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# TANGENTS TO A MULTIPLE PLANE CURVE

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**The limiting behavior of the tangents and the flexes are computed as a reduced plane curve degenerates into a multiple plane curve.**

**0. Introduction.** In this paper, we consider the degeneration of a reduced irreducible plane curve to a multiple plane curve. We study the associated degeneration of tangent lines by viewing a line as a linear imbedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^2$ and studying deformations of this imbedding. We compute the limiting behavior of the dual curve and the flexes. A similar computation yields the limiting behavior of the bitangents; this will appear later in a separate paper. The main result is stated as Proposition (2.1).

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**1. The dual of a multiple curve.** Let $C \subset \mathbf{P}^2_{\mathbf{C}}$ be a smooth curve of degree $d$. $C^* \subset \mathbf{P}^{2*}$ will denote the dual curve of tangents to $C$.

Let $n$ be a positive integer, $n \geq 2$. Let

$$G^n + tF = 0 \tag{1.1}$$

be a generic pencil of plane curves, with $\deg G = d$, $\deg F = nd$. We will freely abuse notation by using the same letter to denote a polynomial or its zero locus. Here, generic means that $G$, $F$ are smooth, and meet transversely at their $nd^2$ points of intersection, the base points of the pencil. $G^*$ is assumed to have only nodes and cusps as singularities. The pencil (1.1) will be denoted by $C_t$. Let $C_0^* = \lim_{t\to 0} C_t^*$. The goal of this section is to prove the following.

(1.2) PROPOSITION. *$C_0^*$ is the union of $G^*$ with multiplicity $n$, together with the $nd^2$ pencils of lines through the base points, each pencil having multiplicity $(n-1)$.*

REMARKS. (1) Proposition (1.2) is quite elementary. It is not much more difficult than the case $n = 2$, $d = 1$ implicitly worked out in [4]. The value in this method of proof lies purely in its expository value as a prelude to §2.

(2) By a standard formula for plane curves ([2], for example) $\deg C_t^* = nd(nd-1)$ for $t \neq 0$, while $\deg C_0^* = nd(d-1) + (n-1)nd^2 = nd(nd-1)$.

The techniques used are a variant of the techniques of [3], which were inspired by the work of Clemens. Given a line $L \subset \mathbf{P}^2$, we look for a family of lines $L_s$ with $L_0 = L$ and $L_s$ tangent to $C_t$ with $t = s^r$ for a positive integer $r$. Then $L$ would correspond to a general point of a multiplicity $r$ component of $C_0^*$ with cyclic local monodromy.

We choose an isomorphism $\alpha\colon \mathbf{P}^1 \to L$ given by three homogeneous linear forms $\alpha = (\alpha_0(u,v), \alpha_1(u,v), \alpha_2(u,v))$, where $(u,v)$ are homogeneous coordinates on $\mathbf{P}^1$. We single out $(1,0) \in \mathbf{P}^1$ as the candidate for a point of tangency of $L$ with $C_0$. We look for an extension of $\alpha$ to $\alpha(s)$, holomorphic in $s$ for $|s| < \varepsilon$, with $\alpha(0) = \alpha$, and satisfying

$$(1.3) \qquad (G^n + s^r F) \circ \alpha(s) \equiv 0 \; (v^2) \quad \text{for } |s| < \varepsilon.$$

We attempt to solve (1.3) by power series in $s$. We show that this is possible when either $L$ is tangent to $G$ or when $L$ passes through a base point. In the former case, for general $L$, we must take $r = n$, while in the latter case, we take $r = n - 1$. By consideration of degrees, i.e. Remark (2), no other components are present, proving Proposition (1.2).

