THE SYMBOLS OF AN ALGEBRA
OF PSEUDODIFFERENTIAL OPERATORS

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Atiyah and Singer constructed a Fréchet operator algebra $\mathscr{P}_\infty$ by closing a set of zero order pseudodifferential operators on a compact manifold X in the topology generated by all the norms of the spaces $L(H^s)$, s real. Each operator in $\mathscr{P}_\infty$ has a 'symbol', a function in $C(S^*X)$. Contradicting a statement of Atiyah and Singer and establishing the manifold analogue of a conjecture by H. O. Cordes, it will be shown that each function in $C(S^*X)$ is the symbol of an operator in $\mathscr{P}_\infty$.

0. Introduction The algebra $\mathscr{P}^0$ was defined by Atiyah and Singer [1] as a subalgebra of the zero order pseudodifferential operators on the compact manifold X: they required the operator symbol $p(x, \xi)$ to approach a limit along each ray in ξ-direction. $\mathscr{P}_s$ is the closure of $\mathscr{P}^0$ in $L(H^s)$, the space of bounded operators on the Sobolev space $H^s = H^s(X)$ for real s. $\mathscr{P}_\infty$ is obtained by closing $\mathscr{P}^0$ under the Fréchet topology induced by all the $L(H^s)$-norms. These definitions turn out to be equivalent to transferring the $\mathbf{R}^n$ concept of H. O. Cordes and E. Hermann [2], Ch. IV, [4], to the case of a compact manifold. $\mathscr{P}_s/K(H^s)$ is a commutative C^*-algebra with unit (here $K(H^s)$ denotes the ideal of compact operators). By the Gelfand-Naimark theorem it is therefore isometrically isomorphic to the space of continuous functions on a compact Hausdorff space, which is known to be homeomorphic to S^*X, the unit sphere in the cotangent bundle. The symbol of an operator A in $\mathscr{P}_s$ is defined as the function corresponding to $A + K(H^s)$. An interesting feature is that neither the symbols nor the symbol space depend on s.

Since $\mathscr{P}_\infty$ is contained in each $\mathscr{P}_s$, the symbols of its operators certainly form a subspace of $C(S^*X)$. In fact, for the somewhat different case $X = \mathbf{R}^n$, H. O. Cordes and D. A. Williams [6] proved that at least all those functions are symbols, which are C^∞ on the corresponding symbol space. Using a slightly modified version of Egorov's theorem and a technique developed by H. O. Cordes, Theorem 2.1 shows that for each function a in $C(S^*X)$ we can find an operator A in $\mathscr{P}_\infty$ with symbol a. This contradicts an assertion of Atiyah and Singer [1], p. 513. They stated without proof that the symbol map from $\mathscr{P}_\infty$ to $C(S^*X)$ is *not* surjective.

On the other hand, in the case $X = \mathbf{R}^n$, this result was conjectured by H. O. Cordes (for a slightly different algebra) and proven by Cordes and the author [5].

The symbol of an operator A in $\mathscr{P}_s$ governs the Fredholm properties of A. Reducing the order to zero does not result in a loss of generality. The case of elliptic (pseudo-) differential operators is easily reduced to studying operators in $\mathscr{P}_s$ or $\mathscr{P}_\infty$. Moreover, $\mathscr{P}_\infty$ shares one of the most important properties of C^*-algebras: if $A \in \mathscr{P}_\infty$ is invertible in $L(H^s)$, then its inverse is also in $\mathscr{P}_\infty$. This makes $\mathscr{P}_\infty$ a Ψ-subalgebra of $L(H^s)$ as defined by B. Gramsch [7].

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1. Definitions and preliminary results. Throughout this paper X denotes a compact n-dimensional manifold. We cover X by a finite set V_j, $j = 1, \ldots, N$ of coordinate neighborhoods which are mapped diffeomorphically onto open, relatively compact sets U_j in $\mathbf{R}^n$ by charts χ_j. We further choose a fixed partition of unity $\{\phi_j\}$ subordinate to $\{V_j\}$ and functions $\psi_j \in \mathscr{D}(V_j)$ with $\phi_j \cdot \psi_j = \phi_j$. We choose a Riemannian metric $g = (g_{jk})_{j,k=1,\ldots,n}$ on X and define $(g^{jk}) = (g_{jk})^{-1}$, where the matrices correspond to the choice of the coordinates. Unless stated otherwise, i.e. in 1.1, 1.2 or 1.3, all notions connected with pseudodifferential operators will be used as in M. Taylor's book [10].

1.1. DEFINITION. (a) Let $H^s(\mathbf{R}^n)$ be the usual Sobolev space of order s on $\mathbf{R}^n$ and $H^s = H^s(X)$ the Sobolev space on X defined via the fixed partition of unity and the coordinate maps. H^s is a Hilbert space.

(b) $L(H^s)$ is the set of all bounded linear operators on H^s. We will also write L_s. $K(H^s)$ or K_s denotes the ideal of compact operators on H^s.

(c) $CB^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ is the set of all C^∞ functions on $\mathbf{R}^n \times \mathbf{R}^n$ with all derivatives bounded.

