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ON PELCZYNSKI'S PROPERTIES (V) AND (V*)

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It is shown that a Banach lattice X has Pelczynski's property (V*) if and only if X contains no subspace isomorphic to c_0. This result is used to show that there is a Banach space E that has Pelczynski's property (V*) but such that its dual E^* fails Pelczynski's property (V), thus answering in the negative a question of Pelczynski.

In his fundamental paper [7], Pelczynski introduced two properties of Banach spaces, namely property (V) and property (V*). For a Banach space X we say that *X has property* (V*) if any subset $K \subset X$ such that $\lim_n \sup_{x \in K} x_n^*(x) = 0$ for every weakly unconditionally Cauchy series (w.u.c.) $\sum_{n=1}^{\infty} x_n^*$ in X^*, then K is relatively weakly compact. We say that *X has property* (V) if any subset $K \subset X^*$ such that $\lim_n \sup_{x^* \in K} x_n(x^*) = 0$ for every weakly unconditionally Cauchy series (w.u.c.) $\sum_{n=1}^{\infty} x_n$ in X then K is relatively weakly compact. In [7] Pelczynski noted that it follows directly from the definition that if X^* has property (V) then X has property (V*), and he asked [7, Remark 3, p. 646] if the converse is true. As we shall soon show Example 5 below will provide a negative answer to Pelczynski's question.

In this paper we will concentrate on property (V*) and we shall refer the reader to [4] and [7] for more on property (V). Among classical Banach spaces that have property (V*), L^1-spaces are the most notable ones. In [7] Pelczynski showed that if a Banach space has property (V*), then it must be weakly sequentially complete. He also-noted that for a closed subspace X of a space with unconditional basis, the space X has property (V*) if and only if X contains no subspace isomorphic to c_0. This prompted the following natural question:

Problem 1. *Let $(\Omega, \Sigma, \lambda)$ be a probability space, and let X be a closed subspace of a Banach space with unconditional basis. Does the Banach space $L^1(\lambda, X)$ of Bochner integrable X-valued functions have property* (V*) *whenever X has* (V*)?

In this paper we shall give an affirmative answer to this question, in fact we shall prove a more general result, namely if X is a separable subspace of an order continuous Banach lattice, then $L^1(\lambda, X)$ has property (V*) if and only if X has (V*). It will also be shown that for a Banach lattice X, the space X has property (V*) if and only if X contains no subspace isomorphic to c_0.

First, let us fix some notations and terminology. We say that a series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is weakly unconditionally Cauchy (w.u.c.) if for every $x^* \in X^*$, the series $\sum_{n=1}^{\infty} x^*(x_n)$ is unconditionally convergent or equivalently if

$$\sup\left\{\left\| \sum_{i\in\sigma} x_i \right\| : \sigma \text{ finite subset of } I\!N\right\} < \infty .$$

If $(\Omega, \Sigma, \lambda)$ is a probability space and X is a Banach space, then $L^1(\lambda, X)$ will stand for the Banach space of all (classes of) Bochner integrable X-valued functions defined on Ω. For a compact Hausdorff space T, we shall denote by $M(T, X)$ the space of all countably additive X-valued measures defined on the σ-field of Borel subsets of T, and that are of bounded variation. The space $M(T, X)$ is a Banach space under the variation norm.

Recall that a Banach space X has the separable complementation property if every separable subspace E of X is contained in a separable complemented subspace F of X. In this paper we shall need the fact that any order continuous Banach lattice has the separable complementation property [**6**, p. 9].

Any other notation or terminology used and not defined can be found in [**5**] or [**6**].

1. The main result. The next theorem gives a characterization of those separable subspaces of an order continuous Banach lattice that have Pelczynski's property (V*).

THEOREM 2. *Let X be a separable subspace of an order continuous Banach lattice Y. Then X has property* (V*) *if and only if X contains no subspace isomorphic to c_0.*

Proof. Of course if X has (V*), then X is weakly sequentially complete [**7**], hence one direction is obvious.

