

LIE'S FUNDAMENTAL THEOREMS FOR LOCAL ANALYTICAL LOOPS

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A central piece of classical Lie theory is the fact that with each local Lie group, a Lie algebra is associated as tangent object at the origin, and that, conversely and more importantly, every Lie algebra determines a local Lie group whose tangent algebra it is. Up to equivalence of local groups, this correspondence is bijective.

Attempts at the development of a Lie theory for analytical loops have not been entirely satisfactory in this direction, since they relied more or less on certain associativity assumptions. Here we associate with an arbitrary local analytical loop a unique tangent algebra with a ternary multiplication in addition to the standard binary one, and we call this algebra an Akivis algebra. Our main objective is to show that, conversely, for every Akivis algebra there exist many inequivalent local analytical loops with the given Akivis algebra as tangent algebra. We shall give a good idea about the degree of non-uniqueness. It is curious to note that, on account of this non-uniqueness, the construction is more elementary than in the case of analytical groups.

The First and Third Fundamental Theorems of Sophus Lie assert the following statements (Cf. [18], IV, Kap. 15, pp. 365–404):

THE FIRST THEOREM. *Every local analytical group determines on its tangent vector space at the identity the structure of a unique Lie algebra with the Lie bracket given by the formula*

$$(1) \quad [x, y] = \lim_{t \rightarrow 0} t^{-2} \left(\frac{tx \circ ty}{ty \circ tx} \right)$$

where the local group operation \circ is transported into the tangent space, and where g/h is written for $g \circ h^{-1}$.

THE THIRD THEOREM. *If L is a finite dimensional real Lie algebra, then there exists a local analytical group whose tangent Lie algebra according to the First Theorem is isomorphic to L . Moreover, local analytical groups with this property are locally isomorphic.*

A reasonable proof of the Third Theorem proceeds along these lines (see e.g. [10], Chap. II, [26]): In the algebra of formal power series in two non-commuting variables X and Y , there is a well-defined power series $X * Y$ defined by the equation $\exp(X * Y) = \exp X \exp Y$. If $X * Y = \sum_{n=1} H_n(X, Y)$, where $H_n(X, Y)$ are the homogeneous polynomials of degree n , being defined as the linear combinations of the monomials of degree n occurring in the series, then it turns out that $H_n(X, Y)$ is always contained in the Lie algebra $L[X, Y]$ generated by the variables X and Y in the Lie algebra of all polynomials relative to the commutator bracket $[p, q] = pq - qp$. In particular $H_1(X, Y) = X + Y$, $H_2(X, Y) = \frac{1}{2}[X, Y]$, $H_3(X, Y) = \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]])$. One shows further, that $\text{Lie}[X, Y]$ is in fact the free Lie algebra generated by X and Y so that for every given Lie algebra L the polynomials $H_n(x, y)$ are well defined for any pair of elements $x, y \in L$. If we now consider a finite dimensional real Lie algebra L we can introduce a norm which is compatible with the topology and which satisfies $\|[[x, y]]\| \leq \|x\| \|y\|$. Then one shows that the Campbell-Hausdorff series converges absolutely on the open ball B with radius $\frac{1}{2} \log 2$. Thus one obtains an analytical function $(x, y) \mapsto x * y: B \times B \rightarrow L$ which defines on B the desired structure of a local group.

A loop is a group except that one allows the multiplication to be non-associative. Specifically: A loop is a set G with a distinguished element e and three binary operations $(x, y) \mapsto xy, x/y, y \setminus x: G \times G \rightarrow G$ such that $xe = ex = x$ and $y(y \setminus x) = (x/y)y = x$ for all $x, y \in G$. Topological and analytical loops are easily defined, and with the usual circumspection one defines local analytical loops (see §3 below or [2]). In this paper we prove Lie's fundamental theorems for local analytical loops.

Why bother? There are several reasons. Firstly, Lie loops have been considered in the literature at least since that paper by A. I. Malcev [20] which is considered the start of the theory. The area of Moufang Lie loops is in fact a highly developed theory for which most results of classical Lie group theory have been verified mostly through the work of Kuzmin and Kerdman [14, 15, 16, 17] and through Sagle [22, 23, 24]. In his extensive study of webs in differential geometry, Akiwis and his school [1 through 7, 21] contributed substantially to the general foundations of Lie loop theory. However, Lie's Third fundamental theorem was never established (see [3]). Holmes and Sagle studied local power associative Lie loops and dealt with the question to which extent the Campbell-Hausdorff formalism could be brought to bear on the theory [13]. A similar discussion is to be found in a paper by Akiwis [1].

However, apart from the applications to Lie loop theory we feel that a discussion of Lie's fundamental theorems in the absence of associativity

sheds new light on the classical associative case. In fact, before we can deal with the non-associative case we first have to find the appropriate generalization of the concept of a Lie algebra. The way in this direction was shown by Akiwis. In (local) Lie group theory, the Lie bracket $[x, y]$ introduced in (1) measures the degree of commutativity of the local operation. It is Lie's most remarkable insight that the bracket is determined by the degree two terms in the Taylor expansion of the product, and that it suffices as a basis for the entire local theory. Akiwis [3] has observed that, in the absence of associativity, one must measure the degree of associativity by a ternary operation called associator bracket:

$$(2) \quad \langle x, y, z \rangle = \lim_{t \rightarrow 0} t^{-3} \left(\frac{(tx \circ ty) \circ tz}{tx \circ (ty \circ tz)} \right)$$

and that the two are linked by an identity which we shall call the Akiwis identity (cf. [3], [7])

$$(A) \quad \sum_{g \in S_3} \text{sgn}(g) \langle x_{g(1)}, x_{g(2)}, x_{g(3)} \rangle \\ = [[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2].$$

One remarks immediately that in the case of an associative local multiplication the associator bracket vanishes and the Akiwis identity reduces to the classical Jacobi identity.

We shall say that an algebra with a bilinear anticommutative multiplication $(x, y) \mapsto [x, y]$ and a trilinear ternary multiplication is an Akiwis algebra if it satisfies the Akiwis identity (A). (Akiwis calls these algebras W -algebras [3].) This concept is quite natural: If one considers an algebra E over a field F with a binary bilinear multiplication $(x, y) \mapsto xy$ and defines two new operations by $[x, y] = xy - yx$ and $\langle x, y, z \rangle = (xy)z - x(yz)$, then the resulting algebra $(E, [\cdot, \cdot], \langle \cdot, \cdot, \cdot \rangle)$ is an Akiwis algebra.

Thus a problem is whether for any finite dimensional real Akiwis algebra we find a local analytical loop whose associated tangent Akiwis algebra is the given one. Our answer is affirmative and reads, perhaps a bit surprisingly, as follows:

THEOREM. *Let W be a completely normable vector space over the field of real or complex numbers and suppose that we are given continuous multilinear maps $[\cdot, \cdot]: W^2 \rightarrow W$ and $\langle \cdot, \cdot, \cdot \rangle: W^3 \rightarrow W$ making W into an Akiwis algebra. Then there are continuous trilinear maps $r, s: W^3 \rightarrow W$ such*

that the multiplication

$$(x, y) \mapsto x \circ y = x + y + \frac{1}{2}[x, y] + r(x, x, y) + s(x, y, y): W^2 \rightarrow W$$

defines near 0 the structure of a local analytical loop in such a way that the commutator and associator bracket defined for it according to formulae (1) and (2) is the given one (so that the associated Akinis algebra is indeed the given one).