(d) For real m let $S^m = S^m(\mathbf{R}^n \times \mathbf{R}^n)$ be the set of all C^∞ functions a on $\mathbf{R}^n \times \mathbf{R}^n$ with

$$\langle x^\alpha \rangle \left| D_x^\beta D_\xi^\gamma a(x, \xi) \right| \leq C_{\alpha\beta\gamma} \langle \xi \rangle^{m-|\gamma|}$$

for all x, ξ in $\mathbf{R}^n$ and all multiindices α, β, γ with a suitable constant $C_{\alpha\beta\gamma}$. Similarly, $S^m(U \times R^n)$ is defined for an open set U in $\mathbf{R}^n$: we

require a to have its x-support in U. Note that we used the notation $\langle x \rangle = (1 + |x|^2)^{1/2}$.

1.2. DEFINITION. (a) Let $a(x, \xi) \in CB^{\infty}(\mathbf{R}^n \times \mathbf{R}^n)$. We define the operator $\mathrm{Op}(a)$ on $\mathscr{D}(\mathbf{R}^n)$ by

$$(1.1) \qquad \mathrm{Op}(a)f(x) = (2\pi)^{-n/2} \int e^{ix\xi} a(x, \xi) \hat{f}(\xi) \, d\xi$$

where $\hat{f}$ denotes the Fourier transform of f. By a slight extension of a theorem due to Calderon and Vaillancourt (cf. [3], Thm. B_ρ, p. 117) this extends to a bounded linear operator on each $H^s(\mathbf{R}^n)$. We even have the estimate for its norm

$$\|\mathrm{Op}(a)\|_{s,s} \le c_s \max\left\{ \left| D_x^\alpha D_\xi^\beta a(x, \xi) \right| : |\alpha|, |\beta| \le n_s \right\},$$

where c_s, n_s depend on s but not on a.

(b) Similarly we may substitute a function $a \in S^m$ into (1.1). $\mathrm{Op}(a)$ will then define a bounded operator from $H^{s+m}(\mathbf{R}^n)$ to $H^s(\mathbf{R}^n)$ for any real s.

(c) Whenever we can represent an operator A in the form $A = \mathrm{Op}(a)$ with a function a in $CB^{\infty}(\mathbf{R}^n \times \mathbf{R}^n)$ or a in $S^m(\mathbf{R}^n \times \mathbf{R}^n)$, we will call a the operator symbol of A in order to avoid confusion with the "symbol" of A which will be defined in 1.4.

(d) This concept is transferred to the manifold case in the usual way. In particular, we will write $P \sim \mathrm{Op}(p)$ if

(i) $p = \{ p_j : p_j \in S^m(U_j \times \mathbf{R}^n) \}$ and

(ii) $\Psi_j P \Phi_j \sim M_{\psi_j \circ \chi_j^{-1}} \mathrm{Op}(p_j) M_{\phi_j \circ \chi_j^{-1}}$.

where Φ_j: $C^{\infty}(\mathbf{R}^n) \to \mathscr{D}(V_j)$ is defined by $(\Phi_j f)(x) = \phi_j(x) f(\chi_j(x))$ and Ψ_j: $C^{\infty}(X) \to \mathscr{D}(U_j)$ by $(\Psi_j f)(y) = \psi_j(\chi_j^{-1}(y)) f(\chi_j^{-1}(y))$. the " $\sim$ " means that the difference is a regularizing operator on $\mathbf{R}^n$.

1.3. PROPOSITION. (a) *On T^*X, the cotangent bundle of X, a norm is defined by*

$$|\xi| = \left(\sum g^{jk}(x) \xi_j \xi_k \right)^{1/2},$$

where $\xi \in T_x^* X$ has the representation $\xi = \sum \xi_j \, d\chi^j$, and χ stands for one of the coordinate maps χ_j.

(b) *The Laplace-Beltrami operator on X is given in local coordinates by* $\Delta = (1/v) \sum_{j,k} \partial/\partial x_j \, v g^{jk}(x) \partial/\partial x_k$, *with $v = (\det g_{jk}(x))^{1/2}$.*

(c) *The principal symbol of $-\Delta$ is $p(x, \xi) = \sum g^{jk}(x) \xi_j \xi_k$. Here we use the notion "principal symbol" in the sense that $(-\Delta) - \mathrm{Op}(p)$ is an*

*operator of order ≤ 1. It follows from the transformation law for pseudodif-
ferential operators on a manifold that each principal symbol may be consid-
ered as a function on the cotangent bundle T^*X.*

(d) *The operators Λ, M on X are given by*

$$\Lambda = (Q - \Delta)^{1/2}$$

$$M = (Q - \Delta)^{1/4}.$$

Q denotes the projection onto the constants.