We now fix some more notation. Let $P_k$ denote the vector space of homogeneous forms of degree $k$ on $\mathbf{P}^1$. There is a linear map

$$(1.4) \qquad \Phi_G\colon P_1^3 \to P_d, \; \Phi_G(\sigma_0, \sigma_1, \sigma_2) = \sum_{j=0}^{2} \sigma_j \left( \frac{\partial G}{\partial X_j} \circ \alpha \right)$$

and for each integer $k \geq 0$, the related map

$$(1.5) \qquad \Phi_G^{(k)}\colon P_1^3 \overset{\Phi_G}{\to} P_d \to P_d/(v^{k+1}).$$

(1.6) LEMMA. *For any $L$, $\Phi_G^{(1)}$ is surjective (hence also $\Phi_G^{(0)}$).*

*Proof*. Since $G$ is smooth, we may change coordinates so $\psi = \partial G/\partial X_0 \circ \alpha \not\equiv 0$ (v), so that $\psi$ is a unit in the graded ring $R_1 = \bigoplus_j P_j/(v^2)$. Thus any $Q \in P_d/(v^2)$ can be divided by $\psi$ (mod $v^2$) to yield $\sigma \in P_1$; then $\Phi_G^{(1)}(\sigma, 0, 0) = Q$. □

We introduce some more notation to facilitate higher order computations. Let

$$\alpha^{(r)} = \left( \left. \frac{d^r \alpha_i}{ds^r} \right|_{s=0} \right)_{i=0,1,2}, \quad G_{ij}\alpha^{(r)}\alpha^{(s)} = \sum_{i,j} \left( \frac{\partial^2 G}{\partial X_i \partial X_j} \right) \alpha_i^{(r)} \alpha_j^{(s)}.$$

We also note that homogeneous polynomials of degree $j$ in $(u, v)$ can be viewed as polynomials of degree $\leq j$ in $v$; we will hence usually view $P_j/(v^{k+1}) \subset \mathbf{C}[v]/(v^{k+1})$, and speak of constant terms, linear terms, etc. We also freely divide truncated polynomials.

We start by specializing to the case $n = 2$ to fix ideas.

(1.7) PROPOSITION. (1.2) *is true for* $n = 2$.

*Proof.* We set $n = 2$, $r = 1$ (so that $s = t$) in (1.3), and let $t = 0$ to obtain

$$G^2 \equiv 0\ (v^2) \tag{1.8}$$

where we have abused notation by viewing $G$ as a form on $\mathbf{P}^1$ via $\alpha$. This gives

$$G \equiv 0\ (v). \tag{1.9}$$

We continue by differentiating (1.3) with respect to $t$ and setting $t = 0$.

$$2G\Phi_G\alpha' + F \equiv 0\ (v^2) \tag{1.10}$$

Using (1.9), (1.10) forces $F = 0\ (v)$, i.e.

(1.11) $L$ passes through a base point.

To show that the pencil containing $L$ indeed has multiplicity 1 in $C_0^*$, we may take $L$ general, and so assume $G$ is not tangent to $L \simeq \mathbf{P}^1$ at $(1, 0)$. We then obtain from (1.10)

$$\Phi_G^{(0)}\alpha' = -F/2G. \tag{1.12}$$

and Lemma 1.6 implies that we can solve (1.12) for $\alpha'$. Thus the pencils through the base points deform to first order; these pencils are the only candidates for a multiplicity 1 component of $C_0^*$.

For the second order obstruction, we take the second derivative of (1.3) with respect to $t$ and set $t = 0$ to obtain

$$2G\Phi_G\alpha'' + 2GG_{ij}\alpha'\alpha' + 2\left(\Phi_G\alpha'\right)^2 + 2\Phi_F\alpha' \equiv 0\ (v^2). \tag{1.13}$$

In order for (1.13) to have a solution for $\alpha''$, we must require that

$$2\left(\Phi_G\alpha'\right)^2 + 2\Phi_F(\alpha') \equiv 0\ (v). \tag{1.14}$$

This can be accomplished by the following lemma.

