ON SOLUTIONS OF DIFFERENTIAL EQUATIONS WHICH SATISFY CERTAIN ALGEBRAIC RELATIONS

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In the following, we provide another proof (Theorem 3.1 below) of recent results of Harris-Sibuya, using some elementary commutative algebra. Our purpose is to give a uniform treatment for their results which also permits some generalization. We note that the study of differential equations under the hypothesis that the solutions satisfy an algebraic relation is not new. Fano, among others, made a systematic study of this situation in the last century. Also Lamé equations in which two solutions have a rational function as their product have proved to be a good source of examples for unusual arithmetic behavior. But in the case of Harris-Sibuya, as well as the present paper, the solutions need not be solutions of the same linear equation. In the treatment below the differential equation only enters in dilineating a type of recursion.

1. Let $R = K[\{\Lambda_i\}_{i=1}^N]$ be a polynomial ring over a field K. Assume that it is graded by assigning to each variable Λ_i a natural number, $w(\Lambda_i) = w_i \in \mathbf{N}$, and then assigning

$$w\left(\prod_{i=1}^N \Lambda_i^{\mu_i}\right) = \sum_{i=1}^N \mu_i w_i.$$

The following result makes use of some elementary commutative algebra.

(1.1) THEOREM. *Let I be the ideal of R generated by a collection of polynomials, $\{f_\beta(\Lambda)\}_{\beta \in \Gamma} \subseteq R$. Let $\tilde{f}_\beta(\Lambda)$ be the leading homogeneous form of $f_\beta(\Lambda)$ with respect to the above grading; let J be the homogeneous ideal generated by $\{\tilde{f}_\beta(\Lambda)\}_{\beta \in \Gamma}$. If $Z(J) = \{(0,\ldots,0)\}$, then $Z(I)$ is a zero-dimensional variety.*

Proof. Since $Z(J) = \{(0,\ldots,0)\}$, the Nullstellansatz implies that $\sqrt{J}$, the radical of J, satisfies

$$\sqrt{J} = (\Lambda_1,\ldots,\Lambda_N).$$

Thus, for each $i \in \{1,2,\ldots,N\}$, there exists an $m_i \in \mathbf{N}$ such that $\Lambda_i^{m_i} \in J$. The sequence $\{\Lambda_1^{m_1},\ldots,\Lambda_N^{m_N}\}$ forms a regular sequence (in any order). Let

us denote $\tilde{h}_i(\Lambda) = \Lambda_i^{m_i}$ and write

$$\tilde{h}_i(\Lambda) = \sum_{\beta \in \Gamma} \tilde{A}_{i,\beta}(\Lambda) \tilde{f}_\beta(\Lambda)$$

where $\tilde{A}_{i,\beta}(\Lambda)$ is homogeneous of weight equal to $m_i w_i - w(\tilde{f}_\beta)$.

Define

$$h_i(\Lambda) = \sum_{\beta \in \Gamma} \tilde{A}_{i,\beta}(\Lambda) f_\beta(\Lambda).$$

Clearly the leading form of $h_i(\Lambda)$ is $\tilde{h}_i(\Lambda)$ so that

$$h_i(\Lambda) = \tilde{h}_i(\Lambda) + \hat{h}_i(\Lambda)$$

with $w(\hat{h}_i(\Lambda)) < m_i w_i$. We assert that $\{h_1(\Lambda), \ldots, h_N(\Lambda)\}$ is a regular sequence (in any order). Suppose

$$\sum_{i=1}^{k} \xi_i(\Lambda) h_i(\Lambda) = 0.$$

Let w be the weight of the highest weighted terms that appear on the left-side (before cancellation). We write

$$\xi_i(\Lambda) = \tilde{\xi}_i(\Lambda) + \hat{\xi}_i(\Lambda)$$

where $w(\xi_i) = w(\tilde{\xi}_i) = w - m_i w_i > w(\hat{\xi}_i)$. We claim there exists a skew-symmetric set $\alpha_{i,j}(\Lambda) \in R$ such that

$$\xi_i(\Lambda) = \sum_{j=1}^{k} \alpha_{ij}(\Lambda) h_j(\Lambda)$$

for each i, $1 \le i \le k$. We proceed by induction on w. Since

$$\sum_{i=1}^{k} \tilde{\xi}_i \tilde{h}_i = 0,$$

there is a skew-symmetric set of homogeneous polynomials $\{\tilde{\eta}_{ij}\}_{i \le i, j \le k} \subseteq R$ such that