(e) *By [9, §4, Thm. 2 and Thm. 3], we know that Λ and M are
invertible elliptic pseudodifferential operators of order 1 and $\frac{1}{2}$, resp. Their
principal symbols are*

$$\lambda_0(x, \xi) = \left(\sum g^{jk}(x)\xi_j\xi_k \right)^{1/2} = |\xi| \quad and$$

$$\mu_0(x, \xi) = \left(\sum g^{jk}(x)\xi_j\xi_k \right)^{1/4} = |\xi|^{1/2}.$$

(f) *Although λ_0 and μ_0 are not smooth for $\xi = 0$, we can still use them
as principal symbols: if we restrict λ_0, μ_0 to $\{|\xi| \geq c\}$ and then extend
them smoothly to $\{|\xi| < c\}$ by $\tilde{\lambda}$, $\tilde{\mu}$, then $\mathrm{Op}(\lambda_0) - \mathrm{Op}(\tilde{\lambda})$ and $\mathrm{Op}(\mu_0) -
\mathrm{Op}(\tilde{\mu})$ are regularizing.*

1.4. DEFINITION. (a) In [1, p. 509], Atiyah and Singer define the class
$\mathscr{P}^0$. It denotes the set of operators P in $L(H^0)$ with $P \sim \mathrm{Op}(p)$ for an
operator symbol $p = \{ p_j \colon p_j \in S^0(U_j \times \mathbf{R}^n) \}$, where each p_j has a limit

$$\sigma(p_j)(x, \xi) = \lim_{t \to \infty} p_j(x, t\xi), \qquad \xi \neq 0.$$

(b) By patching coordinate neighborhoods together, $\sigma(p)$ is well-de-
fined as a function on $T^*X - \{0\}$. It is called the symbol of P. We will
also write $\sigma(P)$. This is justified by the following lemma which states the
basic properties of these symbols.

1.5. LEMMA. *The notation is as in 1.4.*

(a) $\sigma(p_j) \in C^\infty(U_j \times (\mathbf{R}^n - \{0\}))$.

(b) *In fact, the assumptions in 1.4(a) imply that all derivatives
$D^\alpha\sigma(p_j)(x, \xi)$ exist on $U_j \times (\mathbf{R}^n - \{0\})$ and that $D^\alpha\sigma(p_j)(x, \xi) =
\lim_{t \to \infty} D^\alpha p_j(x, t\xi)$, $\xi \neq 0$.*

(c) *Convergence in (b) is uniform: given $\varepsilon > 0$ and a fixed pair
(x_0, ξ_0) in $U_j \times (\mathbf{R}^n - \{0\})$, there is a $t_0 > 0$ and a neighborhood U of
(x_0, ξ_0) such that*

$$\left| D^\alpha p_j(x, t\xi) - D^\alpha\sigma(p_j)(x, \xi) \right| < \varepsilon$$

for all (x, ξ) in U and $t \geq t_0$.

(d) $\sigma(p_j)$ is homogeneous of degree 0 in ξ.

(e) By patching together the coordinate neighborhoods with the partition of unity, $\sigma(p)$ is well-defined on $T^*X - \{0\}$.

(f) For A, B in $\mathcal{P}^0$, $t \in \mathbf{C}$, we have

$$\sigma(A + B) = \sigma(A) + \sigma(B)$$

$$\sigma(AB) = \sigma(A)\sigma(B)$$

$$\sigma(tA) = t\sigma(A)$$

$$\sigma(A^*) = \overline{\sigma(A)}.$$

In particular, all operators on the left hand side of these equations are in $\mathcal{P}^0$, and $\mathcal{P}^0$ is a $*$-closed subalgebra of L_0.

(g) If two operators in $\mathcal{P}^0$ have the same symbol, then their difference is compact.

Proof. (a) follows from (b).

(b) is proved by induction. To make things easier, we may assume $n = 1$ and show only

(i) $\partial_x p_j(x, t\xi)$ approaches a limit $q(x, \xi)$ as $t \to \infty$, $\xi \neq 0$, and so do all other derivatives with respect to x or ξ.

(ii) $q = \partial_x \sigma(p_j)$.

First suppose (i) did not hold in (x_0, ξ_0), i.e. $\partial_x p_j(x_0, t\xi)$ did not approach a limit as $t \to \infty$. By the usual simplifications we may assume that p_j is real-valued and that we have a sequence $t_1, t_2, \ldots, \to \infty$ such that $\partial_x p_j(x_0, t_k\xi_0) > 1$ for k even and < -1 for k odd. Since $\partial_{xx} p_j$ is bounded on $U_j \times \mathbf{R}^n$, there is a $c > 0$, such that the above inequalities also hold for all x with $|x - x_0| < c$. We conclude that

$$p_j(x_0 + c, t_k\xi_0) - p_j(x_0, t_k\xi_0) = c \int \partial_x p_j(x_0 + uc, t_k\xi_0) \, du,$$

which is $> c$ for k even and $< -c$ for k odd. Hence one of the sequences on the left hand side cannot converge as $k \to \infty$, contradicting our assumption. A similar argument now holds for all other derivatives.

To prove (ii), we first note that

$$\frac{\left(\sigma(p_j)(x + h, \xi) - \sigma(p_j)(x, \xi)\right)}{h} - q(x, \xi)$$

$$= h \lim_{h \to \infty} \int_0^1 \int_0^1 \partial_{xx} p_j(x + uvh, t\xi) \, du \, dv$$

$$= h \int_0^1 \int_0^1 \lim \partial_{xx} p_j(x + uvh, t\xi) \, du \, dv$$

by Lebesgue's theorem on dominated convergence, since $\partial_{xx} p_j$ is bounded and approaches a limit as $t \to \infty$. Thus we can take the limit $h \to 0$ on the left hand side, and it equals zero.

