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ON INCLUSION RELATIONS FOR ABSOLUTE NÖRLUND SUMMABILITY

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Recently Das gives sufficient conditions for $(N, r_n) \subseteq (N, p_n)(N, q_n)$ or $(N, p_n)(N, q_n) \subseteq (N, r_n)$, and for $|N, P_n| \sim |(N, p_n)(C, 1)|$. The purpose of this paper is to give sufficient conditions for $|N, r_n| \subseteq |(N, p_n)(N, q_n)|$ or $|(N, p_n)(N, q_n)| \subseteq |N, r_n|$. The results obtained here are also absolute summability analogues of Das' theorems.

1. Let $\{p_n\}$ and $\{q_n\}$ be real or complex sequences such that $P_n = \sum_{k=0}^n p_k \neq 0$ and $Q_n = \sum_{k=0}^n q_k \neq 0$. A sequence $\{s_n\}$ is said to be summable (N, p_n) to s, if $t_n^p = \sum_{k=0}^n p_{n-k}s_k/P_n \to s(n \to \infty)$, and summable $(N, p_n)(N, q_n)$ to s, if $t_n^{p,q} = \sum_{k=0}^n p_{n-k}t_k^q/P_n \to s(n \to \infty)$. It is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if $\sum|t_n^p - t_{n+1}^p| < \infty$.

Given two summability methods A and B, we write $A \subseteq B$ if each sequence summable A is summable B. If each includes the other, we write $A \sim B$.

We define the sequence $\{r_n\}$ by $r_n = \sum_{k=0}^n p_{n-k}q_k$ and define the sequence $\{c_n\}$ formally by $1/\sum_{n=0}^\infty p_n x^n = \sum_{n=0}^\infty c_n x^n$. We write $\{p_n\} \in \mathfrak{M}$ if $p_n > 0$, $p_{n+1}/p_n \leq p_{n+2}/p_{n+1} \leq 1$, and also write, for any sequence $\{f_n\}$, $f_n^{(1)} = \sum_{k=0}^n f_k$, $f_n^{(2)} = \sum_{k=0}^n f_k^{(1)}$. And K denotes an absolute constant, not necessarily the same at each occurrence.

On inclusion relations between two summability methods Das gives the following theorems.

THEOREM A [**1**, *Theorem* 2]. *If* $\{p_n\} \in \mathfrak{M}$ *and* $\{q_n\}$ *is positive, then* $(N, r_n) \subseteq (N, p_n)(N, q_n)$.

THEOREM B [**1**, *Theorem* 5]. *If* $\{p_n\} \in \mathfrak{M}$ *and* $\{q_n\}$ *is positive and* $(n+1)q_n = O(Q_n)$, *then* $(N, p_n)(N, q_n) \subseteq (N, r_n)$.

THEOREM C [**2**, *Theorem* 5]. *If* $\{p_n\} \in \mathfrak{M}$, *then* $|N, P_n| \sim |(N, p_n)(C, 1)|$.

The purpose of this paper is to prove the following theorems.

THEOREM 1. *If* $\{p_n\} \in \mathfrak{M}$ *and if* $\{q_n\}$ *is positive and nonincreasing, then* $|N, r_n| \subseteq |(N, p_n)(N, q_n)|$.

This is an absolute summability analogue of Theorem A.

THEOREM 2. *If* $\{p_n\} \in \mathfrak{M}$ *and if* $\{q_n\}$ *is positive and nonincreasing and if* $R_n = \sum_{k=0}^{n} r_k \to \infty (n \to \infty)$, *then* $|(N, p_n)(N, q_n)| \subseteq |N, r_n|$.

This is an absolute summability analogue of Theorem B. Combining Theorem 1 and Theorem 2, we have the following

THEOREM 3. *Under the assumptions of Theorem* 2, *the relation* $|(N, p_n)(N, q_n)| \sim |N, r_n|$ *holds.*

In this Theorem, if we put $q_n = 1$, then we obtain Theorem C.

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2. We require the following lemmas.

LEMMA 1. *Let* $y_n = \sum_{\nu=0}^{n} a_{n\nu} x_\nu$. *If*

$$\sum_{n=\rho}^{\infty} \left| \sum_{\nu=\rho}^{\infty} (a_{n\nu} - a_{n-1,\nu}) \right| \le c < \infty \quad \textit{for all } \rho,$$

then $\sum_{n=0}^{\infty} |\Delta y_n| < \infty$ *whenever* $\sum_{n=0}^{\infty} |\Delta x_n| < \infty$.