(1.15) LEMMA. $\Phi_F^{(0)}|_{\ker\Phi_G^{(1)}}\colon \ker\Phi_G^{(1)} \to P_{nd}/(v)$ *is surjective.*

*Proof*. Since $\dim P_{nd}/(v) = 1$, the lemma can fail to hold only if $\ker\Phi_G^{(1)} \subset \ker\Phi_F^{(0)}$. But since $F$ and $G$ intersect transversally, we can change coordinates in $\mathbf{P}^2$ so that $X_0 = 0$ is tangent to $F$, and $X_1 = 0$ is tangent to $G$ at $\alpha(1,0)$. So we may assume that, in the affine coordinate $v$ near $(1,0) \in \mathbf{P}^1$, $(\partial G/\partial X_0)(\alpha(v)) \equiv av\ (v^2)$, $(\partial G/\partial X_1)(\alpha(v)) \equiv b\ (v)$, where $b \neq 0$. Then $(-bu, av, 0) \in \ker\Phi_G^{(1)} - \ker\Phi_F^{(0)}$.

Now we can replace $\alpha'$ with $\alpha' - \tilde{\alpha}$, where $\tilde{\alpha} \in \ker\Phi_G^{(1)}$ and $\Phi_F^{(0)}\tilde{\alpha} \equiv (\Phi_G\alpha')^2\ (v)$, by the lemma. Then (1.12) still holds, but now the left-hand side of (1.13) is divisible by $G$, since (1.14) now holds. After dividing (1.13) by $G$, we can now solve for $\alpha''$ by using lemma (1.6) again.

For simplicity, we introduce the symbol $Q_j$ to stand for any expression involving $\alpha$ only through $\alpha', \alpha'', \ldots, \alpha^{(j)}$. The higher order obstructions are now handled by the following easily established lemma.

(1.16) LEMMA. *For $n \geq 2$, the nth obstruction to* (1.3) *is*

$$\left.\frac{d^n}{dt^n}(G^2 + tF)\right|_{t=0} \equiv 2G\Phi_G\alpha^{(n)} + n\Phi_F\alpha^{(n-1)} + 2n\Phi_G\alpha'\Phi_G\alpha^{(n-1)} + GQ_{n-1} + Q_{n-2} \equiv 0\ (v^2). \quad \square$$

We inductively complete the power series solution of (1.3). We suppose that we have solved for $\alpha', \ldots, \alpha^{(n-1)}$. Then using Lemma 1.15, we modify $\alpha^{(n-1)}$ so that (1.16) becomes divisible by $G$. After dividing by $G$, we use Lemma (1.6) once more to solve for $\alpha^{(n)}$.

This procedure gives a formal power series solution of (1.3). By Artin's theorem [**1**] there is a holomorphic solution of (1.3) for $|t| < \varepsilon$. Thus, the pencils through the base points are each multiplicity 1 components of $C_0^*$.

REMARK. The solution for $\alpha^{(n)}$ is far from unique; in fact, the computation above shows that the ambiguity lies in $\ker\Phi_G^{(0)} \cap \ker\Phi_F^{(0)}$, a 4-dimensional vector space. Let $B \subset \mathrm{GL}(2)$ denote the isotropy group of

(1, 0), so that $\dim B = 3$. This is the ambiguity arising by representing $L$ as $(\mathbf{P}^1, (1, 0))$. The difference between 4 and 3 reflects that a curve (the pencil) is deforming.

The other component $2G^*$ is found by letting $n = 2$, $t = s^2$ in (1.3). The order zero obstruction again leads to (1.9), which holds for a tangent to $G$ (in fact, $G \equiv 0\ (v^2)$). The first order obstruction is

$$(1.17) \qquad 2G\Phi_G\alpha' \equiv 0\ (v^2)$$

which is again automatic, and puts no restrictions on $\alpha'$.

The second order obstruction is

$$(1.18) \qquad 2G\Phi_G\alpha'' + 2\left(\Phi_G\alpha'\right)^2 + 2GG_{ij}\alpha'\alpha' + 2F \equiv 0\ (v^2).$$

This equation can be solved for $\alpha''$ provided that

$$(1.19) \qquad \left(\Phi_G\alpha'\right)^2 \equiv -F\ (v^2).$$

We can assume that $L$ does not pass through a base point (i.e. $F \not\equiv 0\ (v)$). After taking a square root, Lemma (1.6) ensures that we can find such an $\alpha'$, and (1.18) imposes no conditions on $\alpha''$. For the higher order obstructions we need an easy lemma.