(c) We pick an arbitrary (x_0, ξ_0), $\xi_0 \neq 0$, let $\varepsilon > 0$ be given and write for the moment $\sigma*$ instead of $D^\alpha \sigma(p_j)$ and $p*$ instead of $D^\alpha p_j$. By (b), $\sigma*$ is continuous. This furnishes a neighborhood U_1 of (x_0, ξ_0) such that $|\sigma*(x, \xi) - \sigma*(x_0, \xi_0)| < \varepsilon$ for (x, ξ) in U_1. On the other hand,

$$\left| p*(x, t\xi) - p*(x_0, t\xi_0) \right|$$

$$\leq \int_0^1 \left\| \operatorname{grad}\left(p*(x_0 + u(x - x_0), t\xi_0 + u(\xi - \xi_0)) \right) \right\|$$

$$\times \left| (x - x_0, t(\xi - \xi_0)) \right| du$$

$$\leq \int_0^1 \left\langle t\xi_0 + ut(\xi - \xi_0) \right\rangle^{-1} \left| (x - x_0, t(\xi - \xi_0)) \right| du$$

$$= O\left(t^{-1} |x - x_0| + |\xi - \xi_0| \right) \quad \text{for all } t > 1, \quad x \in U_j, \xi \in \mathbf{R}^n.$$

In the second inequality we used the fact that $p*$ is in S^0 (even in $S^{-|\alpha|}$) and therefore satisfies an inequality of the form stated in 1.1(d). Hence we can find a neighborhood U_2 of (x_0, ξ_0) such that $|p*(x, t\xi) - p*(x_0, t\xi_0)| < \varepsilon$ for all $t \geq 1$, $(x, \xi) \in U_2$. Finally there is a $t_0 \geq 1$ with $|p*(x_0, t\xi_0) - \sigma*(x_0, \xi_0)| < \varepsilon$ for $t \geq t_0$. So, if we choose (x, ξ) in $U = U_1 \cap U_2$ and $t \geq t_0$ we obtain

$$\left| p*(x, t\xi) - \sigma*(x, \xi) \right|$$

$$\leq \left| p*(x, t\xi) - p*(x_0, t\xi_0) \right| + \left| p*(x_0, t\xi_0) - \sigma*(x_0, \xi_0) \right|$$

$$+ \left| \sigma*(x_0, \xi_0) - \sigma*(x, \xi) \right| < 3\varepsilon.$$

This proves (c).

(d) is obvious.

(e) follows from the transformation law for pseudodifferential operators under diffeomorphisms on $\mathbf{R}^n$, cf. [10, §II.5].

(f) is a consequence of the calculus of pseudodifferential operators, cf. [10, §II.4].

(g) By (f) we may assume that $P \in \mathscr{P}^0$ has symbol zero. We have to show that P is compact. Using the fact that $P \sim \operatorname{Op}(p)$, which is made up from a finite number of operators $\operatorname{Op}(p_j)$ on $\mathbf{R}^n$, we may confine ourselves to the case $P = \operatorname{Op}(p_j)$, $p_j \in S^0(U_j \times \mathbf{R}^n)$ with $\sigma(p_j) = 0$. Now choose a sequence of functions $(h_k)_k$ in $\mathscr{D}(\mathbf{R}^n)$ with $h_k(\xi) = 1$ for

$|\xi| \leq k$, and $|h_k^{(\alpha)}(\xi)| \leq 1$ for all $|\alpha| \leq n_0$, where n_0 is the constant in 1.2(a). We write

$$\mathrm{Op}(p_j) = \mathrm{Op}(h_k p_j) + \mathrm{Op}((1 - h_k)p_j).$$

The first operator on the right hand side is regularizing, since it has compact support in both, x and ξ. In particular, it is compact, since X is compact. As for the second operator, we first note that $(1 - h_k)p_j$ is in $CB^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ with

$$\sup\{|D^\alpha((1 - h_k)(\xi)p_j(x, \xi))| : |\alpha| \leq n_0, \, x \in U_j, \, \xi \in \mathbf{R}^n\}$$

$$= \sup\{|D^\alpha p_j(x, \xi)| : |\alpha| \leq n_0, \, x \in U_j, \, |\xi| \geq k\}.$$

This tends to zero as $k \to \infty$: The closure C of the set $\{(x, \xi) \in U_j \times \mathbf{R}^n : p(x, \xi) \neq 0, \, 1 \leq |\xi| \leq 2\}$ is compact. Thus it can be covered by finitely many of the neighborhoods constructed in (c). This gives us a $t_0 > 0$ such that $|D^\alpha p_j(x, t\xi)| < \varepsilon$ for all (x, ξ) in the set C, $t \geq t_0$, $|\alpha| \leq n_0$, or $|D^\alpha p_j(x, \xi)| < \varepsilon$ for all $x \in U_j$, $|\xi| \geq t_0$. Using the Calderon-Vaillancourt theorem, we see that the second operator on the right hand side tends to zero. Therefore, P is compact.