The proof is relatively elementary insofar as it is practically carried out in the domain of multilinear algebra which never goes beyond degree 3. Not only is this proof more elementary than any of the proofs of Lie's Third Theorem, but it also lacks the uniqueness statement. Even among the polynomial multiplications of degree three there are many pairs of trilinear maps (r, s) which satisfy the condition of the Theorem. In fact, the variety of different solutions to the problem in the realm of polynomial maps of degree three is in bijective correspondence to the space of all continuous symmetric trilinear maps $W^3 \rightarrow W$, as we will show. More ambiguity comes in if we modify one of the multiplications exhibited in the Theorem by allowing for additional summands of higher order than the third. All such will always define the same Akinis tangent algebra. In particular, if we start from any of the familiar finite dimensional Lie algebras, there are numerous local analytical loops whose associated Akinis algebra is the given Lie algebra and there are several among them whose multiplication is polynomial of total degree not more than three. However, up to local isomorphism, there will be only one which is associative, namely, the one guaranteed by Lie's Theorem and which we can obtain from the Campbell-Hausdorff series. If we start with an arbitrary Akinis algebra, the Campbell-Hausdorff formula will not in general give a local loop whose associated Akinis algebra is the given one. This is, however, the case if the given Akinis algebra is *alternative*—which means that any pair of its elements is contained in a Lie subalgebra. Since the relation between the proof of Lie's Third Theorem and the Campbell-Hausdorff formalism is so important, we will also comment on the Campbell-Hausdorff formalism. In this context we discuss the property of alternativity. We emphasize, however, that we do not wish to deal primarily with the effect of weak associativity hypotheses; we are interested in the Fundamental Theorems in the absence of any associativity assumption. Under such general circumstances it is not even known whether local one parameter subloops exist (although a recent theorem by Hofmann and Lawson shows that in any local analytical loop one can, perhaps after

making the domain of the multiplication smaller, when necessary, always find a unique smallest closed local multiplicatively closed connected subset having a given half-line in the tangent algebra as its tangent space [11]). Our methods therefore do not rely on the traditional local Lie theory. We proceed as follows: In the first section we provide some background from multilinear algebra. Since the action of the symmetric group on the arguments of multilinear maps is crucial for our proof, we need some information on the representation theory of symmetric groups. The second section gives the proof of the Fundamental Theorems, however, in a purely algebraic setting of formal loops. We also comment on the Campbell-Hausdorff formula in the context of power-associative algebras. Indeed, power-associativity is the very least assumption we have to require to be even able to speak about the exponential function. We indicate a purely algebraic proof of a result of Holmes and Sagle's, according to which one can express the homogeneous components of the Campbell-Hausdorff series, in the context of non-associative power-associative algebras, completely inside the Akiwis algebra associated with the algebra [13]. One can show in this context that a power-associative local Bruck loop (i.e. a loop satisfying $x^{-1}(xy) = (yx)x^{-1} = y$) is alternative. We formulate some problems which appear to be of urgency in the general context of Akiwis algebras and any theory which would parallel that of Lie algebras and Lie groups. In the third section we utilize the accumulated information to give a proof of the Fundamental Theorems; given standard analysis results this is no longer difficult, since all the work is done in the algebraic context of the first two sections.

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1. A Lemma on multilinear algebra. Symmetry and antisymmetry of multilinear maps are governed by the actions of the symmetric group on the arguments. Hence we consider actions of the group S_n (on n elements) on vector spaces. The case that actually interests us is that of $n = 3$. We will take a groundfield of characteristic zero. (A characteristic relatively prime to $n!$ would suffice!) We define R to be the group algebra $F[S_n]$. In R we define the *antisymmetry element* $a \in R$

$$(i) \quad a = \frac{1}{n!} \sum \{ \text{sgn}(g) \cdot g : g \in S_n \}.$$

This element is idempotent, and if A_n is the subgroup of even permutations in S_n , then $ag = a$ for $g \in A_n$ and $ag = -a$ for $g \notin S_n \setminus A_n$. We are interested in the right ideal $(1 - a)R$ and note

(ii) $(1 - a)R = \{x \in R: ax = 0\} = \text{annihilator of } a \text{ on the right.}$

In particular,

(iii) $1 + (S_n \setminus A_n) \subset (1 - a)R,$

(iv) $1 - A_n \subset (1 - a)R.$

As the $n!/2 + (n!/2 - 1)$ elements of $(1 + (S_n \setminus A_n)) \cup (1 - (A_n \setminus \{1\}))$ are linearly independent, and since a does not annihilate a because of $a^2 = a \neq 1$, we know

(iv) $\dim(1 - a)R = n! - 1.$

If H is a subgroup of S_n and we set $e = (1/|H|)\sum H$, then eR is the fixed point set of H on R for left multiplication. If $T \subset S_n$ is a set which meets each coset Hg in precisely one point, then $S_n = HT$ and $eHs = \{es\}$ for each $s \in T$, and thus eR is spanned precisely by the $[G:H]$ linearly independent elements $es, s \in T$. In particular, if t is an involution ($t^2 = 1 \neq t$), then this applies to $H = \{1, t\}$ and shows

(v) $\dim(1 + t)R = \dim \frac{1}{2}(1 + t)R = \frac{n!}{2}.$

We are now ready for the following Lemma:

1.1. LEMMA. *Let t denote a transposition of two elements in S_3 and $c \neq 1$ a cyclic permutation of all elements (i.e., an element of order 3). Then*

(1) $(1 + t)R + (1 + tc)R = (1 - a)R.$

Proof. Since $n = 3$, we know that tc is again an involution. Hence the left hand side of (1) is contained in the right hand side by (iii). By (iv) it now suffices to show that the dimension of the left hand side is $n! - 1$. Since $\dim(1 + t)R = \dim(1 + ct)R = n!/2$ by (v) it suffices to show that $(1 + t)R \cap (1 + ct)R$ is one dimensional. But $(1 + t)R$ is the precise fixed point set of t and $(1 + ct)R$ is the precise fixed point set of ct (under multiplication on the left). Hence the intersection of the two sets is the fixed point set of the subgroup generated by t and tc , which is all of S_n . But this fixed point set is precisely eR with $e = (1/n!)\sum S_n$, and hence is one dimensional. \square

We observe that Lemma 1.1 generalizes to more general groups G in place of S_3 : Indeed if G is dihedral, i.e., contains a cyclic normal subgroup $C = \langle c \rangle$ of order $m \geq 3$ and of index 2 such that G/C acts on C by inversion, then we let $\text{sgn}: G \rightarrow \{1, -\}$ denote the homomorphism with kernel C and note that every element $t \in G \setminus C$ is an involution. If we replace $n!$ by $|G| = 2m$, then the Lemma remains intact.

The identity (1) in Lemma 1.1 is of the form $I + J = eR$ for two right ideals I and J and an idempotent e in a ring R . The following Lemma generalizes it to the corresponding relation $IV + JV = eV$ for any arbitrary left R -module V . Just as Lemma 1.1, we formulate it, however, in the more narrow frame which we need in Corollary 1.5 below, and this in turn is all that is used later in the paper.

1.2. LEMMA. *Let V be an arbitrary left S_3 -module over F , hence an $R = F[S_3]$ -left-module. Then*

$$(2) \quad (1 + t)V + (1 + tc)V = (1 - a)V$$

for any element $t, c \in S_3$ of order 2 and 3, respectively.

Proof. We want to reduce the proof to the information gathered in Lemma 1. We recall that the left R -module V is isomorphic to the left module $R \otimes_R V$ under the map $v \mapsto 1 \otimes v: V \rightarrow R \otimes_R V$ (with the inverse map $x \otimes v \mapsto xv$), where the R -left-multiplication on $R \otimes_R V$ is given by $x(y \otimes v) = xy \otimes v$, and where, in the formation of the tensor product R is considered as a R -right-module. Since every submodule of an R -module splits by Maschke's theorem, the functor $-\otimes_R V$ from the category of R -right-modules to the category of F -vector spaces is exact. We now consider the sequence of R -right-modules.

$$(3) \quad R \oplus R \xrightarrow{f} R \xrightarrow{f'} R, \quad f(x, y) = (1 + t)x - (1 + ct)y, \quad f'(x) = ax.$$

To say that (3) is exact is precisely the statement of Lemma 1.1. Hence upon applying $-\otimes_R V$ to (3) we obtain the exact sequence of F -vector spaces

$$(4) \quad (R \oplus R) \otimes_R V \xrightarrow{f \times 1} R \otimes_R V \xrightarrow{f' \times 1} R \otimes_R V.$$

The function

$$h: (R \oplus R) \otimes_R V \rightarrow (R \otimes_R V) + (R \otimes_R V),$$

$$h((x, y) \otimes v) = (x \otimes v, y \otimes v)$$

$v: W_1 \times \cdots \times W_1 \rightarrow W_2$. Then S_n acts on V on the left by permutation of the arguments as follows:

(6) For $x \in S_n$ and $v \in V$ let $xv(w_1, \dots, w_n) = v(w_{\bar{x}(1)}, \dots, w_{\bar{x}(n)})$, $\bar{x} = x^{-1}$. (If we write $w'_k = w_{\bar{x}(k)}$, then

$$\begin{aligned} y(xv)(w_1, \dots, w_n) &= yv(w'_1, \dots, w'_n) = v(w'_{\bar{y}(1)}, \dots, w'_{\bar{y}(n)}) \\ &= v(w_{\bar{x}\bar{y}(1)}, \dots, w_{\bar{x}\bar{y}(n)}) = v(w_{\overline{yx}(1)}, \dots, w_{\overline{yx}(n)}) \\ &= (yx)(w_1, \dots, w_n). \end{aligned}$$

If we set $m = (1/n!) \sum S_n$ and $e = (2/n!) \sum A_n$, we notice $am = ma = 0$, $m + a = e$. A multilinear map v is called *symmetric* iff $gv = v$ for all $g \in S_n$, equivalently $mv = v$. It is called *antisymmetric* iff $tv = -v$ for all transpositions t ; this means $gv = \text{sgn}(g)v$ for all $g \in S_n$, equivalently $av = v$.