$$\tilde{\xi}_i(\Lambda) = \sum_{j=1}^{k} \tilde{\eta}_{ij}(\Lambda) \tilde{h}_j(\Lambda),$$

$$w(\tilde{\eta}_{ij}(\Lambda)) = w - m_i w_i - m_j w_j.$$

We may rewrite

$$\tilde{\xi}_i(\Lambda) = \sum_{j=1}^{k} \tilde{\eta}_{ij} h_j - \sum_{j=1}^{k} \tilde{\eta}_{ij} \hat{h}_j$$

so that

$$0 = \sum_{i=1}^{k} \xi_i h_i = \sum_{i=1}^{k} \tilde{\xi}_i h_i + \sum_{i=1}^{k} \hat{\xi}_i h_i = \sum_{i=1}^{k} \left(\hat{\xi}_i - \sum_{j=1}^{k} \tilde{\eta}_{ij} \hat{h}_j \right) h_i.$$

If we write,

$$\zeta_i = \hat{\xi}_i - \sum_{j=1}^{k} \tilde{\eta}_{ij} \hat{h}_j,$$

then $w(\zeta_i) < w(\xi_i)$ so that in

$$\sum_{i=1}^{k} \zeta_i h_i = 0$$

the highest weight term appearing on the left is strictly less than w. By induction

$$\zeta_i = \hat{\xi}_i - \sum_{j=1}^{k} \tilde{\eta}_{ij} \hat{h}_j = \sum_{j=1}^{k} \gamma_{ij} h_j$$

so that

$$\hat{\xi}_i = \sum_{j=1}^{k} \gamma_{ij} h_j + \sum_{j=1}^{k} \tilde{\eta}_{ij} \hat{h}_j$$

and

$$\xi_i = \sum_{j=1}^{k} (\tilde{\eta}_{ij} + \gamma_{ij}) h_j$$

as required.

In a similar manner, we can show that

$$\big(h_1(\Lambda), \ldots, h_N(\Lambda) \big) \neq R.$$

For suppose a relation $1 = \sum_{i=1}^{N} \alpha_i h_i$ exists with $\alpha_i \in R$ and $M = \max_{1 \leq i \leq N} \{ w(\alpha_i) + m_i w_i \}$. As before we write $\alpha_i = \tilde{\alpha}_i + \hat{\alpha}_i$ with $w(\alpha_i) = w(\tilde{\alpha}_i) > w(\hat{\alpha}_i)$ and $\tilde{\alpha}_i$ homogeneous. We may assume the α_i's chosen so that M is minimal. Clearly $\sum_{i=1}^{N} \tilde{\alpha}_i \tilde{h}_i = 0$ where $\tilde{\alpha}_i(\Lambda)$ is the leading form of $\alpha_i(\Lambda)$. This implies the existence of a skew-symmetric set of homogeneous polynomials $\{ \tilde{\gamma}_{ij} \}_{i,j=1}^{N} \subseteq R$ satisfying

$$\tilde{\alpha}_i = \sum_{j=1}^{N} \tilde{\gamma}_{ij} \tilde{h}_j = \sum_{j=1}^{N} \tilde{\gamma}_{ij} h_j - \sum_{j=1}^{N} \tilde{\gamma}_{ij} \hat{h}_j,$$

$$w(\tilde{\gamma}_{ij}) = w(\alpha_i) - w(\tilde{h}_j).$$

Using this

$$1 = \sum_{i=1}^{N} \left(\hat{\alpha}_i - \sum_{j=1}^{N} \tilde{\gamma}_{ij} \hat{h}_j \right) h_i$$

and

$$\max_{1 \leq i \leq N} \left\{ w \left(\hat{\alpha}_i - \sum_{j=1}^{N} \tilde{\gamma}_{ij} \hat{h}_j \right) + m_i w_i \right\} < M.$$

This contradicts the choice of $\{ \alpha_i \}_{i=1}^{N} \subseteq R$.