1.6. DEFINITION (cf. [1], p. 512). $\mathscr{P}_s$ denotes the closure of $\mathscr{P}^0$ in L_s, $\mathscr{P}_\infty$ the closure of $\mathscr{P}^0$ in the topology generated by the set of norms $\{\| \cdot \|_{s,s} : s \in \mathbf{R}\}$ of operators on all spaces H^s. It follows from Calderon's Interpolation Theorem (cf. [10], p. 22) that this is indeed a Fréchet topology.

1.7. REMARK. Let M be the operator defined in 1.3(d).
(a) For $P \in \mathscr{P}^0$, k an integer, we have $M^k P M^{-k} - P \in K_s$ for all s.
(b) $M^k \mathscr{P}_s M^{-k} = \mathscr{P}_{s-k/2}$, $k \in \mathbf{Z}$, $s \in \mathbf{R}$
   $M^k K_s M^{-k} = K_{s-k/2}$ $k \in \mathbf{Z}$, $s \in \mathbf{R}$.
(c) Let $P \in \mathscr{P}_0$. Then
   $P \in \mathscr{P}_\infty$ if and only if $M^k P M^{-k} - P \in K_0$ for all $k \in \mathbf{Z}$.
(d) Let $P \in \mathscr{P}_0$. Then
   $P \in \mathscr{P}_\infty$ if and only if $\mathrm{ad}^k M(P) M^{-k}$, $M^{-k} \mathrm{ad}^k M(P) \in K_0$ for $k \in \mathbf{N}$. Here $\mathrm{ad}^k M(P)$ denotes the k-fold commutator of P with M: $[M, \ldots, [M, P] \ldots]$.
(e) $\mathscr{P}_s$ contains K_s for all s.

Proof. (a) follows from the calculus of pseudodifferential operators: the difference is an operator of order ≤ -1 and hence compact.

(b) For each real k, M^k is an isomorphism of H^s onto $H^{s-k/2}$. This leads to an estimate of the norms of the operators involved on H^s and $H^{s-k/2}$. Since $\mathscr{P}_s$ is defined as the closure of $\mathscr{P}^0$ and K_s is the closure of the finite dimensional operators, this yields the desired result.

(c) The proof is a slight modification of the proof given in [2], Lemma IV 5.3 for the $\mathbf{R}^n$-case.

(d) follows from (c) by induction.

(e) cf. [1], p. 512.

1.8. REMARK. Similar as in [5], we might introduce the algebra $\mathscr{A}^0$ generated by

(i) The multiplication operators M_ϕ, $\phi \in C^\infty(X)$ and

(ii) the $n + 1$ operators $\sum \Psi_j \mathrm{Op}(\xi_k/\langle \xi \rangle) \Phi_j$, $\sum \Psi_j \mathrm{Op}(1/\langle \xi \rangle^2) \Phi_j$, $k = 1, \ldots, n$. The algebras $\mathscr{A}_s$ are the closure of $\mathscr{A}^0$ in L_s, $\mathscr{A}_\infty$ is the closure in the Fréchet topology. Using 1.5 and the Stone-Weierstrass theorem it can be shown that $\mathscr{A}_s = \mathscr{P}_s$, $\mathscr{A}_\infty = \mathscr{P}_\infty$.

1.9. THEOREM. *$\mathscr{P}_s/K_s$ is a commutative C^*-algebra with unit for each real s. $\mathscr{P}_s/K_s \simeq C(S^*X)$, the space of continuous functions on the unit sphere in the cotangent bundle.*

Proof. [1], p. 512 or [8].

1.10. REMARK. For A in $\mathscr{P}_s$, the function a associated with $A + K_s$ via the Gelfand isomorphism is called the symbol of A. For A in $\mathscr{P}^0$ it turns out that a is just the restriction of $\sigma(A)$ as defined in 1.4 to S^*X. Since we know that $\sigma(A)$ is homogeneous of degree 0, we may ignore this difference (cf. [1], [8])

1.11. DEFINITION. We consider the strictly hyperbolic time-independent pseudodifferential equation

(1.2a) $$\partial_t u = iMu$$

(1.2b) $$u(s, \cdot) = g \in H^k(X), \qquad k \text{ real.}$$

(a) By $S(t, s)$ we denote the operator on $H^k(X)$ taking $g = u(s, \cdot)$ to $u(t, \cdot)$ and by $S(t)$ the operator $S(t, 0)$.

(b) H is the Hamiltonian vector field on T^*X induced by the principal symbol μ_0 of M, homogeneously extended to zero, cf. 1.3(e), (f):

$$H = \left(\frac{\partial \mu_0}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial \mu_0}{\partial x_j} \frac{\partial}{\partial \xi_j} \right).$$

(c) F_t is the flow generated by H. For (x_0, ξ_0) in T^*X we shall also write $F_t(x_0, \xi_0) = (x_t(x_0, \xi_0), \xi_t(x_0, \xi_0)) = (x_t, \xi_t)$.

1.12. LEMMA. (a) *The solution to equations* (1.2) *exists and is uniquely determined in* $H^k(X)$. *The operator* $S(t)$ *is bounded on each* $H^k(X)$, $t, k \in \mathbf{R}$.