For each multilinear map v we have $v = av + (1 - a)v = mv + (1 - m)v$; thus it can be represented as a sum of its *antisymmetric part* av and a complement $(1 - a)v$ which is annihilated by a , and it can be written as a sum of its *symmetric part* mv and a complement $(1 - m)v$ which is annihilated by m . The symmetric parts of v and $(1 - a)v$ agree because of $am = 0$; likewise the antisymmetric parts of v and $(1 - m)v$ agree.

From our preceding Lemmas we now readily derive the following result:

1.4. PROPOSITION. *If V denotes the F -vector space $\text{Hom}(W_1^3; W_2)$ of all 3-linear maps $v: W_1^3 \rightarrow W_2$, then the space $(1 - a)V$ of all 3-linear maps with zero antisymmetric part is $(1 + t)V + (1 + tc)V$ for each transposition t and each 3-cycle c in S_3 .*

This follows immediately from Lemma 1.2.

We now formulate 1.4 in the form we shall need it later:

1.5. COROLLARY. *Every trilinear map $v: W_1^3 \rightarrow W_2$ with zero antisymmetric part can be written in the form $v(x, y, z) = r(x, y, z) + r(y, x, z) - s(x, y, z) - s(x, z, y)$ with two trilinear maps $r, s: W_1^3 \rightarrow W_2$.*

Proof. In 1.4 we take for t the transposition $t = (1\ 2)$, and for c the cyclic permutation $(1\ 2\ 3)$. Then $tc = (2\ 3)$, and the assertion is a direct consequence of 1.4. \square

In this very context we now consider the F -vector space $U = \text{Pol}^3(W_1, W_2)$ all homogeneous polynomial functions $p: W_1 \times W_1 \rightarrow W_2$ of degree 3 with the additional property that $p(x, 0) = p(0, x) = 0$ for all x . This means that each $p \in U$ is of the form $p(x, y) = r(x, x, y) + s(x, y, y)$ with a pair $(r, s) \in V \times V$. We define the linear function $G: V \times V \rightarrow U$ by $G(r, s)(x, y) = r(x, x, y) + s(x, y, y)$ and notice that $(r, s) \in \ker G$ iff $r(x, x, y) + s(x, y, y) = 0$ for all x, y ; this condition implies $r(x + y, x + y, z) + s(x + y, z, z) = 0$ for all x, y, z . A quick calculation shows that this implies $r(x, y, z) + r(y, x, z) = 0$ for all x, y, z , and this is equivalent to $r(x, x, y) = 0$ for all x, y . Similarly we obtain $s(x, y, y) = 0$ for all x, y . Clearly, these conditions in turn imply that $(r, s) \in \ker G$. If we again denote the transposition (1 2) in S_3 with t and the transposition (2 3) with tc , we have shown that

$$(7) \quad \ker G = \ker(1 + t) \oplus \ker(1 + tc),$$

where we use the expression $1 + t$ for the V -endomorphism $v \mapsto (1 + t)v$, and similarly for $1 + tc$. In particular, we have $\ker G \subseteq \ker F$ with $F(v, w) = (1 + t)v - (1 + tc)w$ as in 1.2 and 1.3. Now G induces an isomorphism $G^*: V/\ker(1 + t) \oplus V/\ker(1 + tc) \rightarrow U$, and F induces a morphism $F^*: V/\ker(1 + t) \oplus V/\ker(1 + tc) \rightarrow (1 - a)V$. This allows us to define the morphism $\phi = F^*G^{*-1}: U \rightarrow (1 - a)V$ which we know to be surjective after 1.4. and 1.5. The kernel of F^*G^{*-1} is isomorphic to $\ker F^*$. We may replace $V/\ker(1 + t)$ by its isomorphic copy $(1 + t)V$, and $V/\ker(1 + tc)$ by its isomorphic copy $(1 + tc)V$; under this replacement, we have to replace F^* by the map $D: (1 + t)V \oplus (1 + tc)V \rightarrow (1 - a)V$, $D(v, w) = v - w$. But the kernel of this morphism is isomorphic to $(1 + t)V \cap (1 + tc)V$ under the map $v \mapsto (v, v)$. We have observed earlier that $(1 + t)V \cap (1 + tc)V$ is the fixed point space of t and of tc , which generate S_3 . Hence it is the fixed point space mV of all symmetric trilinear maps. We have shown:

1.6. PROPOSITION. *Let V be the vector space of all trilinear maps $W_1^3 \rightarrow W_2$, and consider the following vector subspaces:*

- (a) *the subspace aV of all antisymmetric maps,*
- (b) *the subspace $(1 - a)V$ of all trilinear maps without antisymmetric parts, and*
- (c) *the subspace mV of all symmetric trilinear maps.*

If further U denotes the F -vector space of all homogeneous polynomial maps $p: W_1^2 \rightarrow W_2$ of degree 3 with $p(0, x) = p(x, 0)$, then there are exact

sequences

$$(8) \quad 0 \rightarrow mV \rightarrow U \xrightarrow{\phi} (1 - a)V \rightarrow 0,$$

$$(9) \quad 0 \rightarrow mV \rightarrow U \xrightarrow{\phi} V \xrightarrow{a} aV \rightarrow 0.$$

If W_1 and W_2 are finite dimensional F -vector spaces of dimension j and k , respectively, then elementary calculations show that mV is the space of homogeneous polynomials of degree 3 in j commuting variables, tensored with W_2 ; thus mV is $(j^2 + \binom{j}{3})k$ -dimensional, while aV may be identified with $\text{Hom}(W_1 \wedge W_1 \wedge W_1, W_2)$, hence is $\binom{j}{3}k$ -dimensional. From the fact $(1 - a)V$ is the kernel of $a: V \rightarrow aV$, it then follows that

$$\dim_F(1 - a)V = \dim V - \dim aV = j^3k - \binom{j}{3}k.$$

Thus

$$\begin{aligned} \dim U &= \dim mV + \dim(1 - a)V \\ &= (j^2 + \binom{j}{3})k + j^3k - \binom{j}{3}k = j^2(j + 1)k. \end{aligned}$$

2. Formal power series in two variables, Akinis algebras. We consider a field F of arbitrary characteristic and a vector space W over F . A *monomial function* $f: W \times W \rightarrow W$ in x and y of degree $n = 1, 2, \dots$ is a function for which there is a multilinear map $\Phi: W^p \times W^q \rightarrow W$ with $f(x, y) = \Phi((x, \dots, x), (y, \dots, y))$, $p + q = n$. We say that $f: W \times W \rightarrow W$ is a *homogeneous polynomial* in x and y of degree n if it is a finite linear combination of monomial functions of degree n . We consider constants as degree 0 polynomials. We say that a sequence (f_0, f_1, \dots) of homogeneous polynomials of x and y of degree $\deg f_n = n$ is a *formal power series* in x and y which we write $f = f_0 + f_1 + \dots$. Clearly, formal power series in x and y form an F -vector space. (It should be clear how one defines formal power series in any finite set of variables.)

We shall consider a formal power series in x and y which we shall denote $x \circ y$, and for which we postulate

$$(1) \quad x \circ 0 = 0 \circ x = x \quad \text{for all } x.$$

We then note

$$(2) \quad x \circ y = x + y + q(x, y) + r(x, x, y) + s(x, y, y) + f_4 + \dots$$

with a bilinear monomial q and trilinear monomials r and s .

