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ON THE KATO-ROSENBLUM THEOREM

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The Kato-Rosenblum Theorem has no straightforward generalization to operators with non-absolutely continuous spectra. For example, if A is a bounded selfadjoint operator such that the singular continuous parts of H and $H + A$ are unitarily equivalent for every selfadjoint operator H, then $A = 0$.

1. Introduction. The classical theorem of Kato and Rosenblum (1957) asserts the invariance of absolutely continuous parts under tracse class perturbations. [**5**, p. 540; **6**, p. 26]

THEOREM (*Kato-Rosenblum*). *If H and A are selfadjoint, and A is trace class, then the absolutely continuous parts of H and $H + A$ are unitarily equivalent.*

It is notable that the theorem gives a unitarily invariant condition on the perturbation A alone, and that Lebesgue measure plays a distinguished role.

That the trace condition cannot be radically improved, follows from the Weyl-von Neumann theorem [**5**, p. 523], which states that given any selfadjoint operator H, there is a selfadjoint perturbation A of arbitrarily small Hilbert-Schmidt norm, such that $H + A$ has pure point spectrum—a phenomenon often termed *curdling*. Moreover, according to Kuroda, the Hilbert-Schmidt norm may be replaced by any cross-norm *except* the trace norm. [**5**, p. 525]

For singular measures, there are a few, largely negative, results. Donoghue [**2**], following earlier work of Aronszajn, gave examples in which a purely singular continuous spectrum is curdled by a perturbation of rank one. He also obtained the following result, which we shall use [**2**, p. 565; **4**, Cor. 1].

THEOREM. (*Donoghue*). *Let H be selfadjoint and $A = c\langle \cdot, \phi\rangle \phi$ where ϕ if cyclic for H and c is real and non-zero. Then the singular parts of H and $H + A$ are supported on disjoint sets (i.e. are mutually singular).*

A generalization was proved in [**4**].

Following Donoghue's approach, Carey and Pincus [**1**] proved that the spectrum of any operator with *purely singular* spectrum can be curdled by a perturbation of arbitrarily small *trace* norm. A proof of this fact following the Weyl-von Neumann construction has recently been given by Eugene Wayne [**6**].

These results leave it difficult to imagine a unitarily invariant condition on A alone which might guarantee that A preserves singular continuous parts. Indeed, as we shall prove, there is no such condition: if H and $H + A$ have unitarily equivalent singular parts for every H, then $A = 0$.

We shall, in fact, prove that it is impossible to generalize the Kato-Rosenblum theorem to other measures in the following sense. Let μ be a non-zero Borel measure, and $A \neq 0$ a bounded operator. If the parts of H and $H + A$ which are absolutely continuous with respect to μ are unitarily equivalent for all selfadjoint H, then μ is absolutely continuous with respect to Lebesgue measure, and, moreover, the entire absolutely continuous parts of $H$and $H + A$ are unitarily equivalent.

We shall also prove that A is necessarily compact. The Weyl-von Neumann-Kuroda result strongly suggests that A is trace class, but we know of no proof.

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2. Preservation of measures. Let $\mathscr{H}$ be a separable Hilbert space, and $H = \int \lambda\, E(d\lambda)$ a selfadjoint operator on $\mathscr{H}$. We shall assume throughout that *all operators are bounded*. For Borel measures m and μ on $\mathbf{R}$, write $u \ll \mu$ if m is absolutely continuous with respect to μ. For $x \in \mathscr{H}$, let m_x be the Borel measure

$$m_x(d\lambda) = \langle E(d\lambda)x, x\rangle.$$

The set

$$\mathscr{H}_\mu(H) = \{x \in \mathscr{H}\colon m_x \ll \mu\}$$

is a closed reducing subspace of H, called the *absolutely continuous subspace* of H with respect to μ. Its orthogonal complement is

$$\mathscr{H}_\mu^s(H) = \{x \in \mathscr{H}\colon m_x \text{ and } \mu \text{ are mutually singular}\}.$$

(See [**5**, p. 516.] The proof is given for Lebesgue measure, but holds in general without change.)

For any Borel measure μ, define H_μ to be the *restriction* of H to $\mathscr{H}_\mu(H)$. If $\nu \ll \mu$, then

$$(H_\mu)_\nu = H_\nu. \tag{2.1}$$

For real t, define the translated measure

$$\mu_t(S) = \mu(S - t). \tag{2.2}$$

Then

$$(H + t)_{\mu_t} = H_\mu + t.$$

Write $A \cong B$ to mean that A and B are unitarily equivalent.

2.1 DEFINITION. *Let μ be a Borel measure on $\mathbf{R}$. A selfadjoint operator A preserves μ iff*

$$(H + A)_\mu \cong H_\mu$$

for every selfadjoint operator H.