Conversely, assume X contains no subspace isomorphic to c_0, hence X is weakly sequentially complete, and let $K \subset X$ such that

$$(') \qquad \lim_n \sup_{x\in K} x_n^*(x) = 0$$

for every w.u.c. series $\Sigma_{n=1}^{\infty} x_n^*$ in X^*. Let $\{x_n)_{n \geq 1}$ be a sequence in K. By [**6**, p. 9] let X_0 be a band with weak order unit in Y such that $X \subset X_0$. By [**6**, p. 25] there exists a probability space (Ω, Σ, ν) such that X_0 is an order ideal of $L^1(\nu)$ and such that

$$L_{\infty}(\nu) \subset X_0 \subset L^1(\nu),$$

and if $f \in L_{\infty}(\nu)$, then

$$\frac{1}{2}\|f\|_1 \leq \|f\|_{X_0} \leq \|f\|.$$

Since K satisfies (′) and $L^1(\nu)$ has property (V*), one can find a subsequence $\{x_{n_k}\}_{k \geq 1}$ of $\{x_n\}_{n \geq 1}$ such that $\{x_{n_k}\}_{k \geq 1}$ is weakly convergent in $L^1(\nu)$. We claim that $\{x_{n_k}\}_{k \geq 1}$ is in fact weak Cauchy in X. For this, let $g \in X^*$, by the Hahn-Banach theorem, let $h \in X_0^*$ such that $h = g$ on X and $\|h\|_{X_0^*} = \|g\|_{X^*}$. Since $h \in X_0^*$, we know [**6**, p. 25] that we can consider h as a measurable mapping on Ω and

$$\|h\|_{X_0^*} = \sup\left\{\int hf{:}\,d\nu\colon f \in X_0, \|f\|_{X_0} \leq 1\right\} < \infty.$$

Without loss of generality we may assume that $h \geq 0$. For each $n \geq 1$ let $\Omega_n = \{\omega \in \Omega \,|\, n - 1 < h(\omega) \leq n\}$. Then it follows from the duality between X_0 and X_0^* that the series $\Sigma_{n=1}^{\infty} h \cdot 1_{\Omega_n}$ converges weak* to h, moreover the series $\Sigma_{n=1}^{\infty} h \cdot 1_{\Omega_n}$ is a w.u.c. series in X_0^*. To see that $\Sigma_{n=1}^{\infty} h 1_{\Omega_n}$ is a w.u.c. series, note that if σ is a finite subset of N and $Z = \bigcup_{n \in \sigma} \Omega_n$, then

$$\begin{aligned}\left\|\sum_{n\in\sigma} hX_{\Omega_n}\right\| &= \sup\left\{\sum_{n\in\sigma}\int_{\Omega_n} hf\,d\nu; f \geq 0, \|f\|_{X_0} \leq 1\right\}\\ &= \sup\left\{\int_Z hf\,d\nu; f \geq 0, \|f\|_{X_0} \leq 1\right\}\\ &\leq \|h\|_{X_0^*}.\end{aligned}$$

Hence $\sup\{\|\Sigma_{n\in\sigma} h1_{\Omega_n}\|;\ \sigma$ finite subset of $IN\} < \infty$; and the series $\Sigma_{n=1}^{\infty} h1\Omega_n$ is a w.u.c. series in X_0^*. Therefore $\Sigma_{n=1}^{\infty} h1_{\Omega_n}$ when restricted to X is a w.u.c. series in X^*. Since K satisfies (′), it follows that the series $\Sigma_{n=1}^{\infty} h1_{\Omega_n}$ converges unconditionally uniformly on K. If not, one can find $\delta > 0$ $p_1 < p_2 < \cdots < p_n < \cdots$ such that for every $n \geq 1$

$$\sup_{x\in K}\left(\sum_{j=p_n+1}^{p_{n+1}} < h1_{\Omega_n}, x >\right) > \delta.$$

For each $n \geq 1$, let $y_n^* = \sum_{j=p_n+1}^{p_{n+1}} h1_{\Omega_j}$, the series $\sum_{n=1}^{\infty} y_n^*$ is also w.u.c. but $\lim_{n\to\infty} \sup_{x\in K} y_n^*(x) \neq 0$, thus contradicting ($'$). This implies that for $\varepsilon > 0$, there exists $m > 0$ such that for all $n \geq 1$

$$\left| \sum_{j=m+1}^{\infty} \int_{\Omega_j} h x_n \, d\nu \right| < \varepsilon.$$

Let $e^* = \sum_{j=1}^{m} h1_{\Omega_j}$, then $e^* \in L^\infty(\nu)$. Since the sequence $\{e^*(x_{n_k})\}_{k\geq 1}$ is Cauchy, it follows that there exists $N > 0$ such that for $p, q > N$

$$\left|e^*(x_{n_p} - x_{n_q})\right| < \varepsilon,$$

this of course implies that for $p, q > N$

$$\left|g(x_{n_p} - x_{n_q})\right| < 3\varepsilon.$$

This shows that K is weakly precompact, and hence K is relatively weakly compact since X is weakly sequentially complete.