From §1 we recall that f_3 does not determine the pair $(r, s) \in \text{Hom}(W^3; W)$ uniquely, but only up to pairs (r, s) such that r is antisymmetric in the first two variables and s is antisymmetric in the second two.

If $f = f_1 + f_2 + \dots +$ is a power series with zero constant term, then substitutions of the type $x \circ f(x, y)$ and $f(x, y) \circ y$ are well defined. By induction, we construct unique power series x/y and $x \setminus y$ satisfying the equations

$$(3) \quad (x/y) \circ y = x \quad \text{and} \quad x \circ (x \setminus y) = y.$$

(Indeed, if e.g. $x/y = p_1 + p_2 + \dots$ with homogeneous polynomials p_n which are to be determined, we obtain equations $p_1 + y = x$, $p_2 + q(p_1, y) = 0$, $p_3 + q(p_2, y) + r(p_1, p_1, y) + s(p_1, y, y) = 0, \dots$ which allow a unique successive solution

$$\begin{aligned} p_1 &= x - y, & p_2 &= -q(x, y) + q(y, x), \\ p_3 &= -q(q(x, y), y) + q(q(y, x), x) - r(x - y, x - y, y) \\ &\quad - s(x - y, y, y), \dots \end{aligned}$$

On account of (1), (2) and (3) we shall call the pair (W, \circ) a *formal loop*.

By straightforward calculation we obtain

2.1. LEMMA. *The power series $(x \circ y)/(y \circ x)$, $(y \circ x) \setminus (x \circ y)$, and $(x \circ y) - (y \circ x)$ all have vanishing degree 1 term and the same degree 2 component $q(x, y) - q(y, x)$.*

The following is a bit more technical, but presents no essential difficulty:

2.2. LEMMA. *The power series*

$$((x \circ y) \circ z)/(x \circ (y \circ z)), (x \circ (y \circ z)) \setminus ((x \circ y) \circ z)$$

and $(x \circ y) \circ z - x \circ (y \circ z)$ have all vanishing degree 1 and 2 components and the same degree 3 component

$$(4) \quad q(q(x, y), z) - q(x, q(y, z)) + r(x, y, z) + r(y, x, z) \\ - s(x, y, z) - s(x, z, y).$$

Proof. We record the calculation for the last power series: Firstly, we calculate up to degree 3:

$$\begin{aligned} u &= (x \circ y) \circ z = x + y + q(x, y) + r(x, x, y) + s(x, y, y) + z \\ &\quad + q(x + y, z) + q(q(x, y), z) + r(x + y, x + y, z) + s(x + y, z, z) \\ &= x + y + z + q(x, y) + q(x, z) + q(y, z) + q(q(x, y), z) \\ &\quad + r(x, x, y) + r(x, x, z) + r(x, y, z) + r(y, x, z) + r(y, y, z) \\ &\quad + s(x, y, y) + s(x, z, z) + s(y, z, z). \end{aligned}$$

Secondly and similarly we calculate up to degree 3:

$$\begin{aligned} v = x \circ (y \circ z) &= x + y + z + q(x, y) + q(x, z) + q(y, z) \\ &+ q(x, q(y, z)) + r(x, x, y) + r(x, x, z) + r(y, y, z) \\ &+ s(x, y, y) + s(x, y, z) + s(x, z, y) + s(x, z, z) + s(y, z, z). \end{aligned}$$

Subtracting the two polynomials gives us precisely the vanishing of the degree 1 and 2 components and the degree 3 component (4) of $u - v$, the last of the three power series in the Lemma.

If we consider the first of the three power series, then the linear term $u - v$ in the expansion of u/v give us (4). The quadratic term $-q(u, v) + q(v, u)$ yields vanishing components of order 2 and 3, since the linear and quadratic terms of u and v agree, and the higher order components do not concern us here. In a similar vein, the third order term $f_3(u, v)$ of u/v cannot bring any contribution since only the degree 1 components of u and v play a role here, and they are equal: indeed $f_3(w, w)$ is clearly 0. \square

Through the preceding calculation we have associated with the degree 2 and 3 components of the given power series $x \circ y$ a bilinear map in Lemma 2.1 and a trilinear map in Lemma 2.2. These measure the “degree of commutativity and associativity” of the power series operation $x \circ y$ and thus deserve special attention.

2.3. DEFINITION. With a power series $x \circ y = x + y + q(x, y) + r(x, x, y) + s(x, y, y) + \dots$ we *associate* (a) the *commutator* $[x, y] = q(x, y) - q(y, x)$ and (b) the *associator*

$$\begin{aligned} \langle x, y, z \rangle &= q(q(x, y), z) - q(x, q(y, z)) + r(x, y, z) \\ &+ r(y, x, z) - s(x, y, z) - s(x, z, y). \end{aligned}$$

Thus we have equipped the vector space W with a binary and ternary operation. If, by chance, W is a Lie algebra over a field of characteristic 0 and if $q(x, y) = \frac{1}{2}[x, y]$, $r(x, y, z) = s(y, z, x) = \frac{1}{12}[x, [y, z]]$, $f_n = H_n$ so that $x \circ y$ is the Campbell-Hausdorff series $x * y$, then we have by a well-known basic property of this series the associative identity $(x \circ y) \circ z = x \circ (y \circ z)$, whence $\langle x, y, z \rangle = 0$ by Lemma 2.2, and the commutator is the given Lie bracket by Lemma 2.1. What is the general situation? As a guiding principle, let us consider an F -algebra E with a binary bilinear multiplication $(x, y) \mapsto xy: E \times E \rightarrow E$. We then define

$$(5) \quad [x, y] = xy - yx \quad \text{and} \quad \langle x, y, z \rangle = (xy)z - x(yz),$$

thereby creating a new binary multiplication and a new ternary trilinear operation. We define two further ternary trilinear operations from the bracket $[x, y]$ by setting

$$(6) \quad K = K(x, y, z) = [[x, y], z] \quad \text{and} \\ J = J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y].$$

We will also abbreviate $\langle x, y, z \rangle$ by $A_0 = A(x, y, z)$. Let V again denote the F -vector space of all trilinear maps $E^3 \rightarrow E$. Then S_3 acts on V as in §1, paragraph (6). We can rewrite the definition of J as

$$(7) \quad J = (1 + c + c^2)K \text{ with the cyclic permutation } c = (1 \ 2 \ 3).$$

We note that the element $6a = \sum_{g \in S_3} \text{sgn}(g)g$ is well-defined for any characteristic.

2.4. LEMMA. *Let $p \in (1 - a)V$ be any trilinear map $E^3 \rightarrow E$ without non-trivial antisymmetric part and define $A = A_0 + p$ (i.e., $A(x, y, z) = \langle x, y, z \rangle + p(x, y, z)$). Then*

(A) $6aA = J$, equivalently,

$$\sum_{g \in S_3} \text{sgn}(g)A(x_{g(1)}, x_{g(2)}, x_{g(3)}) = J(x_1, x_2, x_3).$$

Proof. Since $p \in (1 - a)V$ we have $ap = 0$, so it obviously suffices to verify (A) with A_0 in place of A .

If we set $u(x, y, z) = (xy)z$ and $v(x, y, z) = x(yz)$, then $A_0 = u - v$, and we note that with $t = (1 \ 2)$ and $c = (1 \ 2 \ 3)$ in S_3 , we have $K = u - tu - (cv - ctv)$, for $u - tu = (xy)z - (yx)z$ and $cv - ctv = z(xy) - z(yx)$. Then

$$J = (1 + c + c^2)K \\ = (1 + c + c^2 - t - ct - c^2t) - (c + c^2 + 1 - ct - c^2t - t)v \\ = 6a(u - v) = 6aA_0. \quad \square$$

This leads us to the following Definition:

2.5. DEFINITION. An *Akivis algebra* L over a field F of any characteristic is a F -vector space together with a binary bilinear antisymmetric multiplication $(x, y) \mapsto [x, y]$ and a ternary, trilinear multiplication $(x, y, z) \mapsto \langle x, y, z \rangle$ such that the identity (A) links the two operations. The relation (A) is called the *Akivis identity*. The Akivis algebra is called *alternative* iff the ternary multiplication $A = \langle x, y, z \rangle$ is antisymmetric,

i.e., is in aV , where V is the space of all trilinear maps $L \times L \times L \rightarrow L$. This is tantamount to saying that $aA = A$ or, in other words, that $\langle x, x, y \rangle = \langle x, y, x \rangle = \langle x, y, y \rangle = 0$. We call an Akivis algebra *power-associative* iff $\langle x, x, x \rangle = 0$ for all $x \in L$.

