $V_0 \to \mathbf{C}_\infty \setminus \{a\}$, where a equals either 0 or 1. Thus if $g(z) = 1/(z-a)$, then $g \circ l_0$ is analytic on V_0. Also, by Proposition 1.4, M: $\{0 < |\lambda| < 2\} \to \kappa(V_0)$ can be extended to an a.m.v. function M_0: $\{|\lambda| < 2\} \to \kappa(V_0)$. Then $(g \circ l_0) \circ M_0$ is an a.m.v. function on $\{|\lambda| < 2\}$, and if $\lambda \neq 0$ then

$$(g \circ l_0) \circ M_0(\lambda) = g \circ l \circ M(\lambda) = g \circ K(\lambda).$$

Case 2. $V = \{s < |z| < 1\}$, where $s > 0$.

In this case, Proposition 1.4 shows that M: $\{0 < |\lambda| < 2\} \to \kappa(V)$ can be extended to an a.m.v. function M_0: $\{|\lambda| < 2\} \to \kappa(V)$. Then $l \circ M_0$ is a.m.v. on $\{|\lambda| < 2\}$ and equals $K(\lambda)$ if $\lambda \neq 0$. □

We conclude by giving examples to show that both connectedness and polynomial convexity are indispensable assumptions in Theorem 6.2.

EXAMPLE 1. Define K: $\{0 < |\lambda| < 1\} \to \kappa(\mathbf{C})$ by

$$K(\lambda) = \{z: z^2 = 1 + \exp(1/\lambda)\} \qquad (0 < |\lambda| < 1).$$

Then K is a.m.v., polynomially convex (indeed, two-valued) and $K(\{0 < |\lambda| < 1\}) \subset \mathbf{C} \setminus \{-1, 1\}$, but any u.s.c. extension of K to 0 would be unbounded.

EXAMPLE 2. Define K: $\{0 < |\lambda| < 1/2\} \to \kappa(\mathbf{C})$ by

$$K(\lambda) = \{z: |z| \leq 1/|\lambda|, |z-1| \geq |\lambda|, |z+1| \geq |\lambda|\} \quad (0 < |\lambda| < 1/2).$$

Then K is a.m.v., connected, and $K(\{0 < |\lambda| < 1/2\}) \subset \mathbf{C} \setminus \{-1, 1\}$, but again any u.s.c. extension of K to 0 would be unbounded.

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Received February 23, 1985.

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