The point of this is to establish that

$$\mathrm{depth}(I) \doteq N$$

so that, since we are in a Cohen-Macauley ring,

$$\mathrm{height}(I) = \mathrm{depth}(I) = N.$$

Since I is an ideal in a polynomial ring, this implies that dimension $(I) = 0$ which completes the proof of the theorem.

2. We now apply this algebraic result to differential equations. Let K be a field of characteristic zero.

(2.1) DEFINITION. Let $(N_1, \ldots, N_r)$ be an r-tuple of non-negative integers. Let F be a field extension of K. A polynomial $G(Y_1, \ldots, Y_r) \in K[Y_1, Y_2, \ldots, Y_r]$ is called $(N_1, \ldots, N_r)$-PF ("polynomial-free") over F provided that $G(y_1, \ldots, y_r) = 0$ with $y_i \in F[t]$, (t transcendental over F), $\deg_t(y_i) \leq N_i$, implies $y_i \in F$.

(2.2) Examples. (i) $Y_1^{m_1} Y_2^{m_2} \cdots Y_r^{m_r} = 1$; $m_i \geq 1$ is $(N_1, \ldots, N_r)$-PF over F for arbitrary $(N_1, \ldots, N_r) \in (\mathbf{Z}_{\geq 0})^r$, F/K.

(ii) $Y_1 = Y_2^{m_2} Y_3^{m_3} \cdots Y_r^{m_r}$; $m_i \geq 1$ is $(N_1, \ldots, N_r)$-PF over F provided $N_1 < \min(m_2, m_3, \ldots, m_r)$.

In both of these examples, the verification is a trivial observation concerning degrees. Other examples can easily be constructed.

(2.3) THEOREM. *Assume for each $i \in \{1, 2, \ldots, r\}$,*

$$y_i(t) = \sum_{j=1}^{\infty} \lambda_j^{(i)} t^j \in F[[t]]$$

is a formal power series solution of a differential equation $L_i y_i = 0$ where $L_i \in K[[t]][d/dt]$ is a linear differential operator of order $N_i + 1$ with an

ordinary point at $t = 0$. *Assume also that* $G(y_1, \ldots, y_r) = 0$ *where* $G(Y) \in$ $K[Y_1, \ldots, Y_r]$ *is* $(N_1, \ldots, N_r)$-*PF over* $\overline{K}$, *where* $\overline{K}$ *is the algebraic closure of* $K(\{\lambda_0^{(i)}\}_{i=1}^r)$. *Then* $\lambda_j^{(i)} \in \overline{K}$ *for all* i *and all* j.

Proof. Replacing K by $K(\{\lambda_0^{(i)}\}_{i=1}^r)$ if necessary, we may assume $\{\lambda_0^{(i)}\}_{i=1}^r \subseteq K$ and are fixed. Let $\{\Lambda_j^{(i)}\}_{1 \le i \le r; \, j \ge 1}$ be algebraically independent over $K(t)$. Let $Z_i = \lambda_0^{(i)} + \sum_{j=1}^\infty \Lambda_j^{(i)} t^j$. We view $\{\lambda_j^{(i)}\}_{1 \le i \le r; \, j \ge 1}$ as a solution of the system of polynomials

$$(2.4) \qquad \left\{ \tilde{f}_k\left(\{\Lambda_j^{(i)}\}_{j \ge 1; 1 \le i \le r}\right) \right\} \in \tilde{R} = K_0\left[\left\{\Lambda_j^{(i)}\right\}_{j \ge 1; 1 \le i \le r}\right]\Big\}_{k \ge 1}$$

where these polynomials are defined as follows

$$G(Z_1, \ldots, Z_r) = G\left(\lambda_0^{(i)}, \ldots, \lambda_0^{(r)}\right) + \sum_{k=1}^\infty \tilde{f}_k\left(\{\Lambda_j^{(i)}\}\right) t^k.$$

(Of course, $G(\lambda_0^{(1)}, \ldots, \lambda_0^{(r)}) = 0$). The hypothesis that $y_i(t)$ satisfies a linear differential equation of order $N_i + 1$ with coefficients in $K[[t]]$ and an ordinary point at $t = 0$ means the $\{\lambda_j^{(i)}\}_{j \ge 1}$ satisfy a recursion of the form

$$(2.5) \qquad \Lambda_{N_i + l}^{(i)} = \gamma_l^{(i)}\left(\lambda_0^{(i)}, \Lambda_1^{(i)}, \ldots, \Lambda_{N_i}^i\right); \qquad l \ge 1$$

where $\gamma_l^{(i)}$ is a linear form in $N_i + 1$ variables defined over K.