We noted in Lemma 2.4 that for any binary algebra E we have constructed an Akivis algebra $E_A = (E, [\cdot, \cdot], \langle \cdot, \cdot, \cdot \rangle)$, which we call the *Akivis algebra associated with E* .

A few simple remarks are in order:

2.6. REMARKS. 1. If L is an alternative Akivis algebra over a field of char $F \neq 2, 3$ then $\langle x, y, z \rangle = \frac{1}{6}J(x, y, z)$. Thus the ternary operation is completely determined by the bracket operation, and the algebra may be considered as a binary algebra whose associator is defined by $\langle x, y, z \rangle = \frac{1}{6}J(x, y, z)$, and which then satisfies (A).

2. If E is a binary algebra which is alternative in the sense that the associativity condition $x(yz) = (xy)z$ holds whenever two of the arguments agree, then the associated Akivis algebra E is alternative. If E is power-associative in the sense that each element generates an associative subalgebra, then E_A is power-associative.

3. If E is binary algebra with an antisymmetric multiplication $(x, y) \mapsto [x, y]$, then the definition $\langle x, y, z \rangle = \frac{1}{6}J(x, y, z)$ turns E into an alternative Akivis algebra $(E, [\cdot, \cdot], \langle \cdot, \cdot, \cdot \rangle)$.

Proof. 1. Since L is alternative, and in view of the Akivis identity, we have $J = 6aA = A$. We leave 2 and 3 as exercises. (Cf. [12], IX.6.5.)

The gist of these observations is that in the context of alternative structures, no harm comes from forgetting the ternary multiplication, because it can be reconstituted from the binary one. In particular, every Lie algebra is an Akivis algebra with $A = 0$, and conversely, an Akivis algebra with $A = 0$ is nothing but a Lie algebra.

Now we have the concepts for clarifying the link between the commutator and the associator associated with a power series (for a differential geometric version see Akivis [3], Theorem 5):

2.7. THEOREM I. *Let W be a vector space over the field F of characteristic $\neq 2, 3$ and let*

$$x \circ y = x + y + q(x, y) + r(x, x, y) + s(x, y, y) + f_4 + \dots$$

be a formal power series with a bilinear map q and two trilinear maps r and s . If $[x, y]$ and $\langle x, y, z \rangle$ denote the commutator and associator associated with the formal loop (W, \circ) according to 2.1, 2.2, and 2.3, then writing $L(W, \circ) := (W, [\cdot, \cdot], \langle \cdot, \cdot, \cdot \rangle)$ we find that $L(W, \circ)$ is an Akivis algebra. If the power series $x \circ y$ is such that $(x \circ y) \circ z = x \circ (y \circ z)$ holds whenever two (respectively, all three) of x, y and z are equal, then $L(W, \circ)$ is alternative (resp., power-associative). If $(x \circ y) \circ z = x \circ (y \circ z)$ for all x, y, z , then $L(W, \circ)$ is a Lie algebra.

Proof. We let $A = \langle x, y, z \rangle$ and write $A_q(x, y, z) = q(q(x, y), z) - q(x, q(y, z))$ for the associator bracket for the binary multiplication q . Then by 2.3 we have

$$(8) \quad A = A_q + (1 + t)r - (1 + tc)s$$

with $t = (1 \ 2)$, $c = (1 \ 2 \ 3)$, $tc = (2 \ 3)$ in S_3 .

If we write $p = (1 + t)r - (1 + tc)s$ ($= F(r, s)$) in the notation of §1! then $ap = a(1 + t)r - a(1 + tc)s = 0$. Then Lemma 2.4 shows that $6aA = J_q$ where $J_q = (1 + c + c^2)K_q$ with $K_q(x, y, z) = [[x, y]_q, z]_q$, where $[x, y]_q = q(x, y) - q(y, x)$. But by Lemma 2.1. we know $q(x, y) - q(y, x) = [x, y]$, whence $J_q = J$. Thus the Akivis identity is proved. If $x \circ (y \circ z) = (x \circ y) \circ z$ whenever two of x, y, z are equal, then by Lemma 2.2, we find that $\langle x, y, z \rangle = 0$ whenever two of x, y, z are equal. The proof of Theorem 1 is complete. \square

In Theorem 1 we associate with a formal loop (W, \circ) an Akivis algebra $L(W, \circ)$. To what extent, if at all, is the converse true: Given an Akivis algebra E , is there a formal loop (W, \circ) such that $L(W, \circ) \cong E$?

2.8. THEOREM II. *Let W be an Akivis algebra over a field F with characteristic $\neq 2, 3$. Then there is a polynomial in x and y of total degree 3:*

$$x \circ y = x + y + \frac{1}{2}[x, y] + r(x, x, y) + s(x, y, y)$$

such that the formal loop (W, \circ) satisfies $L(W, \circ) \cong W$.

Let U denote the vector space of all homogeneous polynomials $p(x, y)$ in x and y of degree 3 with $p(x, 0) = p(0, x) = 0$, and U_W the set of all $p \in U$ such that the definition $x \circ y = x + y + \frac{1}{2}[x, y] + p(x, y)$ implies $L(W, \circ) \cong W$. Then U_W is an affine variety in U whose translation group is isomorphic to the vector space mV of all symmetric trilinear maps $W^3 \rightarrow W$. If $\dim W = j$ is finite, then $\dim mV = j^2(j^2 - \frac{1}{6}(j-1)(j-2))$.

Proof. We let V be the vector space of all trilinear maps $W^3 \rightarrow W$. Again we define $F: V \oplus V \rightarrow V$ by $F(r, s) = (1 + t)r - (1 + tc)s$. According to Lemma 2.4, using the fact that $q(x, y) - q(y, x) = 2q(x, y)$ for any antisymmetric map q , we must find pairs $(r, s) \in V \oplus V$ such that

$$(9) \quad \langle x, y, z \rangle = \frac{1}{2}[\frac{1}{2}[x, y], z] - \frac{1}{2}[x, \frac{1}{2}[y, z]] + r(x, y, z) \\ + r(y, x, z) - s(x, y, z) - s(x, z, y).$$

This is equivalent to

$$(10) \quad A = \frac{1}{4}(1 + c)K + F(r, s); \quad \text{equivalently,} \\ F(r, s) = A - \frac{1}{4}(1 + c)K.$$

The Akivis identity says $6aA = J$. We always have $6a(1 + c)K = 12aK = 2((1 + c + c^2)K - (1 + c + c^2)tK) = 4J$, i.e. $3a(1 + c)K = 2J$. Thus the Akivis identity is equivalent to $6a(A - \frac{1}{4}(1 + c)K) = J - \frac{2}{3}a(1 + c)K = 0$, i.e. to

$$(11) \quad A - \frac{1}{4}(1 + c)K \in (1 - a)V.$$

But by the results of §1, notably 1.3 and 1.4, we know that $(1 - a)V = \text{im } F$, whence the desired existence of (r, s) follows.

Now let $p \in U$ be any polynomial of degree 3 such that $x \circ y = x + y + \frac{1}{2}[x, y] + p(x, y)$ gives us $L(W, \circ) \cong W$. If p' is another one with the same property, then $p - p'$ is in the kernel of the map $\phi: U \rightarrow (1 - a)V$ of 1.6. By 1.6, this kernel is isomorphic to mV , the space of all symmetric trilinear maps $W^3 \rightarrow W$. In the finite dimensional case, the dimension of this space was given at the end of §1. \square

It is precisely in the proof of this Theorem that we use the background information which we prepared in §1. We note that Theorem II is indeed a converse of Theorem I, but it lacks a precise uniqueness statement. The existence of the desired power series $x \circ y$ can be secured even in the realm of polynomials of degree at most 3, but even in this domain there is no uniqueness; the ambiguity is of the magnitude of the space of all symmetric trilinear maps of W^3 into W .