This is due to F. M. Mears ([3, p. 595]).

LEMMA 2. *Let* $\{p_n\} \in \mathfrak{M}$. *Then*

$$\sum_{\rho=0}^{r} P_\rho \sum_{n=r+1}^{\infty} |c_{n-\rho}| \le r + 1, \tag{1}$$

$$\left\{ c_n^{(1)} \right\} \textit{ is nonnegative and nonincreasing and} \tag{2}$$

$$c_n^{(2)} p_n \le 1. \tag{3}$$

This is Lemmas 3 and 4 in [2].

LEMMA 3. *If* $\{p_n\}$ *and* $\{q_n\}$ *are nonnegative, then*

$$P_n^{(1)} \leq K(n+1)P_n \quad \text{and} \tag{4}$$

$$R_n \leq P_n Q_n. \tag{5}$$

Further, if $\{p_n\}$ *and* $\{q_n\}$ *are nonincreasing, then*

$$(n+1)P_n \leq KP_n^{(1)} \quad \text{and} \tag{6}$$

$$R_n \geq KP_n Q_n. \tag{7}$$

Proof. The inequalities (4) and (6) are Lemma 5 in [2]. The inequality (5) is easily established. So we shall prove the inequality (7). Since the sequence $\{P_n/(n+1)\}$ is nonincreasing, and $KQ_n^{(1)} \geq (n+1)Q_n$,

$$\begin{aligned} R_n &= P_0 q_n + P_1 q_{n-1} + \cdots + P_n q_0 \\ &= P_0 q_n + 2\frac{P_1}{2} q_{n-1} + \cdots + (n+1)\frac{P_n}{n+1} q_0 \\ &\geq \frac{P_n}{n+1}(q_n + 2q_{n-1} + \cdots + (n+1)q_0) \\ &= P_n Q_n^{(1)}/(n+1) \geq P_n Q_n/K. \end{aligned}$$

LEMMA 4. *If* $\{p_n\} \in \mathfrak{M}$ *and if* $\{q_n\}$ *is positive and nonincreasing, then*

$$0 \leq \sum_{\rho=\mu}^{\nu} p_{n-\rho} c_{\rho-\mu} \leq p_{n-\mu} c_{\nu-\mu}^{(1)} \qquad (\mu \leq \nu \leq n), \tag{8}$$

$$0 \leq \sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu) p_{n-\nu} c_{\nu-\mu} \leq q_\mu, \tag{9}$$

$$\frac{1}{Q_n} \sum_{\nu=\mu}^{n-1} \frac{Q_n - Q_\nu}{Q_\nu} p_{n-\nu} c_{\nu-\mu} \leq p_{n-\mu} \sum_{\nu=\mu}^{n-1} \frac{q_{\nu+1} c_{\nu-\mu}^{(1)}}{Q_\nu Q_{\nu+1}} \tag{10}$$

and uniformly in $\nu \leq \rho$,

$$\sum_{n=\rho+1}^{\infty} \frac{(Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu})}{Q_n P_{n-1}} = O\left(\frac{1}{\rho+1}\right). \tag{11}$$

Proof. The inequality (8) is Lemma 6(11) in [2].

The inequality (9); Using Abel's transformation, from (3) and (8), we have

$$\sum_{\nu=\mu}^{n-1}(Q_n - Q_\nu)p_{n-\nu}c_{\nu-\mu} = \sum_{\nu=\mu}^{n-1} q_{\nu+1}\sum_{\rho=\mu}^{\nu} p_{n-\rho}c_{\rho-\mu}$$

$$\le \sum_{\nu=\mu}^{n-1} q_{\nu+1}p_{n-\mu}c_{\nu-\mu}^{(1)} \le q_\mu p_{n-\mu}\sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)}$$

$$= q_\mu p_{n-\mu}c_{n-\mu}^{(2)} \le q_\mu .$$

The inequality (10); Using Abel's transformation, from (8), we get

$$\frac{1}{Q_n}\sum_{\nu=\mu}^{n-1}\frac{Q_n - Q_\nu}{Q_\nu}p_{n-\nu}c_{\nu-\mu}$$