Substituting (2.5) into (2.4), $\lambda = (\{\lambda_j^{(i)}\} | 1 \le j \le N_i; 1 \le i \le r)$ satisfies an infinite system of polynomials in a finite number $N = \sum_{i=1}^r N_i$ variables:

$$(2.6) \quad F_k(\Lambda) \in R = K_0\left[\left\{\Lambda_j^{(i)}\right\} | 1 \le j \le N_i; 1 \le i \le r\right], \qquad k \ge 1.$$

In particular λ is a point on the algebraic variety $Z(I)$ where $I \subseteq R$ is the ideal generated by the $F_k(\Lambda)$, $k \ge 1$. It suffices then to show $Z(I)$ is zero dimensional.

We give R and $\tilde{R}$ a grading by assigning $w(\Lambda_j^{(i)}) = j$, for all i, $1 \le i \le r$, and all j, $j \ge 1$. Then $\tilde{f}_k$ is a homogeneous polynomial of weight k. Let $\bar{f}_k(\Lambda) \in R$ be the image of $F_k(\Lambda)$ under the ring-homomorphism $\varphi: \tilde{R} \to R$ defined by sending $\Lambda_j^{(i)} \to 0$, whenever $j \ge N_i + 1$ and $\Lambda_j^{(i)} \to \Lambda_j^{(i)}$ whenever $j \le N_i$. Then, since the substitution (2.5) involves replacing $\Lambda_{N_i + l}^{(i)}$ by a polynomial of strictly smaller weight, it follows that $\bar{f}_k(\Lambda) \in J$ where J is the homogeneous ideal generated by the leading forms $\tilde{F}_k(\Lambda)$ of the polynomials $F_k(\Lambda)$, (2.6).

Since

$$\varphi G(Z_1, \ldots, Z_r) = G(\varphi Z_1, \ldots, \varphi Z_r) = \sum_{k=1}^\infty \bar{f}_k(\Lambda) t^k,$$

the hypothesis that G is $(N_1, \ldots, N_r) - PF$ over $\overline{K}$, the algebraic closure of K, implies that

$$Z(\check{J}) = \{(0, \ldots, 0)\}$$

where $\check{J}$ is the ideal of R generated by the $\bar{f}_k(\Lambda)$. Clearly then $Z(J) = \{(0, \ldots, 0)\}$ and $Z(I)$ is zero-dimensional.

3. As in the case of Harris-Sibuya [3], [4], we utilize a homomorphism of differential rings to obtain our main result.

(3.1) THEOREM. *Let K be a differential field of characteristic zero. Let F/K be an extension of differential fields. Assume for each i, $1 \leq i \leq r$, $y_i \in F$ is annihilated by a linear differential operator $L_i \in K[D]$,*

$$L_i = \sum_{j=0}^{N_i+1} \alpha_{ij} D^j, \qquad \alpha_{ij} \in K,$$

order $L_i = N_i + 1$. Assume also that

$$y_1 = y_2^{m_2} y_3^{m_3} \cdots y_r^{m_r}$$

where $N_1 + 1 \leq \min\{m_2, \ldots, m_r\}$. Then Dy_i/y_i is algebraic over K, for each i, $1 \leq i \leq r$.

Remarks. (i) The result above includes both Harris-Sibuya theorems as special cases. In particular, the case $y_1 y_2 = 1$ is the case $r = 3$, $m_2 = m_3 = 1$, $y_1 = 1$.

(ii) We are indebted to M. Singer for pointing out that the case $y_1 y_2 = 1$ follows immediately from a result of S. Morrison which appears in [**1**, Theorem 3].