We notice that the power series x/y and $y \setminus x$ constructed for $x \circ y$ as in (3) above will not be polynomial even though $x \circ y$ is a polynomial of total degree 3. In view of the fact, that for a Lie algebra W there is one and only one power series $x * y$ with $(x * y) * z = x * (y * z)$ such that $L(W, *) \cong W$, namely, the Campbell-Hausdorff series, it is certainly

desirable to have more information on the relation between Theorems I and II above and these classical results. Our understanding of the situation is limited at this time. However, the following remarks are certainly pertinent.

Firstly, the Campbell-Hausdorff series is inseparably linked with the exponential function. Formally, the handling of the exponential function requires the ability to deal with formal power series on W of the form $a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$ with $a_n \in F$. Here it is an indispensable hypothesis that we can talk about powers x^n . This means that we are operating in an algebra W with a binary multiplication $(x, y) \rightarrow xy$ which is at least power-associative (see [25], p. 128). In a power-associative algebra W we can now indeed define a power series

$$(12) \quad x * y = H_1(x, y) + H_2(x, y) + \dots = L((\exp x \exp y) - 1),$$

where $L(z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - + \dots,$

and where the $H_n(x, y)$ are homogeneous polynomials in x and y of degree n which are linear combinations of monomials in x and y of degree n , i.e., words in the free binary algebra generated by two letters. Thus $H_1(x, y) = x + y$ and $H_2(x, y) = \frac{1}{2}(xy - yx)$ are verified without undue pain. The calculation of $H_3(x, y)$ is already painful, but instructive. In fact we have

2.9. PROPOSITION (*Holmes and Sagle* [13], Theorem 3.6). *Let W be a power-associative algebra over a field F of characteristic 0 and let $[x, y] = xy - yx$ and $\langle x, y, z \rangle = (xy)z - x(yz)$ be the operations in the associated Akivis algebra W_A . Then the homogeneous component of degree three in the Campbell-Hausdorff series has the form*

$$H_3(x, y) = \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + P_3(x, y)$$

with

$$\begin{aligned} P_3(x, y) &= \frac{2}{3}(\langle x, x, y \rangle - \langle x, y, y \rangle - \frac{1}{3}(\langle y, y, x \rangle - \langle y, x, x \rangle)) \\ &= \frac{1}{3}(\langle x, x, y \rangle - \langle x, y, x \rangle - \langle x, y, y \rangle + \langle y, x, y \rangle). \end{aligned}$$

This Proposition was first proved by Holmes and Sagle in the context of calculus in n -dimensional real vector spaces. Our proof is merely a question of book-keeping. The recipe is as follows: We set

$$\begin{aligned} z &= (\exp x \exp y) - 1 \\ &= x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \frac{1}{6}x^3 + \frac{1}{2}xy^2 + \frac{1}{2}x^2y + \frac{1}{6}y^3 \end{aligned}$$

up to terms of higher order. We then have to consider $x * y = z - \frac{1}{2}z^2 + \frac{1}{3}z^3$ up to terms of higher order. This forces us to calculate the homogeneous components of degree 3 in the power series z^2 and z^3 which leads to the following table:

	x^3	y^3	xy^2	x^2y	y^2x	yx^2	$(xy)y$	$(xy)x$	$y(xy)$	$x(xy)$	$x(yx)$	$y(yx)$
$(z)_3$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$								
$(z^2)_3$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1		
$(z^3)_3$	1	1	1			1			1	1	1	1

We must now calculate $(z)_3 - \frac{1}{2}(z^2)_3 + \frac{1}{3}(z^3)_3$. The terms x^3 and y^3 drop out. There is no canonical way to proceed from here: however we suggest to do the accounting as follows: We calculate the polynomial in question as a linear combination of xy^2 , $x(xy)$, $x(yx)$, yx^2 , $y(yx)$, $y(xy)$, $\langle x, x, y \rangle = x^2y - (xy)y$, $\langle x, y, y \rangle = (xy)y - xy^2$, $\langle x, y, x \rangle = (xy)x - x(yx)$, and $\langle y, y, x \rangle = y^2x - y(yx)$. This calls for some rewriting of $(z)_3$ and $(z^2)_3$, but not of $(z^3)_3$. Now we collect the coefficients in the following table:

xy^2	$x(xy)$	$x(yx)$	yx^2	$y(yx)$	$y(xy)$	$\langle x, x, y \rangle$	$\langle x, y, y \rangle$	$\langle x, y, x \rangle$	$\langle y, y, x \rangle$
$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$
4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{4}$			
$\frac{1}{3}$	$-\frac{1}{4}$								
$-\frac{1}{2}$	$\frac{1}{2}$								

Now we have the following information on the series $x * y$:

$$\begin{aligned}
 (13) \quad x * y &= x + y + \frac{1}{2}[x, y] + \frac{1}{12}xy^2 + \frac{1}{12}x(xy) - \frac{1}{6}x(yx) \\
 &\quad + \frac{1}{12}yx^2 + \frac{1}{12}y(yx) - \frac{1}{6}y(xy) + \frac{1}{4}\langle x, x, y \rangle - \frac{1}{2}\langle x, y, y \rangle \\
 &\quad - \frac{1}{2}\langle x, y, x \rangle - \frac{1}{4}\langle y, y, x \rangle + \text{terms of higher order} \\
 &= x + y + \frac{1}{2}[x, y] + \frac{1}{12}(xy^2 + x(xy) + yx^2 + y(yx) \\
 &\quad - x(yx) - x(yx) - y(xy) - y(xy)) \\
 &\quad + \frac{3}{4}\langle x, x, y \rangle - \frac{1}{2}\langle x, y, y \rangle + \frac{1}{2}\langle y, x, x \rangle - \frac{1}{4}\langle y, y, x \rangle \\
 &\quad + \text{terms of higher order,}
 \end{aligned}$$

since

$$(14) \quad \langle x, x, y \rangle + \langle y, x, x \rangle + \langle x, y, x \rangle = 0$$

as a consequence of power-associativity, which yields $\langle x \pm y, x \pm y, x \pm y \rangle = 0$. We now eliminate the remaining monomials xy^2 etc. by introducing brackets in a suitable fashion. We note e.g. that

$$\begin{aligned} x(xy) - x(yx) - x(yx) + yx^2 \\ &= x[x, y] - (xy)x + (yx)x + \langle x, y, x \rangle - \langle y, x, x \rangle \\ &= x[x, y] - [x, y]x + \langle x, y, x \rangle - \langle y, x, x \rangle \\ &= [x, [x, y]] + \langle x, y, x \rangle - \langle y, x, x \rangle \\ &= [x, [x, y]] - (\langle x, x, y \rangle + 2\langle y, x, x \rangle) \end{aligned}$$

(again with the aid of (14)). In a similar fashion, we calculate

$$\begin{aligned} xy^2 + y(yx) - y(xy) - y(xy) &= [y, [y, x]] - \langle x, y, y \rangle + \langle y, x, y \rangle \\ &= [y, [y, x]] - (\langle y, y, x \rangle + 2\langle x, y, y \rangle). \end{aligned}$$

If we substitute these expressions into (13) we obtain the asserted form of $H_3(x, y)$. \square

We notice that $P_3(x, y) = 0$, whenever W is an associative algebra, and then $H_3(x, y)$ becomes the usual homogeneous component of degree 3 in the classical Campbell-Hausdorff formula.