(b) $S(t + s, s) = S(t, 0) = S(t)$ *for all* s, t. *In particular,* $S(-t) = S(-t, 0) = S(0, t)$. $S(t)$ *is often written as* e^{iMt}.

(c) M *and* $S(t)$ *commute.*

(d) *The flow* F_t *exists for all* t, *and for each* $T > 0$ *there are constants* c, C *such that*

$$0 < c \le \langle x_t \rangle / \langle x_0 \rangle, \langle \xi_t \rangle / \langle \xi_0 \rangle \le C \quad \text{for } -T \le t \le T.$$

Proof. (a) Using the notation of Taylor ([10]), iM is symmetric hyperbolic in $\text{OPS}_{1,0}^1$ and $(iM + iM^*) \in \text{OPS}_{1,0}^0$. Applying Theorem [10, IV2.3] we get existence and uniqueness of the solution. Boundedness is stated in equation IV(2.4).

(b) $S(t, s)g = u(t, \cdot)$, where u satisfies equation (1.2), s, t are fixed. So $S(t, 0)g = u(t, \cdot)$, where $u(0, \cdot) = g$. Introduce $v(y, x) = u(y - s, x)$. Then $\partial_t v = \partial_t u(y - s, x) = iMu(y - s, x) = iMv$, and $v(s, x) = u(s - s, x) = g(x)$. Thus $S(t + s, s)g = v(t + s, \cdot) = u(t, \cdot) = S(t, 0)g$.

(c) Let $g \in C^\infty(X)$, u the solution of equation (1.2) with $s = 0$. Then $\partial_t u = iMu$, $u(0, \cdot) = g$. Hence $MS(t)g = Mu(t, \cdot)$. On the other hand, $\partial_t(Mu) = M\partial_t u$, since $u \in C^\infty(X)$, ∂_t and Δ commute, and M is a holomorphic function of Δ. This implies that $\partial_t(Mu) = M\partial_t u = iM(Mu)$, $Mu(0, \cdot) = Mg$. We see that $S(t)Mg = Mu(t, \cdot) = MS(t)g$. The density of $C^\infty(X)$ in each H^k then furnishes the desired result.

(d) For each coordinate neighborhood V_j, $T^*X|_{V_j}$ is mapped diffeomorphically to $U_j \times \mathbf{R}^n$. The flow F_t is the solution of the ODE

$$\dot{x} = \partial_\xi \mu_0, \ x(0) = x_0,$$

$$\dot{\xi} = -\partial_x \mu_0, \ \xi(0) = \xi_0.$$

It is known that the solution to this ODE exists for small $|t|$ and can be continued to the boundary of $\mathbf{R}^n \times U_j \times \mathbf{R}^n$. Since $|\partial_x \mu_0(x, \xi)| \le \text{const.} \langle \xi \rangle^{1/2}$ for all (x, ξ) in $U_j \times \mathbf{R}^n$, ξ_t cannot tend to infinity for finite t. Now $F_{t + t_0}(x_0, \xi_0) = F_t(x_{t_0}, \xi_{t_0})$ and μ_0 is invariantly defined on T^*X (under changes of coordinates). So we can switch to another coordinate neighborhood in case x_t approaches the boundary of U_j. We obtain a

solution for all $t \in \mathbf{R}$. As for the estimate, we first derive on $U_j \times \mathbf{R}^n$

$$\langle x_t \rangle^{\cdot} = x_t \dot{x}_t / \langle x_t \rangle = \partial_\xi \mu_0(x_t, \xi_t) x_t / \langle x_t \rangle = O(\langle x_t \rangle) \quad \text{and}$$

$$\langle \xi_t \rangle^{\cdot} = \xi_t \dot{\xi}_t / \langle \xi_t \rangle = -\partial_x \mu_0(x_t, \xi_t) \xi_t / \langle \xi_t \rangle = O(\langle \xi_t \rangle).$$

We may integrate this to obtain

$$\log(\langle x_t \rangle / \langle x_0 \rangle) = O(1),$$

$$\log(\langle \xi_t \rangle / \langle \xi_0 \rangle) = O(1).$$

with constants depending on the bounds for the derivatives of μ_0 and T. Hence we can keep the constants when switching from one coordinate patch to another and get the global estimate.

1.13. LEMMA. *Let $f \in C^\infty(X)$ be fixed. Then the function $H: \mathbf{R} \to C^\infty(X)$, defined by $H(t) = S(t)f$ is differentiable and $H'(t) = \partial_t S(t)f = iMS(t)f$, where convergence holds in C^∞-topology.*

Proof. This follows from the fact that $S(t)f = u(t, \cdot)$, where u satisfies equation (1.2) with $s = 0$, $g = f$, and that a solution is in $C^m(\mathbf{R}, H^{k-m})(X)$ for each choice of k, m, cf. [**10**, Ch. IV].

1.14. THEOREM. *Let $P \sim \text{Op}(p)$ be a pesudodifferential operator of order m on X. Then the operator $P(t) = S(-t)PS(t)$ is a pseudodifferential operator on X of order m with principal symbol $p_t(x_0, \xi_0) = p(F_t(x_0, \xi_0))$*

Proof. The proof is the same as the one given in [**10**, §VIII.1]: M. Taylor's assumptions on the operator symbol may be reduced to those the operator symbol of M satisfies.