The trivial zero measure is preserved by every A, because the space $\mathscr{H}_\mu$ is then always zero-dimensional. The Kato-Rosenblum theorem says that trace class operators preserve Lebesgue measure.

2.2 PROPOSITION. *Let A and B preserve μ. Then:*

(a) *$A + B$ and cA also preserves μ, if c is real:*

(b) *if $\nu \ll \mu$, then A preserves ν;*

(c) *A preserves μ_t for all t;*

(d) *if $W \cong A$, then W preserves μ; and*

(e) *If P is an orthogonal reducing projection for A, then AP preserves μ on $P\mathscr{H}$;*

Proof.

(a) We have

$$(H + A + B)_\mu \cong (H + A)_\mu \cong H_\mu.$$

and similarly

$$(H + cA)_\mu = c(c^{-1}H + A)_\mu \cong c(c^{-1}H)_\mu = H_\mu.$$

(b) By (2.1),

$$(H + A)_\nu = [(H + A)_\mu]_\nu \cong (H_\mu)_\nu = H_\nu.$$

(c) By (2.2),

$$(H + A)_{\mu_t} = (H + t + A)_\mu - t \cong (H + t)_\mu - t = H_{\mu_t}.$$

(d) If $W = UAU^*$, with U unitary, then

$$(H + W)_\mu = (H + UAU^*)_\mu = [U(U^*HU + A)U^*]_\mu$$
$$\cong (U^*HAU + A)_\mu \cong (U^*HU)_\mu \cong H_\mu$$

(e) Let A be the restriction of A of $P\mathscr{H}$. Writing operator matices for the decomposition $\mathscr{H} = P\mathscr{H} \oplus (I - P)\mathscr{H}$ gives

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

If H is defined on $\mathscr{H}$ by

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix}$$

then $(H + A)_\mu \cong H_\mu$ says that

$$\begin{pmatrix} (H_1 + A_1)_\mu & 0 \\ 0 & (A_2)_\mu \end{pmatrix} \cong \begin{pmatrix} (H_1)_\mu & 0 \\ 0 & 0_\mu \end{pmatrix}$$

which gives $(H_1 + A_1)_\mu \cong (H_1)_\mu$, by equating the first components.

THEOREM 1. *If A preserves a non-zero measure μ, then A is compact.*

Proof. If A is not compact, then, possibly replacing A by $-A$, there is an infinite dimensional reducing projection P of A such that $A_1 = AP \geq \delta P$ for some $\delta > 0$. By restriction and translation (2.2(b) and (c)), we can assume that $[0,1]$ supports μ, and that $\mu[0,\varepsilon] > 0$ for every $\varepsilon > 0$. Choosing H_1 to be an operator on $P\mathscr{H}$ unitarily equivalent to multiplication by λ on $\mathscr{L}^2([0,1], d\mu(\lambda))$, we see that $H_1 \geq 0$ and that the spectrum of $H_1 = (H_1)_\mu$ contains 0. By (e),

$$(H_1 + A_1)_\mu = (H_1)_\mu.$$

But $H_1 + A_1 \geq \delta P > 0$, so 0 is not in the spectrum of $H_1 + A_1$, a contradiction. Hence, A is compact.

THEOREM 2. *If A preserves μ and $A \neq 0$, then μ is absolutely continuous with respect to Lebesgue measure.*

Proof. Choose a vector ϕ of norm one, which is *not* an eigenvector of A, but for which $A\phi \neq 0$. The operator

$$U = 1 - 2\cdot\langle\,\cdot\,,\phi\rangle\phi$$

is unitary, and we compute that

$$B = A - UAU^* = 2\langle\,\cdot\,,A\phi\rangle\phi + 2\langle\,\cdot\,,\phi\rangle A - 4\langle A\phi,\phi\rangle\langle\,\cdot\,,\phi\rangle\phi.$$

Since ϕ and $A\phi$ are independent, B has rank exactly two. By 2.2(a) and (d), B also preserves μ.

Let ϕ_1 and ϕ_2 be the two eigenvectors of B with non-zero eigenvalues. Let $\mathscr{H}_1$ be the orthogonal complement of ϕ_1, which reduces B. By 2.1(e), the restriction B_1 of B to $\mathscr{H}_1$ must preserve μ on $\mathscr{H}_1$. But $B_1 = c\langle \cdot, \phi_2\rangle\phi_2$ has rank one. Since H_μ depends only on a class of mutually absolutely continuous measures, we can assume that μ is finite, with total mass one. Identify $\mathscr{H}_1$ with $L^2(\mu)$ and ϕ_2 with the constant function 1, and let H_1 be multiplication by λ. Since ϕ_2 is cyclic for H_1, Donoghue's theorem says that the singular parts of H_1 and $H_1 + B_1 = H_1 + c\langle \cdot, \phi_2\rangle\phi_2$ are disjointly supported. Thus, if μ had a singular part, B_1 could not preserve μ, so μ must be absolutely continuous.