The preceding proposition suggests that for a given Akivis algebra E we consider that polynomial

$$x \circ y = x + y + \frac{1}{2}[x, y] + r(x, x, y) + s(x, y, y)$$

for which the r and s are chosen so that

$$(15) \quad H_3(x, y) = r(x, x, y) + s(x, y, y), \quad \text{with } H_3(x, y) \text{ as in 2.9.}$$

If we set

$$(16) \quad \begin{aligned} r(x, y, z) &= \frac{1}{12}[x, [y, z]] + \frac{2}{3}\langle x, y, z \rangle + \frac{1}{3}\langle z, x, y \rangle, \quad \text{and} \\ s(x, y, z) &= \frac{1}{12}[y, [z, x]] - \frac{2}{3}\langle x, y, z \rangle - \frac{1}{3}\langle y, z, x \rangle, \end{aligned}$$

then we verify that (16) implies (15). Now the associated Akivis algebra $L(E, \circ)$ has the same commutator as E , but the associator is different. A straightforward but somewhat lengthy calculation based on (9) yields

$$(17) \quad A_0 = \langle x, y, z \rangle_0 = \frac{1}{2}J + (1 + t + ct + c^2t)A \quad \text{with } A = \langle x, y, z \rangle.$$

Equivalently, we can express the deviation of the given Akivis algebra E from the Akivis algebra $L(E, \circ)$ by the difference

$$(18) \quad A_0 - A = (1 + c + c^2)A - \frac{1}{2}J.$$

We recall $e = \frac{1}{3}(1 + c + c^2)$ from §1 and then note that $L(E, \circ) = E$ iff

(AA) $eA = \frac{1}{6}J$, equivalently,

$$\begin{aligned} &\langle x, y, z \rangle + \langle y, z, x \rangle + \langle z, x, y \rangle \\ &= \frac{1}{2}([[x, y], z] + [[y, z], x] + [[z, x], y]), \end{aligned}$$

which is “half of the Akiwis identity (A)”. Of course, (AA) implies (A), but not inversely as is readily verified in the Akiwis algebra associated with the free binary algebra in three generators x, y , and z . Thus, in the absence of condition (AA), the Campbell-Hausdorff formula *does not* yield a power series for which $L(E, *) = E$.

If E is alternative, then $A = \frac{1}{6}J$ by 2.6.1. Then $eA = \frac{1}{6}eJ = \frac{1}{6}J$ and thus (AA) is satisfied.

More generally, if E is an alternative Akiwis algebra, then the Akiwis subalgebra generated by two elements x and y is a Lie algebra. Hence the homogeneous polynomials $H_n(x, y)$ of the Campbell-Hausdorff series are simply the ones known from Lie algebra theory. This gives us the following remark:

2.10. PROPOSITION. *If W is an alternative Akiwis algebra over a field of characteristic 0, then the classical Campbell-Hausdorff series $x * y$ yields a formal loop $(W, *)$ with the property that $L(W, *) = W$. The same holds for any power series $x \circ y$ which agrees with $x * y$ up to homogeneous components of degree 1, 2 and 3.*

Several pressing questions remain open.

2.11. Problems. [12] 1. Let W be the free power-associative algebra in two indeterminates x and y over a field of characteristic 0. In the associated Akiwis algebra W_A we denote by $\text{Ak}[x, y]$ the Akiwis subalgebra generated by x and y . If $H_n(x, y)$ are the polynomials determined as the homogeneous components of the formal power series $x * y = L((\exp x \exp y) - 1)$, is it true that

$$H_n(x, y) \in \text{Ak}[x, y],$$

i.e. that the $H_n(x, y)$ are linear combinations of monomials built up from commutator and associator brackets alone? (The answer is yes for $n = 1, 2, 3$ by 2.9.)

2. Is $\text{Ak}[X]$, the Akiwis subalgebra generated by the set X of free generators of a free algebra W inside the associated Akiwis algebra W_A the free Akiwis algebra over X ?

3. Let $\text{fr}(W)$ denote the free non-associative algebra over a vector space W . If W carries the structure of an Akivis algebra, we denote with I the ideal generated in $\text{fr}(W)$ by the set of all $xy - yx - [x, y]$ and $(xy)z - x(yz) - \langle x, y, z \rangle$ with $x, y, z \in W$ and define $U(W) = \text{fr}(W)/I$ and let $f_W: W \rightarrow U(W)$ be defined by $f_W(w) = w + I$. Then for any (not necessarily associative) algebra E and any morphism of Akivis algebras $h: W \rightarrow E_A$ there is a unique morphism $h': U(W) \rightarrow E$ such that $h'f_W = h$. We call $U(W)$ the enveloping algebra of the Akivis algebra W . Under which circumstances is f injective? Is there a parallel to the theorem of Poincaré, Birkhoff and Witt (cf. [9], §§2, 7) in the context of enveloping algebras of Akivis algebras? Is $\text{fr}(X) = U(\text{Ak}[X])$? (Cf. also [3], Problem 1.)

3. Lie's fundamental theorems for local analytical loops. In this section we let F denote the real field \mathbf{R} or the complex field \mathbf{C} . (We are aware of the fact that other fields such as e.g. \mathbf{Q}_p could be included, too.) Let W be a completely normable F -vector space. A *monomial function* $f: W \times W \rightarrow W$ of degree $n = 1, 2, 3, \dots$ is a function for which there is a *continuous* multilinear map $\Phi: W^p \times W^q \rightarrow W$ with $f(x, y) = \Phi((x, \dots, x), (y, \dots, y))$, $p + q = n$. We say that $f: W \times W \rightarrow W$ is a *homogeneous polynomial function* of degree n if it is a finite linear combination of monomial functions of degree n . We say that a function $f: B \times B \rightarrow W$ is analytic (with an open neighbourhood B of 0 in W) whenever there is a formal power series $f_0 + f_1 + f_2 + \dots$ whose homogeneous components f_n are homogeneous polynomial functions such that the infinite series $f_0 + f_1(x, y) + f_2(x, y) + \dots$ converges absolutely on $B \times B$ relative to a norm on W compatible with the topology.

3.1. DEFINITION. A *local analytical loop* is a completely normable real or complex vector space W together with an open neighbourhood B of 0 and with three analytical function $\circ, /, \backslash: B \times B \rightarrow W$ such that the following identities are satisfied:

- (i) $x \circ 0 = 0 \circ x = x$ for all $x \in B$
- (ii) $x \circ (x \backslash y) = (y/x) \circ x = y$ for all $x, y \in B$ with $x \backslash y, y/x \in B$.

It follows from Definition 3.1 and the early observation (2) of §2, that the local multiplication of a local analytical loop is given by

$$(1) \quad x \circ y = x + y + q(x, y) + r(x, x, y) + s(x, y, y) + f_4(x, y) + \dots$$

for $(x, y) \in B \times B$ with continuous multilinear maps $q: W^2 \rightarrow W$
and $r, s: W^3 \rightarrow W$.

3.2. PROPOSITION. Let W be a completely normable F -vector space and

$$(2) \quad (x, y) \mapsto x \circ y = x + y + q(x, y) + r(x, x, y) + s(x, y, y) \\ + f_4(x, y) + \cdots : B_0 \times B_0 \rightarrow W$$

an analytical function with multilinear continuous maps q, r, s and an open neighbourhood of 0 in W , so that $x \circ 0 = 0 \circ x$ for all x sufficiently close to 0 . Then there is an open neighbourhood B of 0 in B_0 and analytical functions $/, \backslash : B \times B \rightarrow W$ such that $(W, B, \circ_B, /, \backslash)$ is a local analytical loop with the restriction $\circ_B = \circ|(B \times B)$.

Proof. We consider the function $m: B_0 \times B_0 \rightarrow B_0 \times W$, given by $m(x, y) = (x, x \circ y)$. This function is analytic and has the derivative $Dm(0)$ given by $Dm(0)(x, y) = (x, x + y)$ which is an isomorphism of vector spaces. Hence m allows a local analytical inverse by the inverse function theorem (see e.g. [8], 5.7.6.) We note that $m^{-1}(x, y) = (x, u)$ means $(x, y) = m(x, u) = (x, x \circ u)$; thus if we write $u = x \backslash y$, then the analytical function $(x, y) \mapsto x \backslash y$ satisfies the equation $x \circ (x \backslash y) = y$ on its domain. In a similar fashion one obtains x/y . If we choose B small enough so that all functions in question are defined on $B \times B$, we have completed the proof. \square

3.3. PROPOSITION. Let $(W, B, \circ, /, \backslash)$ be a local analytical loop. Then the following limits all exist and satisfy the identities

$$(3) \quad \lim_{t \rightarrow 0} t^{-2}((tx \circ ty)/(ty \circ tx)) = \lim_{t \rightarrow 0} t^{-2}((ty \circ tx) \backslash (tx \circ ty)) \\ = \lim_{t \rightarrow 0} t^{-2}(tx \circ ty - ty \circ tx) = q(x, y) - q(y, x)$$

$$(4) \quad \lim_{t \rightarrow 0} t^{-3}(((tx \circ ty) \circ tz)/(tx \circ (ty \circ tz))) \\ = \lim_{t \rightarrow 0} t^{-3}((tx \circ (ty \circ tz)) \backslash ((tx \circ ty) \circ tz)) \\ = \lim_{t \rightarrow 0} t^{-3}((tx \circ ty) \circ tz - tx \circ (ty \circ tz)) \\ = q(q(x, y), z) - q(x, q(y, z)) + r(x, y, z) \\ + r(y, x, z) - s(x, y, z) - s(x, z, y).$$

Proof. This follows from the definitions in 3.1, the subsequent observation (1), and Lemmas 2.1 and 2.2. \square

3.4. DEFINITION. Under the circumstances of Proposition 3.3 we call the bilinear expression in relation (3) *the commutator bracket* of the loop

and denote it by $[x, y]$, and we call the trilinear expression in relation (4) *the associator bracket* of the loop and denote it by $\langle x, y, z \rangle$.