$$= \frac{1}{Q_n}\sum_{\nu=\mu}^{n-2}\left(\frac{Q_n - Q_\nu}{Q_\nu} - \frac{Q_n - Q_{\nu+1}}{Q_{\nu+1}}\right)\sum_{r=\mu}^{\nu} p_{n-r}c_{r-\mu}$$

$$+ \frac{q_n}{Q_{n-1}Q_n}\sum_{r=\mu}^{n-1} p_{n-r}c_{r-\mu}$$

$$= \sum_{\nu=\mu}^{n-1}\frac{q_{\nu+1}}{Q_\nu Q_{\nu+1}}\sum_{r=\mu}^{\nu} p_{n-r}c_{r-\mu} \le p_{n-\mu}\sum_{\nu=\mu}^{n-1}\frac{q_{\nu+1}c_{\nu-\mu}^{(1)}}{Q_\nu Q_{\nu+1}}.$$

The inequality (11); Since $\{q_n\}$ is nonincreasing, we have

$$\frac{Q_n}{Q_\nu} = 1 + \frac{Q_n - Q_\nu}{Q_\nu} \le 1 + \frac{(n-\nu)q_\nu}{\nu q_\nu} = \frac{n}{\nu}.$$

Hence, $(Q_n - Q_\nu)/Q_n \le (n-\nu)/n$. Therefore using Das' Lemma 7 in [**2**], we obtain the inequality (11).

LEMMA 6. *If* $\{p_n\}$ *is positive and nonincreasing, then uniformly in* $0 \le \mu \le \nu$,

$$\sum_{n=\nu}^{\infty}\frac{p_n p_{n-\mu}}{P_n P_{n-1}} = O\left(\frac{1}{\nu+1}\right). \tag{12}$$

This is Lemma 8 in [**2**].

3. Proof of Theorem 1. Let us write

$$t_n^r = \frac{1}{R_n}\sum_{\nu=0}^{n} r_{n-\nu}s_\nu \quad\text{and}\quad t_n^{p,q} = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}t_\nu^q .$$

Then, following Das' [**1**, pp. 32–33], we have

$$t_n^{p,q} = \sum_{\mu=0}^{n} \lambda_{n\mu} t_\mu^r,$$

where

$$\lambda_{n\mu} = \begin{cases} \dfrac{R_\mu}{P_n} \displaystyle\sum_{\nu=\mu}^{n} \dfrac{p_{n-\nu} c_{\nu-\mu}}{Q_\nu} & (\mu \le n) \\ 0 & (\mu > n). \end{cases}$$

By Lemma 1, it is sufficient to show that

$$J_\rho = \sum_{n=\rho}^{\infty} \left| \sum_{\mu=\rho}^{n} (\lambda_{n\mu} - \lambda_{n-1,\mu}) \right| = O(1) \qquad (\rho = 0, 1, 2, \dots).$$

Noting that

$$\sum_{\nu=\mu}^{n} p_{n-\nu} c_{\nu-\mu} = \begin{cases} 1 & (n = \mu) \\ 0 & (n > \mu), \end{cases} \tag{13}$$

for $n > \mu$, we get

$$\lambda_{n\mu} = \frac{R_\mu}{P_n} \sum_{\nu=\mu}^{n-1} \frac{Q_n - Q_\nu}{Q_n Q_\nu} p_{n-\nu} c_{\nu-\mu},$$

and for $n > \mu + 1$,

$$\lambda_{n-1,\mu} = \frac{R_\mu}{P_{n-1}} \sum_{\nu=\mu}^{n-1} \frac{Q_n - Q_\nu}{Q_n Q_\nu} p_{n-\nu-1} c_{\nu-\mu}.$$

Also it is easily seen that $\sum_{\mu=0}^{n} \lambda_{n\mu} = 1$. Hence, for $n > \rho$,

$$\begin{aligned} \sum_{\mu=\rho}^{n} (\lambda_{n\mu} - \lambda_{n-1,\mu}) &= \sum_{\mu=0}^{\rho-1} (\lambda_{n-1,\mu} - \lambda_{n\mu}) \\ &= \sum_{\mu=0}^{\rho-1} R_\mu \sum_{\nu=\mu}^{n-1} \frac{Q_n - Q_\nu}{Q_n Q_\nu} \left(\frac{p_{n-\nu-1}}{P_{n-1}} - \frac{p_{n-\nu}}{P_n} \right) c_{\nu-\mu}. \end{aligned}$$