(1.20) LEMMA. *For $n \geq 2$, the nth obstruction is*

$$\frac{d^n}{ds^n}\left(G^2 + s^2F\right)\Big|_{s=0} \equiv 2G\Phi_G\alpha^{(n)} + 2n\Phi_G\alpha'\Phi_G\alpha^{(n-1)} + GQ_{n-1} + Q_{n-2} \equiv 0\ (v^2). \qquad \square$$

Since $\Phi_G\alpha'$ is a unit in $P_d/(v^2)$, we can choose $\alpha^{(n-1)}$ to ensure that there is no $n$th obstruction, using Lemma (1.6). Thus, there is a formal power series solution of (1.3) with $t = s^2$, and Artin's Theorem finishes the proof of Proposition (1.7). $\square$

REMARK. In the case of tangents, the ambiguity lies in $\ker \Phi_G^{(1)}$, which is as before a 4-dimensional vector space.

*Proof of Proposition* (1.2). We start by letting $t = s^{n-1}$ in (1.3), and attempt to deform a pencil through a base point. There are clearly no obstructions through order $n - 2$. The $(n - 1)$st obstruction is (since

$G^j \equiv 0\,(v^2)$ for $j \geq 2$)

$$n!G(\Phi_G\alpha')^{n-1} + (n-1)!F \equiv 0\,(v^2). \tag{1.21}$$

We may assume $L$ is not tangent to $G$ or $F$; then we can solve (1.21) for $\alpha'$.

The $n$th order obstruction is seen to be

$$\begin{aligned} n!\binom{n}{2}G(\Phi_G\alpha')^{n-2}\Phi_G\alpha'' + n!(\Phi_G\alpha')^n + n!\Phi_F\alpha' \\ + GQ_n \equiv 0\,(v^2). \end{aligned} \tag{1.22}$$

We now can use Lemma (1.15) to modify $\alpha'$ so that (1.22) is consistent. After dividing (1.22) by $G$, and noting that $\Phi_G^{(1)}\alpha'$ is a unit, we can then solve for $\alpha''$.

For the higher order obstructions, we note that for $r \geq n+1$

$$\begin{aligned} &\frac{d^r}{ds^r}(G^n + s^{n-1}F)\bigg|_{s=0} \\ &\equiv n(n-1)\frac{r!}{(r-n+2)!}G(\Phi_G\alpha')^{n-2}\Phi_G\alpha^{(r-n+2)} \\ &\quad + n\frac{r!}{(r-n+2)!}(\Phi_G\alpha')^{n-1}\Phi_G\alpha^{(r-n+1)} \\ &\quad + \frac{r!}{(r-n+1)!}\Phi_F\alpha^{(r-n+1)} + GQ_{r-n+1} + Q_{r-n} \equiv 0\,(v^2). \end{aligned} \tag{1.23}$$

As before, we can use Lemma (1.15) inductively to modify $\alpha^{(r-n+1)}$ to ensure the consistency of (1.23), then solve for $\alpha^{(r-n+2)}$ using Lemma (1.6). Finally, Artin's Theorem shows that a pencil through a base point is a multiplicity $(n-1)$ component of $C_0^*$.

Turning next to the tangents to $G$ (so that $G \equiv 0\,(v^2)$), we let $t = s^n$ in (1.3). There are clearly no obstructions through order $(n-1)$.

The $n$th order obstruction yields

$$n!(\Phi_G\alpha')^n + n!F \equiv 0\,(v^2). \tag{1.24}$$

Assuming that $L$ does not pass through a base point, we can solve (1.24) for $\alpha'$.