1.15. LEMMA. *Let $t \in \mathbf{R}$, $(x_0, \xi_0) \in T^*X - \{0\}$ be fixed. If we have the solution*

$$F_t(x_0, \xi_0) = (x_t(x_0, \xi_0), \xi_t(x_0, \xi_0))$$

for small $|t|$, then

$$F_t(x_0, \rho\xi_0) = (x_0 + u(\rho^{-1/2}t), \rho(\xi_0 + v(\rho^{-1/2}t))),$$

where $u(t) = x_t - x_0$, $v(t) = \xi_t - \xi_0$, and ρ is sufficiently large.

Proof. For any pair of initial values (x_0, ξ_0), x_t and ξ_t are uniquely determined by solving the ODE stated in the proof of 1.12(d). Define $u(t)$ and $v(t)$ for small values of $|t|$ as in the statement (we only restrict t,

because we do not want the solution to leave the coordinate neighborhood). For given $\rho > 0$, define

$$\tilde{x}(t) = x_0 + u(\rho^{-1/2}t),$$

$$\tilde{\xi}(t) = \xi_0 + v(\rho^{-1/2}t),$$

which is possible for small $\rho^{-1/2}|t|$. Then $(\tilde{x}(t), \tilde{\xi}(t))$ solves the corresponding ODE with the initial value $(x_0, \rho\xi_0)$. By uniqueness, $\tilde{x}(t) = x_t(x_0, \rho\xi_0)$, $\tilde{\xi}(t) = \xi_t(x_0, \rho\xi_0)$.

2. Result and Proof.

2.1. THEOREM. *Given a in* $C(S^*X)$, *there is an operator A in* $\mathscr{P}_\infty$ *with* $\sigma(A) = a$.

The proof is broken down into a series of lemmata.

2.2. LEMMA. *For A in* $\mathscr{P}^0$
  (a) $A(t) = S(-t)AS(t) \in \mathscr{P}^0$, $\sigma(A(t)) = \sigma(A)$.
  (b) $t \mapsto A(t)$ *is differentiable from* **R** *into* $\mathscr{P}^0$ *equipped with the* L_0-*norm.* $\partial_t A(t) = i[A(t), M]$, $\sigma(\partial_t A(t)) = -i\{\mu_0, a\} = -iHa$.

Proof. Let $A \sim \text{Op}(a)$, t fixed. The principal symbol of $A(t)$ is $a \circ F_t$ by 1.14. Pick (x_0, ξ_0), $\xi_0 \neq 0$ in local coordinates, and choose any sequence $\rho_1 \leq \rho_2 \leq \cdots \to \infty$ with ρ_1 already so large that Lemma 1.15 is applicable.

Define $(x_k, \xi_k) = (x_t(x_0, \rho_k\xi_0), \xi_t(x_0, \rho_k\xi_0))$, $r_k = |\xi_k|$, $\theta_k = \xi_k/|\xi_k|$. By Lemma 1.15, $x_k \to x_0$, $r_k \to \infty$, $\theta_k \to \xi_0/|\xi_0|$. It follows that

$$|a \circ F_t(x_0, \rho_k\xi_0) - \sigma(a)(x_0, \xi_0/|\xi_0|)|$$

$$= |a(x_k, \xi_k) - \sigma(a)(x_0, \xi_0/|\xi_0|)|$$

$$= |a(x_k, r_k\theta_k) - \sigma(a)(x_0, \xi_0/|\xi_0|)| \to 0$$

by Lemma 1.5(c). Since we know that $\sigma(a)(x_0, \xi_0/|\xi_0|) = \sigma(a)(x_0, \xi_0)$, we get the desired result.
  (b) First recall that $\partial_t S(t)f = iMS(t)f$ for arbitrary $f \in C^\infty(X)$. Thus $\partial_t A(t)f(x) = i[A(t), M]f(x) = i[A, M](t)f(x)$ and

$$(A(t + h) - A(t))f(x) = h\int_0^1 \partial_t A(t + uh)f(x)\,du$$

$$= ih\int_0^1 [A, M](t + uh)f(x)\,du.$$

$S(t)$ is bounded and $[A, M]$ is in $\mathscr{P}^0$, so the L^2-norm of the right hand side can be estimated by $h \cdot \text{const.} \cdot \|[A, M]\|_{0,0} \cdot \|f\|_0$. This shows that $t \mapsto A(t)$ is continuous for $A \in \mathscr{P}^0$. In the same way, $[A, M](t)$ is in $\mathscr{P}^0$ and continuous in t, and so is $[[A, M], M](t)$. Now a similar argument as before proves differentiability: $\partial_t^2 A(t) f(x) = -[[A, M], M](t) f(x)$. Hence

$$((A(t + h) - A(t))/h - i[A, M](t)) f(x)$$

$$= h \int_0^1 \int_0^1 \partial_t^2 A(t + uvh) f(x) \, du \, dv$$

$$= -h \int_0^1 \int_0^1 [[A, M], M](t + uvh) f(x) \, du \, dv.$$

By estimating the L^2-norm and letting $h \to 0$ we get $\partial_t A(t) = i[A, M](t)$. This also leads to the desired statement on the symbol.