Proof of Theorem. The map

$$\theta \colon F \to F[[t]]$$

defined by

$$\theta(\alpha) = \sum_{l=0}^{\infty} \frac{D^l \alpha}{l!} t^l$$

has been studied by Robba [5]. It is an injective homomorphism of differential rings:

$$\theta(D\alpha) = \frac{d}{dt} \theta(\alpha).$$

Since y_i satisfies an $(N_i + 1)$st order linear differential equation defined over K, it follows easily that $\theta(y_i) \in F[[t]]$ is a power-series solution of an $(N_i + 1)$-st order linear differential equation with coefficients in $K[[t]]$ having an ordinary point at $t = 0$. Since $y_i \in F$ is a constant with respect to the derivation d/dt of $F[[t]]$ it follows that

$$u_i = \theta(y_i)/y_i = 1 + \sum_{l=1}^{\infty} \frac{D^l(y_i)}{l!\, y_i} t^l$$

is also a solution of an $(N_i + 1)$-st order linear differential equation with coefficients in $K[[t]]$ having an ordinary point at $t = 0$. In addition, the relation $y_1 = y_2^{m_2}, \ldots, y_r^{m_r}$ yields $\theta(y_1) = \theta(y_2)^{m_2} \cdots \theta(y_r)^{m_r}$ and therefore $u_1 = u_2^{m_2} \cdots u_r^{m_r}$. As a consequence of the previous section, the coefficients of the u_i are algebraic over K as required.

4. The method above has wider applications than those given above although in most cases the conclusions that can be immediately drawn are significantly weaker than the conclusions of Harris-Sibuya.

(4.1) THEOREM. *Let K be a differential field of characteristic zero. Let F/K be an extension of differential fields. Assume for each $i = 1, 2$, $y_i \in F$ is annihilated by a linear differential operator $L_i \in K[D]$ of order $N_i + 1$. Let $K_0 = $ constants of $K = \{ \alpha \in K | D(\alpha) = 0 \}$. Assume that $G(y_1, y_2) = 0$ where $G(Y_1, Y_2) \in K_0[Y_1, Y_2]$ is $(N_1, N_2) - PF$ over $\bar{K}$, the algebraic closure of K. Then y_i satisfies a first-order algebraic differential equation with coefficients in K.*

Proof. We proceed as in (3.1) working with $\theta(y_i) = \sum_{j=0}^{\infty} \lambda_j^{(i)} t^j$ (with $\lambda_j^{(i)} = D^j(y_i)/j!$) instead of $\theta(y_i)/y_i$. The conclusion in this case is that the coefficients of $\theta(y_i)$ (including $\lambda_1^{(i)} = Dy_i$) are algebraic over $K(\lambda_0^{(1)}, \lambda_0^{(2)}) = K(y_1, y_2)$. Since y_1 and y_2 are algebraically dependent, the desired conclusion follows.

It is clear that in the above treatment, the role played by the linear differential operator is to provide recursions (as in (2.5)) having the properties that they are polynomial in nature and they reduce the given weight (the weight being dictated by the algebraic relation satisfied by the solutions). In particular, no special use was made of the linearity of the recursion in (2.5).

(4.2) DEFINITION. Let $\mathscr{K}$ be a differential ring. Let $\mathscr{F}$ be a differential extension ring of $\mathscr{K}$. We say $y \in \mathscr{F}$ is weighted monic over $\mathscr{K}$ if y

satisfies an algebraic differential equation of the following type

$$y^{(n)} = H(y, \ldots, y^{(n-1)})$$

where $H(z_1, \ldots, z_n) \in \mathscr{K}[z_1, \ldots, z_n]$ and where the monomials $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ of H satisfy

$$\sum_{i=1}^{n} (i - 1)\alpha_i < n.$$

The importance for us of this definition is that when y is weighted monic over K then $\theta(y) = \sum_{k=0}^{\infty} \lambda_k t^k$ is a power series in t which is weighted monic over $K[[t]]$ and the recursion satisfied by the coefficients has the form

$$\lambda_{k+n} = Q_k(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$$

where $Q_k \in K[\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}]$, and if $\Lambda^\beta = \Lambda_0^{\beta_0} \Lambda_1^{\beta_1} \cdots \Lambda_n^{\beta_n}$ is a monomial of Q_k then

$$w(\Lambda_0^{\beta_0} \Lambda_1^{\beta_1}, \ldots, \Lambda_n^{\beta_n}) < k + n.$$

(4.3) REMARK. It follows from the above discussion that the hypothesis of (3.1) and (4.1) that y_t satisfies a linear differential equation over K can be replaced by the weaker hypothesis that the y_t are weighted monic over K.

Finally, it is not difficult to extend the above results to solutions of certain overdetermined systems of partial differential equations.

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