For the higher order obstructions, we note that for $r \geq n+1$

$$\begin{aligned} \frac{d^r}{ds^r}(G^n + s^nF)\bigg|_{s=0} &\equiv n\frac{r!}{(r-n+1)!}(\Phi_G\alpha')^{n-1}\Phi_G\alpha^{(r-n+1)} \\ &\quad + Q_{r-n} \equiv 0\,(v^2). \end{aligned} \tag{1.25}$$

As $\Phi_G \alpha'$ is a unit, we can solve for $\alpha^{(r-n+1)}$. Artin's Theorem completes the proof. □

**2. Flexes on a multiple curve.** In the situation of §1, we look at the limiting behavior of the flexes of $C_t$.

(2.1) PROPOSITION. *The flexes of $C_t$ degenerate to the flexes of $G$, the tangents to $F$ at a base point, and the tangents to $G$ at a base point, with multiplicities $n$, $n-2$, $2n-1$ respectively.*

*Proof*. By a standard formula for plane curves [2], $C_t$ has $3nd(nd-2)$ flexes; $g$ has $3d(d-2)$ flexes and $nd^2$ base points. Also $3nd(nd-2) = n(3d(d-2)) + (n-2)nd^2 + (2n-1)nd^2$. So as in §1, it suffices to construct deformations of the claimed limits with the indicated multiplicities.

We now need to solve

$$(2.2) \qquad (G^n + s^r F) \circ \alpha(s) \equiv 0\ (v^3) \quad \text{for } |s| < \varepsilon$$

for $r = n$ in the case of a flex of $G$, for $r = n-2$ in the case of a tangent to $F$ at a base point, and for $r = 2n-1$ in the case of a tangent to $G$ at a base point.

We first check the flexes of $G$, starting with a lemma.

(2.3) LEMMA. *If $L$ is an ordinary inflectional tangent to $G$, then $\Phi_G^{(2)}$ is surjective.*

*Proof*. We can change coordinates so that $L$ has equation $X_1 = 0$, and $G$ has an equation of the form $X_1 f + X_0^3 g$, where $f(0,0,1)$, $g(0,0,1) \neq 0$. We may as well let $\alpha\colon \mathbf{P}^1 \to L$ be $\alpha(u,v) = (v,0,u)$. Then, using subscript notation for partial derivatives, we find that

$$G_0 \circ \alpha = 3v^2 g + v^3 g_0 \qquad G_1 \circ \alpha = f + v^3 g_1$$

and so $\Phi_G^{(2)}$ is surjective by inspection. □

The proof of the case of flexes is now completed by mimicking the computation of the component $nG^*$ of §1, using Lemma (2.3) in place of Lemma (1.6).

We turn next to the case of a tangent to $F$ at a base point, i.e. $G \equiv 0\ (v)$, $F \equiv 0\ (v^2)$, $t = s^{n-2}$.

There are clearly no obstructions through order $n-3$. For the order $n-2$ obstruction, we note that

$$(2.4)\qquad \left.\frac{d^{n-2}}{ds^{n-2}}(G^n+s^{n-2}F)\right|_{s=0}\equiv\frac{n!}{2}G^2(\Phi_G\alpha')^{n-2}+(n-2)!F\equiv 0\ (v^3)$$

and since $F$, $G$ have order exactly 2, 1 respectively as polynomials in $v$, $F/G^2$ is a unit, so we can extract an $(n-2)$ root and solve for $\Phi_G^{(0)}\alpha'$ in (2.4).

The higher order obstructions are given by

$$(2.5)\qquad \left.\frac{d^k}{ds^k k}(G^n+s^{n-2}F)\right|_{s=0}$$
$$=\frac{n(n-1)(n-2)}{2}\frac{k!}{(k+3-n)!}G^2(\Phi_G\alpha')^{n-3}\Phi_G\alpha^{(k+3-n)}$$
$$+G^2Q_{k+2-n}+n(n-1)\frac{k!}{(k+2-n)!}G(\Phi_G\alpha')^{n-2}\Phi_G\alpha^{(k+2-n)}$$
$$+\frac{k!}{(k+2-n)!}\Phi_F\alpha^{(k+2-n)}+Q_{k+1-n}\ (v^3).$$

This case is finished by a couple of lemmas.