THEOREM 3. *If A preserves a non-zero absolutely continuous measure, then A preserves Lebesgue measure.*

Denote by $\chi_S(\lambda)$ the characteristic function of the Borel set S, by $|S|$ its Lebesgue measure, and by μ_S, the measure

$$\mu_S(d\lambda) = \chi_S(\lambda)\, d\lambda$$

Write H_S for H_{μ_S}. Let

$$\mathscr{B} = \{S\colon A \text{ preserves } \mu_S,\ S \text{ Borel}\}.$$

2.3 LEMMA. (i) *$\mathscr{B}$ contains every set of measure zero.*
(ii) *$\mathscr{B}$ contains a set of positive measure.*
(iii) *If $S \in \mathscr{B}$, then $S + t \in \mathscr{B}$ for every t*
(iv) *If $S \in \mathscr{B}$ and $F \subset S$, then $F \in \mathscr{B}$. Hence, $\mathscr{B}$ is closed under intersection and difference.*
(v) *$\mathscr{B}$ is closed under countable unions.*

Proof. (i) If $|S| = 0$, μ_S is the zero measure, which is always preserved.
(ii) If A preserves the measure $f(\lambda)\, d\lambda$, with density $f(\lambda)$, then $S = \{\lambda\colon f(\lambda) > 0\}$ is in $\mathscr{B}$ and has positive measure.
(iii) follows from 2.2(c), and (iv) from 2.2(b).
(v) Let $S = S_1 \cup S_1 \cup \cdots$, with $S_j \in \mathscr{B}$.

Writing

$$S = S_1 \cup (S_2 \cup S_1) \cup (S_2 \sim [S_1 \cup S_2]) \cup \cdots$$

and noting (iv) permits us to assume that S_1, $S_2 \ldots$ are *disjoint*. In that case

$$(H+A)_S \cong \bigoplus \sum_{j\geq 1} (H+A)_{S_j} \cong \bigoplus \sum_{j \geq 1} H_{S_j} \cong H_S$$

so that $S \in \mathscr{B}$.

Proof of Theorem 3. We wish to show that $\mathscr{B}$ contains the whole line **R**. By (iii) and (v), it suffices that $\mathscr{B}$ contain $[0,1]$. Let

$$\mathscr{B}_0 = \{S \in \mathscr{B}: S \subset [0,1]\}.$$

If we can prove that

$$\sup\{|S|: S \in \mathscr{B}_0\} = 1 \tag{2.3}$$

then (cf. [**3**, p. 75]) the union F of a sequence of sets $F_n \in \mathscr{B}_0$ with $|F_\mu| \to 1$, is in $\mathscr{B}_0$ and has measure 1. Hence $[0,1]$, which is the union of F with a null set, is also in $\mathscr{B}_0$.

It remains to prove (2.3). Let $\varepsilon > 0$ and $0 < \alpha < 1$ be arbitrary. Use (i), (ii) and (iii) to find an $S \in \mathscr{B}_0$ with $0 < |S| < \varepsilon$, and then an interval I with

$$|I \cap S| > \alpha |I|$$

[**3**, p. 68]. Note that $|I| < \varepsilon/\alpha$.

Lay off on $[0,1]$ consecutive intervals $I_1, I_2, \ldots$ of the same length as I, starting at 0 and continuing until $I_1, \ldots, I_{n+1}$ just cover $[0,1]$. Each I_j is a translate $I_j = I + t_j$ of I. If $F_j = (I \cap S) + t_j$, and $F = F_1 \cup \cdots \cup F_n$, then $F, F_j \in \mathscr{B}_0$ and

$$|F| = |F_1 \cup \cdots \cup F_n| = N|I \cap S| > N\alpha|I|$$

$$= \alpha|I_1 \cup \cdots \cup I_n| > (1 - |I_{n+1}|) > \alpha(1 - \varepsilon/\alpha).$$

The right side can be made arbitrarily close to 1 by choosing ε small and α close to 1.

References

[1] R. W. Carey and J. D. Pincus, *Unitary equivalence modulo the trace class for selfadjoint operators*, Amer. J. Math., **98** (1976), 481–514.

[2] W. F. Donoghue, *On the perturbation of spectra*, Comm. Pure Appl. Math., **18** (1965), 559–579.

[3] P. R. Holmes, *Measure Theory*, Van Nostrand, Princeton, N. J., 1950.

[4] J. S. Howland, *On a theorem of Aronszajn and Donaghue on singular spectra*, Duke J. Math., **41** (1974), 141–143.
[5] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.
[6] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, v. III. Academic Press, New York, 1979.
[7] E. Wayne, private communication.

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