Thus

$$\begin{aligned} J_\rho &= |\lambda_{\rho\rho}| + \sum_{n=\rho+1}^{\infty} \left| \sum_{\mu=0}^{\rho-1} (\lambda_{n-1,\mu} - \lambda_{n\mu}) \right| \\ &\le |\lambda_{\rho\rho}| + \sum_{n=\rho+1}^{\infty} \frac{p_n}{Q_n P_n P_{n-1}} \left| \sum_{\mu=0}^{\rho-1} R_\mu \sum_{\nu=\mu}^{n-1} p_{n-\nu} c_{\nu-\mu} \left(\frac{Q_n - Q_\nu}{Q_\nu} \right) \right| \\ &\quad + \sum_{n=\rho+1}^{\infty} \frac{1}{Q_n P_{n-1}} \left| \sum_{\mu=0}^{\rho-1} R_\mu \sum_{\nu=\mu}^{n-1} \frac{Q_n - Q_\nu}{Q_\nu} (p_{n-\nu-1} - p_{n-\nu}) c_{\nu-\mu} \right| \\ &= J_\rho^{(1)} + J_\rho^{(2)} + J_\rho^{(3)}, \quad \text{say.} \end{aligned}$$

From (5),

$$J_\rho^{(1)} = |\lambda_{\rho\rho}| = \frac{R_\rho p_0 c_0}{P_\rho Q_\rho} \leq 1.$$

By Lemma 4(10),

$$J_\rho^{(2)} \leq \sum_{n=\rho+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\mu=0}^{\rho-1} R_\mu p_{n-\mu} \sum_{\nu=\mu}^{n-1} \frac{q_{\nu+1} c_{\nu-\mu}^{(1)}}{Q_\nu Q_{\nu+1}}$$

$$= \sum_{\mu=0}^{\rho-1} R_\mu \sum_{n=\rho+1}^{\infty} \frac{p_n p_{n-\mu}}{P_n P_{n-1}} \left(\sum_{\nu=\mu}^{\rho-1} + \sum_{\nu=\rho}^{n-1} \right) \frac{q_{\nu+1} c_{\nu-\mu}^{(1)}}{Q_\nu Q_{\nu+1}}$$

$$= J_{\rho 1}^{(2)} + J_{\rho 2}^{(2)}, \quad \text{say.}$$

Using the identity

$$\sum_{\mu=0}^{\nu} P_\mu c_{\nu-\mu}^{(1)} = \nu + 1,$$

(5), (12) and the monotonicity of $\{p_n\}$, $\{q_n\}$ and $\{Q_n\}$, we have

$$J_{\rho 1}^{(2)} \leq \sum_{n=\rho+1}^{\infty} \frac{p_n p_{n-\rho}}{P_n P_{n-1}} \sum_{\nu=0}^{\rho-1} \frac{q_{\nu+1}}{Q_\nu Q_{\nu+1}} \sum_{\mu=0}^{\nu} R_\mu c_{\nu-\mu}^{(1)}$$

$$\leq \sum_{n=\rho+1}^{\infty} \frac{p_n p_{n-\rho}}{P_n P_{n-1}} \sum_{\nu=0}^{\rho-1} \frac{q_{\nu+1} Q_\nu}{Q_\nu Q_{\nu+1}} \sum_{\mu=0}^{\nu} P_\mu c_{\nu-\mu}^{(1)}$$

$$= \sum_{n=\rho+1}^{\infty} \frac{p_n p_{n-\rho}}{P_n P_{n-1}} \sum_{\nu=0}^{\rho-1} \frac{(\nu+1) q_{\nu+1}}{Q_{\nu+1}}$$

$$= O(\rho+1) \sum_{n=\rho+1}^{\infty} \frac{p_n p_{n-\rho}}{P_n P_{n-1}} = O(1).$$