3.5. DEFINITION. A *topological Akiwis algebra* is a topological F -vector space W together with two continuous multilinear maps $[\cdot, \cdot]: W^2 \rightarrow W$ and $\langle \cdot, \cdot, \cdot \rangle: W^3 \rightarrow W$ which satisfy the Akiwis identity.

3.6. PROPOSITION. Let $(W, B, \circ, /, \backslash)$ be a local analytical loop. Then the commutator bracket and associator bracket give continuous multilinear maps $[\cdot, \cdot]: W^2 \rightarrow W$ and $\langle \cdot, \cdot, \cdot \rangle: W^3 \rightarrow W$, respectively, and $(W, [\cdot, \cdot], \langle \cdot, \cdot, \cdot \rangle)$ is a topological Akiwis algebra.

Proof. From the fact that q, r and s are continuous multilinear maps and from Proposition 3.3 it is immediate that the commutator and associator brackets are continuous multilinear maps. From Theorem 2.7 we know that $(W, [\cdot, \cdot], \langle \cdot, \cdot, \cdot \rangle)$ is an Akiwis algebra whenever we have the relations $[x, y] = q(x, y) - q(y, x)$ and

$$\begin{aligned} \langle x, y, z \rangle &= q(q(x, y), z) - q(x, q(y, z)) + r(x, y, z) \\ &\quad + r(y, x, z) - s(x, y, z) - s(x, z, y) \end{aligned}$$

with multilinear functions q, r , and s . □

If $(W, B, \circ, /, \backslash)$ is a local analytical loop, then we say that the topological Akiwis algebra $(W, [\cdot, \cdot], \langle \cdot, \cdot, \cdot \rangle)$ is *the Akiwis algebra associated with the loop* and write $L(W, B, \circ, /, \backslash) = (W, [\cdot, \cdot], \langle \cdot, \cdot, \cdot \rangle)$. If no confusion arises we will briefly write $L(B, \circ)$.

We are now ready for the fundamental theorems.

3.7. LIE'S FUNDAMENTAL THEOREMS FOR LOCAL ANALYTICAL LOOPS.

A. Every local analytical loop $(W, B, \circ, /, \backslash)$ determines a unique associated topological Akiwis algebra $L(B, \circ) = (W, [\cdot, \cdot], \langle \cdot, \cdot, \cdot \rangle)$ (whose commutator and associator brackets are given by (3) and (4) above).

B. Let $W = (W, [\cdot, \cdot], \langle \cdot, \cdot, \cdot \rangle)$ be a topological Akiwis algebra on a completely normable real or complex vector space. Then there is a local analytical loop $(W, B, \circ, /, \backslash)$ such that $L(B, \circ) = W$. Moreover, there exist such loops for which the multiplication \circ is polynomial of the form (5) $x \circ y = x + y + \frac{1}{2}[x, y] + r(x, x, y) + s(x, y, y)$ with continuous multilinear maps r, s . The set of differences $x \circ_2 y - x \circ_1 y$ of such multiplications is in bijective correspondence with the vector space of all symmetric continuous trilinear maps $W^3 \rightarrow W$.

Proof. Part A is proved by Proposition 3.3. Part B follows from Theorem 2.8 which gives us two continuous trilinear maps r and s such that, with $q(x, y) = \frac{1}{2}[x, y]$ for $x \circ y$ given by (5), we have $\langle x, y, z \rangle$ equal to the expression in equation (4). Proposition 2.3 gives us the analytical local quotient functions $/$ and \backslash , which will make $(W, B, \circ, /, \backslash)$ a local analytical loop. The remainder now follows from Theorem 2.8. \square

3.8. DEFINITION. A local analytical loop $(W, B, \circ, /, \backslash)$ is said to be a *local analytical group* if (6) $(x \circ y) \circ z = x \circ (y \circ z)$ holds whenever all products in this formula are defined. A subset $C \subseteq B$ is called a *local subloop* iff $(C \circ C) \cap B \subseteq C$, $(C/C) \cap B \subseteq C$ and $(C \backslash C) \cap B \subseteq C$. Also C is called a *local subgroup* if it is a local subloop and (6) holds whenever $x, y, z \in C$ and all products in the formula are defined.

We say that $(W, B, \circ, /, \backslash)$ is an *alternative (power-associative) local analytical loop* if every set $X \subseteq B$ of at most two elements (respectively, one element) lies in a local subgroup.

3.9. PROPOSITION. A. *The Akivis algebra of a local analytical alternative loop is alternative.* B. *If W is an alternative topological Akivis algebra on a completely normable vector space, then there is a local analytical alternative loop $(W, B, *, /, \backslash)$ where $x * y = x + y + \frac{1}{2}[x, y] + H_3(x, y) + \dots$ is given by the classical Campbell-Hausdorff series converging absolutely on B (which we may take an open ball of radius $\frac{1}{2} \log 2$; here we single out a norm compatible with the topology and which satisfies $\| [x, y] \| \leq \|x\| \|y\|$) and where $x/y = y \backslash x = x - y$, and $L(B, *) = W$.*

Proof. A. Proposition 3.3 shows immediately that $\langle x, y, z \rangle$ vanishes whenever $\{x, y, z\}$ contains at most two elements. B. follows from Proposition 2.10 and the classical circle of ideas around the Campbell-Hausdorff formula. \square

There is a considerable theory on Moufang Lie loops and the corresponding Akivis algebras which are known as *Malcev algebras* in the literature [12 through 16, 19 through 23]. Moufang Lie loops are alternative and Malcev algebras are alternative (here the expression *binary Lie algebra* is also used in the literature instead of “alternative Akivis algebra”).

We recall that in the classical local Lie group theory the Lie algebra determines a local Lie group up to local isomorphism. This fact was first extended to alternative Lie loops by Malcev [20]. Through the work of

Holmes and Sagle (see [13]) one knows that a power-associative local analytical loop may always be parametrized in such a fashion that it takes the form $(W, B, \circ, /, \backslash)$ with $x \circ y = x + y$, whenever x and y are linearly dependent. This then holds, in particular, for alternative local analytical loops. If $(W, B, \circ, /, \backslash)$ is an alternative local analytical loop with $x \circ y = x + y$ for linearly dependent $x, y \in W$ then $x \circ y$ must agree with the Campbell-Hausdorff product $x * y$ for all sufficiently small x, y . For if x and y are two elements in B and if we assume that $u * v$ is defined for all $u, v \in B$, then there is a closed vector subspace V of W containing x and y on which the Akiwis algebra structure $L(W, \circ)$ of W induces the structure of a Lie algebra, and for which \circ induces on $V \cap B$ the structure of a local analytical group with $x \circ y = x + y$ for linearly dependent x and y . Classical Lie group theory then yields $u \circ v = u * v$ on V for sufficiently small u and v . Since \circ and $*$ are both analytic, they agree on their common domain. Since x and y were arbitrary in B , the claim is established.

There are other applications of the canonical coordinates introduced by Holmes and Sagle for power-associative local analytical loops.

3.10. REMARK. (Cf. [12], IX.6.23). If $(W, B, \circ, /, \backslash)$ is a power-associative local analytical loop satisfying the identity $x^{-1}(xy) = (yx)x^{-1} = y$, then W is alternative.

It is no problem to carry out the theory outlined in this section for local loops of class C^k , $k = 3, 4, \dots$, instead of class C^ω which we discussed here.

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