(2.6) LEMMA. *$\Phi_F^{(1)}|_{\ker\Phi_G^{(0)}}$: $\ker\Phi_G^{(0)}\to P_{nd}/(v^2)$ is surjective.*

*Proof*. Lemma 1.15 says that $\dim\ker\Phi_G^{(1)}\cap\ker\Phi_F^{(0)}=3$. Reversing the roles of $F$ and $G$ yields the lemma. □

(2.7) LEMMA. *After solving for the kth obstruction, we have $\infty^3$ solutions for $\alpha^1,\dots,\alpha^{(k+2-n)}$, and $\Phi_G^{(0)}\alpha^{(k+3-n)}$ is determined.*

*Proof*. Inductively, we equate the linear plus constant term of (2.5) to $0\ (v^2)$, using Lemma (2.6) to modify $\alpha^{(k+2-n)}$. $\Phi_G^{(0)}\alpha^{(k+3-n)}$ is now found by Lemma (1.6). □

An application of Artin's Theorem completes the proof of the case of a tangent to $F$ at a base point.

Finally, we turn to a tangent to $G$ at a base point, i.e. $G\equiv 0\ (v^2)$, $F\equiv 0\ (v)$, $t=s^{2n-1}$.

There are clearly no obstructions through order $n-2$. The order $n-1$ obstruction is

$$(2.8)\qquad \left.\frac{d^{n-1}}{ds^{n-1}}(G^n+s^{2n-1}F)\right|_{s=0}\equiv n!G(\Phi_G\alpha')^{n-1}\equiv 0\ (v^3)$$

which forces

$$\Phi_G^{(0)}\alpha' = 0. \tag{2.9}$$

We change notation slightly, putting $G^{(j)} = d^j(G \circ \alpha(s))/ds^j|_{s=0}$, noting that $G^{(j)} = \Phi_G \alpha^{(j)} + Q_{j-1}$. With the additional information (2.9), we now see that there are no obstructions through order $2n - 3$. The order $2n - 2$ obstruction is given by

$$\begin{aligned} &\frac{d^{2n-2}}{ds^{2n-2}}(G^n + s^{2n-1}F)\bigg|_{s=0} \\ &= \frac{n(2n-2)!}{2^{n-1}} G(G'')^{n-1} + \frac{n(n-1)(2n-2)!}{2^{n-1}}(G')^2(G'')^{n-2} \\ &\equiv 0\ (v^3). \end{aligned} \tag{2.10}$$

This leads to

$$G'' \equiv -(n-1)(G')^2/G\ (v). \tag{2.11}$$

The order $2n - 1$ obstruction is

$$\begin{aligned} &\frac{d^{2n-1}}{ds^{2n-1}}(G^n + s^{2n-1}F)\bigg|_{s=0} \\ &= \frac{n(n-1)(2n-1)!}{6 \cdot 2^{n-2}} G(G'')^{n-2} G''' \\ &\quad + \frac{n(n-1)(n-2)(2n-1)!}{6 \cdot 2^{n-2}} (G')^2 (G'')^{n-3} G''' \\ &\quad + \frac{n(2n-1)!}{2^{n-1}} G'(G'')^{n-1} + (2n-1)!F \\ &\equiv 0\ (v^3) \end{aligned} \tag{2.12}$$

looking at the linear term, and using (2.11), we find

$$(G')^{2n-1}/G^{n-1} \equiv (-1)^n \frac{2^{n-1}}{n(n-1)^{n-1}} F\ (v^2). \tag{2.13}$$

(2.12) implies that we can solve for $\Phi_G^{(1)}\alpha'$, and that $G''$ is a unit, using (2.11) again.

Turning to the quadratic term of (2.12), we see that we must solve for $G'''(v)$, or equivalently, for $\Phi_G^{(0)}\alpha'''$. This is possible exactly when the expression multiplying $G'''$ in (2.12) is divisible by $v^2$, but not by $v^3$. But this expression is a multiple of

$$(G'')^{n-3}\left[GG'' + (n-2)(G')^2\right] \tag{2.14}$$

which satisfies the indicated requirement, by (2.11) and the fact that $G''$ is a unit.