Using (2), (5), (12) and (13), since $\{q_n\}$ and $\{Q_n\}$ are monotone, we get

$$J_{\rho 2}^{(2)} = \sum_{\mu=0}^{\rho-1} R_\mu \sum_{\nu=\rho}^{\infty} \frac{q_{\nu+1} c_{\nu-\mu}^{(1)}}{Q_\nu Q_{\nu+1}} \sum_{n=\nu+1}^{\infty} \frac{p_n p_{n-\mu}}{P_n P_{n-1}}$$

$$\leq K \sum_{\mu=0}^{\rho-1} Q_\mu P_\mu \sum_{\nu=\rho}^{\infty} \frac{q_{\nu+1} c_{\nu-\mu}^{(1)}}{Q_\nu Q_{\nu+1} (\nu+1)}$$

$$\leq K \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\rho}^{\infty} \frac{q_{\nu+1} c_{\nu-\mu}^{(1)}}{Q_{\nu+1} (\nu+1)}$$

$$\leq K \sum_{\mu=0}^{\rho-1} P_\mu c_{\rho-\mu}^{(1)} \sum_{\nu=\rho}^{\infty} \frac{1}{(\nu+1)^2} = O(1).$$

Next,

$$J_\rho^{(3)} \leq \sum_{n=\rho+1}^{\infty} \frac{1}{Q_n P_{n-1}} \left| \sum_{\mu=0}^{\rho-1} R_\mu \sum_{\nu=\mu}^{\rho-1} \frac{Q_n - Q_\nu}{Q_\nu} (p_{n-\nu-1} - p_{n-\nu}) c_{\nu-\mu} \right|$$

$$+ \sum_{\mu=0}^{\rho-1} R_\mu \sum_{n=\rho+1}^{\infty} \frac{1}{Q_n P_{n-1}} \left| \sum_{\nu=\rho}^{n-1} \frac{Q_n - Q_\nu}{Q_\nu} (p_{n-\nu-1} - p_{n-\nu}) c_{\nu-\mu} \right|$$

$$= J_{\rho 1}^{(3)} + J_{\rho 2}^{(3)}, \quad \text{say.}$$

By Lemma 4(11), we obtain

$$J_{\rho 1}^{(3)} = \sum_{n=\rho+1}^{\infty} \frac{1}{Q_n P_{n-1}} \left| \sum_{\nu=0}^{\rho-1} \frac{Q_n - Q_\nu}{Q_\nu} (p_{n-\nu-1} - p_{n-\nu}) \sum_{\mu=0}^{\nu} R_\mu c_{\nu-\mu} \right|$$

$$= \sum_{n=\rho+1}^{\infty} \frac{1}{Q_n P_{n-1}} \sum_{\nu=0}^{\rho-1} (Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu})$$

$$= \sum_{\nu=0}^{\rho-1} \sum_{n=\rho+1}^{\infty} \frac{(Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu})}{Q_n P_{n-1}} = O(1).$$

Next, using (5), we get

$$J_{\rho 2}^{(3)} \leq \sum_{\mu=0}^{\rho-1} R_\mu \sum_{n=\rho+1}^{\infty} \frac{1}{Q_n P_{n-1}} \sum_{\nu=\rho}^{n-1} \frac{Q_n - Q_\nu}{Q_\nu} (p_{n-\nu-1} - p_{n-\nu}) |c_{\nu-\mu}|$$

$$= \sum_{\mu=0}^{\rho-1} R_\mu \sum_{\nu=\rho}^{\infty} \frac{|c_{\nu-\mu}|}{Q_\nu} \sum_{n=\nu+1}^{\infty} \frac{(Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu})}{Q_n P_{n-1}}$$

$$\leq K \sum_{\mu=0}^{\rho-1} R_\mu \sum_{\nu=\rho}^{\infty} \frac{|c_{\nu-\mu}|}{Q_\nu (\nu + 1)}$$

$$= O\left(\frac{1}{\rho + 1} \right) \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\rho}^{\infty} |c_{\nu-\mu}| = O(1),$$

by Lemma 4(11) and Lemma 2(1).

This completes the proof of Theorem 1.

4. Proof of Theorem 2. First, we have, following Das' [**1**, p. 37],

$$t_n^r = \frac{1}{R_n} \sum_{\mu=0}^{n} P_\mu \left(\sum_{\nu=\mu}^{n} p_{n-\nu} Q_\nu c_{\nu-\mu} \right) t_\mu^{p,q} = \sum_{\mu=0}^{n} \alpha_{n\mu} t_\mu^{p,q},$$

where

$$\alpha_{n\mu} = \begin{cases} \dfrac{P_\mu}{R_n} \displaystyle\sum_{\nu=\mu}^{n} p_{n-\nu} Q_\nu c_{\nu-\mu} & (\mu \le n) \\ 0 & (\mu > n). \end{cases}$$