PROPOSITION 3. *Let X have the separable complementation property. Then X has property* (V*) *if and only if every separable subspace of X has property* (V*).

Proof. Since property (V*) is easily seen to be stable by subspaces one implication is immediate.

Conversely, assume that every separable subspace of X has (V*) and let $K \subset X$ such that $\lim_n \sup_{x\in K} x_n^*(x) = 0$ for every w.u.c. series $\sum_{n=1}^{\infty} x_n^*$ in X^*. Let $\{x_n\}_{n\geq 1}$ be a sequence in K. Since X has the separable complementation property there exists a separable complemented subspace Z of X such that $\{x_n\}_{n\geq 1} \subset Z$. Since $\lim_n \sup_m x_n^*(x_m) = 0$ for every w.u.c. series $\sum_{n=1}^{\infty} x_n$ in X^* and since Z is complemented in X, it follows that $\lim_n \sup_m z_n^*(x_m) = 0$ for every w.u.c. series $\sum_{n=1}^{\infty} z_n^*$ in Z^*. By hypothesis the space Z has property (V*), hence it follows that there exists a subsequence $\{x_{n_k}\}_{n\geq 1}$ of $\{x_n\}_{n\geq 1}$ which is weakly convergent in Z and therefore is weakly convergent in X. This completes the proof and shows that X has property (V*).

THEOREM 4. *If X is a Banach lattice, then X has property* (V*) *if and only if X contains no subspace isomorphic to c_0.*

Proof. If X is a Banach lattice that contains no subspace isomorphic to c_0, then X has an order continuous norm. By Theorem 2 every separable subspace of X has property (V*), since X has the separable complementation property, it follows from Proposition 3 that X has property (V*).

We are now in a position to answer Pelczynski's question [7, Remark 3, p. 646].

EXAMPLE 5. A Banach space E such that E has property (V*) but E^* fails property (V).

Proof. To answer Pelczynski's question one needs to take a weakly sequentially complete Banach lattice E such that E^{**} is not weakly sequentially complete. This space will have the property (V*) by Theorem 4 but its dual E^* does not have property (V) [7]. An example of such a Banach lattice can be provided by the space constructed by M. Talagrand in [9]. Indeed the space E exhibited in [9] is weakly sequentially complete but is such that the space $M([0,1], E)$ contains a subspace isomorphic to c_0. This in particular shows that the space $M([0,1], E)$ cannot be weakly sequentially complete, therefore it follows from [8] that E^{**} cannot be weakly sequentially complete.

The next theorem gives a positive answer to Problem 1 stated at the beginning of this paper.

THEOREM 6. *Let X be a separable subspace of an order continuous Banach lattice Y. If $(\Omega, \Sigma, \lambda)$ is a probability space, then $L^1(\lambda, X)$ has property* (V*) *if and only if X has property* (V*).

Proof If $L^1(\lambda, X)$ has (V*), then X has (V*) since it is easily checked that property (V*) is stable by subspace.

Conversely, let X be a separable subspace of an order continuous Banach lattice Y. If X has property (V*), then X contains no subspace isomorphic to c_0. Of course $L^1(\lambda, X)$ is a subspace of $L^1(\lambda, Y)$ which is an order continuous Banach lattice [3]. The proof now follows from Theorem 2 and from a result of [8] (see also [1]) which guarantees that $L^1(\lambda, X)$ contains no subspace isomorphic to c_0.

2. Notes and remarks.

REMARK A. Theorem 4 fails for arbitrary Banach spaces. Indeed not every weakly sequentially complete Banach space has property (V*) the first Delbaen-Bourgain space [2] DBI is an example of a weakly sequentially complete Banach space that fails (V*). Indeed, the space DBI has the Schur property (weakly compact sets are compact), its dual is isomorphic to an L^1-space, but DBI fails (V*) due to the following easy proposition.

PROPOSITION 7 *For an non-reflexive Banach space X, if X^* is weakly sequentially complete, then X fails* (V*).

REMARK B. In [7] Pelczynski noted that if a Banach space X has property (V) then X^* has property (V*), and he asked [7, Remark 3, p. 646] if the converse is true. Here the first Delbaen-Bourgain space DBI provides a counter example to Pelczynski's question, for DBI fails property (V) since it has the Schur property, but its dual has property (V*) since it is isomorphic to an L^1-space.

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