Notice that $\Phi_G^{(0)}\alpha''$ depends only on $\alpha'$, while $\Phi_G^{(0)}\alpha'''$ depends on $\Phi_G^{(1)}\alpha''$ and $\alpha'$; however, it is a non-trivial linear expression in the linear term of $\Phi_G^{(1)}\alpha''$, as revealed by a examination of our solution of (2.12).

The higher order obstructions are given by

$$\begin{aligned}(2.15)\quad &\frac{d^k}{ds^k}(G^n + s^{2n-1}F)\Big|_{s=0} \\ &= \frac{n(n-1)k!}{2^{n-2}(k+4-2n)!}G(G'')^{n-2}G^{(k+4-2n)} \\ &+ \frac{n(n-1)(n-2)k!}{2^{n-2}(k+4-2n)!}(G')^2(G'')^{n-3}G^{(k+4-2n)} + GQ_{k+3-2n} \\ &+ (G')^2\tilde{Q}_{k+3-2n} + \frac{n(n-1)k!}{2^{n-2}(k+3-2n)!}G'(G'')^{n-2}G^{(k+3-2n)} \\ &+ G'Q_{k+2-2n} + \frac{nk!}{2^{n-1}(k+2-2n)!}(G'')^{n-1}G^{(k+2-2n)} \\ &+ Q_{k+1-2n} + \frac{k!}{(k+1-2n)!}\Phi_F(\alpha^{(k+1-2n)}) \\ &\equiv 0\ (v^3).\end{aligned}$$

Equation (2.15) can be solved inductively.

(2.16) LEMMA. *After solving for the kth obstruction, we have* $\infty^3$ *solutions for* $\alpha', \ldots, \alpha^{(k+1-2n)}$, *we have found* $\Phi_G^{(1)}\alpha^{(k+2-2n)}$, *and we have found* $\Phi_G^{(0)}\alpha^{(k+4-2n)}$. *This last depends non-trivially and linearly on the linear term of* $\Phi_G\alpha^{(k+3-2n)}$, *and on terms of lower order.*

*Proof*. By induction. We start by examining the constant term of (2.15). We observe that the constant term of $G^{(k+2-2n)}$ depends on $\Phi_G^{(1)}\alpha^{(k+1-2n)}$ and lower derivatives of $\alpha$. Also we note that the expression $Q_{k+1-2n}$ in (2.15) depends on $\Phi_G^{(0)}\alpha^{(k+1-2n)}$ and lower derivatives of $\alpha$. So Lemma (1.15) applies to allow for the modification of $\alpha^{(k+1-2n)}$ as before.

Next, we consider the linear term of (2.15). We observe that the constant term of $Q_{k+2-2n}$ depends on $\Phi_G^{(0)}\alpha^{(k+2-2n)}$ and lower derivatives of $\alpha$, while inductively the constant term of $G^{(k+3-2n)}$ depends non-trivially and linearly on the linear term of $\Phi_G\alpha^{(k+2-2n)}$ and on lower order terms, so that after equating the linear term of (2.15) to 0, we can first solve for the linear term of $\Phi_G\alpha^{(k+2-2n)}$ (hence for $\Phi_G^{(1)}\alpha^{(k+2-2n)}$, as we inductively know the constant term). Lemma (1.16) allows us to solve for $\alpha^{(k+2-2n)}$.

Finally, we turn to the quadratic term. Exactly as in the order $2n - 1$ obstruction, we see that $\Phi_G^{(0)}\alpha^{(k+4-2n)}$ is multiplied by a constant multiple of (2.14), which we have seen is divisible by $v^2$, but not by $v^3$. So we can solve for $\Phi_G^{(0)}\alpha^{(k+4-2n)}$, and apply Lemma (1.6). Note that the quadratic term of (2.15) involves $\alpha^{(k+3-2n)}$ only non-trivially and linearly through the linear term of $\Phi_G\alpha^{(k+3-2n)}$, completing the induction. □

An application of Artin's Theorem now finishes the case of tangents to $G$ through a base point, as well as the proof of Proposition (2.1). □

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