By Lemma 1, it is sufficient to show that

$$J_\rho = \sum_{n=\rho}^{\infty} \left| \sum_{\mu=\rho}^{n} (\alpha_{n\mu} - \alpha_{n-1,\mu}) \right| = O(1) \qquad (\rho = 0, 1, 2, \ldots).$$

By (13), we get for $n > \mu$,

$$\alpha_{n\mu} = -\frac{P_\mu}{R_n} \sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu) p_{n-\nu} c_{\nu-\mu}$$

and for $n > \mu + 1$,

$$\alpha_{n-1,\mu} = -\frac{P_\mu}{R_{n-1}} \sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu) p_{n-\nu-1} c_{\nu-\mu}.$$

Hence, for $n > \rho$,

$$\begin{aligned} \sum_{\mu=\rho}^{n} (\alpha_{n\mu} - \alpha_{n-1,\mu}) &= \sum_{\mu=0}^{\rho-1} (\alpha_{n-1,\mu} - \alpha_{n\mu}) \\ &= \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu) \left(\frac{p_{n-\nu}}{R_n} - \frac{p_{n-\nu-1}}{R_{n-1}} \right) c_{\nu-\mu}. \end{aligned}$$

Thus,

$$\begin{aligned} J_\rho \le |\alpha_{\rho\rho}| &+ \sum_{n=\rho+1}^{\infty} \frac{1}{R_{n-1}} \left| \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu}) c_{\nu-\mu} \right| \\ &+ \sum_{n=\rho+1}^{\infty} \frac{r_n}{R_n R_{n-1}} \left| \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu) p_{n-\nu} c_{\nu-\mu} \right| \\ &= J_\rho^{(1)} + J_\rho^{(2)} + J_\rho^{(3)}, \quad \text{say.} \end{aligned}$$

Using (7),

$$J_\rho^{(1)} = |\alpha_{\rho\rho}| = \frac{P_\rho}{R_\rho} p_0 Q_\rho c_0 \le K.$$

By Lemma 4(9), we have

$$J_\rho^{(3)} \le \sum_{n=\rho+1}^{\infty} \frac{r_n}{R_n R_{n-1}} \sum_{\mu=0}^{\rho-1} P_\mu q_\mu$$

$$\le P_\rho Q_\rho \sum_{n=\rho+1}^{\infty} \left(\frac{1}{R_{n-1}} - \frac{1}{R_n} \right) \le \frac{P_\rho Q_\rho}{R_\rho} = O(1).$$

Next,

$$J_\rho^{(2)} \le \sum_{n=\rho+1}^{\infty} \frac{1}{R_{n-1}} \left| \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\mu}^{\rho-1} (Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu}) c_{\nu-\mu} \right|$$

$$+ \sum_{n=\rho+1}^{\infty} \frac{1}{R_{n-1}} \left| \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\rho}^{n-1} (Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu}) c_{\nu-\mu} \right|$$

$$= J_{\rho 1}^{(2)} + J_{\rho 2}^{(2)}, \quad \text{say.}$$

Since $\sum_{\mu=0}^{\nu} P_\mu c_{\nu-\mu} = 1$, using (7) and (11), we get

$$J_{\rho 1}^{(2)} = \sum_{n=\rho+1}^{\infty} \frac{1}{R_{n-1}} \left| \sum_{\nu=0}^{\rho-1} (Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu}) \sum_{\mu=0}^{\nu} P_\mu c_{\nu-\mu} \right|$$

$$= \sum_{n=\rho+1}^{\infty} \frac{1}{R_{n-1}} \sum_{\nu=0}^{\rho-1} (Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu})$$

$$\le K \sum_{\nu=0}^{\rho-1} \sum_{n=\rho+1}^{\infty} \frac{(Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu})}{Q_n P_{n-1}} = O(1).$$

Lastly, using (1), (7) and (11), we obtain

$$J_{\rho 2}^{(2)} \le \sum_{n=\rho+1}^{\infty} \frac{1}{R_{n-1}} \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\rho}^{n-1} (Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu}) |c_{\nu-\mu}|$$

$$= \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\rho}^{\infty} |c_{\nu-\mu}| \sum_{n=\nu+1}^{\infty} \frac{(Q_n - Q_\nu)(p_{n-\nu-1} - p_{n-\nu})}{R_{n-1}}$$

$$= O\left(\frac{1}{\rho+1} \right) \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\rho}^{\infty} |c_{\nu-\mu}| = O(1).$$

This completes the proof of Theorem 2.

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