2.3. COROLLARY. *Let* $A \in \mathscr{P}_0$.
   (a) $A(t) \in \mathscr{P}_0$, $\sigma(A(t)) = \sigma(A)$. *Note that* $A(t)$ *is well-defined as an operator in* $L(H^0)$.
   (b) $t \mapsto A(t)$ *is continuous.*

Proof. (a) $\|A(t) - B(t)\| = \|(A - B)(t)\| \leq \text{const.} \|A - B\|$. $\mathscr{P}_0$ is the closure of $\mathscr{P}^0$. Thus (a) follows from 2.2(a).
   (b) Take a sequence A_k in $\mathscr{P}^0$ converging to A. Since $t \mapsto A_k(t)$ is continuous (even differentiable), and the convergence is uniform by the estimate used in the proof of (a), $t \mapsto A(t)$ is also continuous.

2.4. DEFINITION. Let s be an arbitrary function in $\mathscr{S}(\mathbf{R})$, the Schwartz space of rapidly decreasing functions, with $\int s(t) \, dt = 1$, e.g. $s(t) = \pi^{-1/2} \exp(-t^2)$, the actual choice of s does not matter. Now define the operator B on $\mathscr{P}_0$ by

$$(2.1) \qquad\qquad B(A) = \int A(t) s(t) \, dt,$$

where all the integrals are taken over $\mathbf{R}$.

2.5. LEMMA. (a) *The integral in* (2.1) *exists as an improper Riemann integral, hence* $B(A) \in \mathscr{P}_0$ *for* $A \in \mathscr{P}_0$.
   (b) *The map* $A \mapsto B(A)$ *is continuous on* $\mathscr{P}_0$.
   (c) $\sigma(B(A)) = \sigma(A)$.
   (d) *The commutator* $[M, B(A)]$, *first considered as an operator in* $L(H^{1/2}, H^{-1/2})$ *extends to an operator in* $L(H^0)$. *In fact,* $[M, B(A)] = -i \int A(t) s'(t) \, dt$.

Proof. (a) is a consequence of the boundedness of $S(t)$.

(b) $\|B(A)\| \leq \int \text{const.} \cdot \|A\| s(t)\, dt \leq \text{const.} \cdot \|A\|$.

(c) follows from the continuity of the symbol map:

$$\sigma(B(A)) - \sigma(A) = \int \sigma(A(t) - A) s(t)\, dt = 0,$$

where we have used that $\int s(t)\, dt = 1$.

(d) By continuity of the integral and Lemma 2.2(b), we have for $A \in \mathscr{P}^0$

$$[M, B(A)] = \int [M, A(t)] s(t)\, dt = i \int \partial_t A(t) s(t)\, dt$$

$$= -i \int A(t) s'(t)\, dt,$$

using integration by parts. Letting $A \in \mathscr{P}_0$, $A_k \in \mathscr{P}^0$, $A_k \to A$, we get from (b) that $B(A_k) \to B(A)$ in $L(H^0)$. Thus, on the left hand side, $[M, B(A_k)] \to [M, B(A)]$ in $L(H^{1/2}, H^{-1/2})$. On the right hand side, $-i \int A_k(t) s'(t)\, dt$ tends to $-i \int A(t) s'(t)\, dt$ (the integrals exist and define operators in $L(H^0)$, because s' is in $\mathscr{S}(\mathbf{R})$). This shows that $[M, B(A)] = -i \int A(t) s'(t)\, dt$ is in $L(H^0)$.

2.6. COROLLARY. *Let* $j \in \mathbf{N}$, $A \in \mathscr{P}_0$.

(a) $\mathrm{ad}^j M(B(A))$ *extends to an operator in* $L(H^0)$. *We have* $\mathrm{ad}^j M(B(A)) = (-i)^j \int A(t) s^{(j)}(t)\, dt$.

(b) $\sigma(\mathrm{ad}^j M(B(A))) = 0$, *i.e.* $\mathrm{ad}^j M(B(A))$ *is compact.*

Proof. (a) follows by induction from Lemma 2.5(d).

(b) Using (a) and Lemma 2.3(a), we have $\sigma(\mathrm{ad}^j M(B(A))) = (-i)^j \int \sigma(A) s^{(j)}(t)\, dt = 0$, after integration by parts.

2.7. *Proof of Theorem* 2.1. Let $a \in C(S^*X)$. Pick any operator $P \in \mathscr{P}_0$ with $\sigma(P) = a$. Define $A = B(P)$. Then $\sigma(A) = a$ by 2.5(c), and $\mathrm{ad}^j M(A)$ is compact for each $j \in \mathbf{N}$ by 2.6(b). The operators M^{-j}, $j \in \mathbf{N}$, are also compact. Hence $M^{-j} \mathrm{ad}^j M(A)$ and $\mathrm{ad}^j M(A) M^{-j}$ are compact operators on H^0 for each $j \in \mathbf{N}$. By 1.7(d), A is in $\mathscr{P}_\infty$.

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