A FUNCTIONAL CALCULUS FOR BANACH PI-ALGEBRAS

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Let A be a unital Banach algebra. Take $a_1, \ldots, a_n \in A$ and let B be the closed subalgebra of A they generate. The algebras $\mathscr{D}(\Omega)$ of entire matrix-valued functions were introduced by J. L. Taylor, who asked if they led to a functional calculus, generalizing the Shilov-Waelbroeck-Arens-Calderon theorem. We show that a necessary condition for a functional calculus map $\mathscr{D}(\Omega) \to A$ to exist is that B satisfy a polynomial identity; sufficient conditions are that B be a topological subquotient of a Banach Azumaya algebra, or that $n = 2$ and B satisfy all identities of 2×2 matrices. For closed subalgebras of Banach Azumaya algebras, we obtain a functional calculus on polynomial polyhedra containing the joint spectrum. Various properties of algebras of matrix-valued functions are studied, including domains of holomorphy.

Introduction. In the early 1950's, Shilov, Waelbroeck and Arens-Calderon constructed a functional calculus for several variables in a commutative Banach algebra with unit (all algebras are assumed to be complex). If A is such an algebra, and $a_1, \ldots, a_n \in A$, we have a homomorphism $P_n \to A$ that sends $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$, where P_n is the polynomial algebra in $X_1, \ldots, X_n$. This map extends to a continuous homomorphism $\mathcal{O}(U) \to A$, i.e., we have a commutative diagram

$$
\begin{array}{ccc}
P_n & \to & A \\
& \searrow \quad \nearrow & \\
& \mathcal{O}(U) &
\end{array}
$$

provided the open subset U of $\mathbf{C}^n$ contains the joint spectrum of $(a_1, \ldots, a_n)$ [8].

Is there a reasonable equivalent of this result for noncommutative algebras? Let A be any unital Banach algebra, and take $a_1, \ldots, a_n \in A$. We have a homomorphism $F_n \to A$ that sends $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$, where F_n is the free algebra in $X_1, \ldots, X_n$. Can we extend this map to some algebra $\mathscr{A}$ of "noncommutative functions"

$$
\begin{array}{ccc}
F_n & \to & A \\
& \searrow \quad \nearrow & \\
& \mathscr{A} &
\end{array}
$$

the existence of the extension being subordinate to a condition on a (yet to be defined) joint spectrum of $(a_1, \ldots, a_n)$? This is the functional calculus problem for noncommutative algebras.

Our first task is to define the algebras of noncommutative functions that will play the role of the $\mathcal{O}(U)$, $U \subseteq \mathbb{C}^n$. Let us again consider a commutative situation: entire functions of one variable. We can regard them as "holomorphic functions", i.e., continuously differentiable functions $\mathbb{C} \to \mathbb{C}$ that satisfy the Cauchy-Riemann equations. Or we can view them as "analytic functions", i.e., power series with infinite radius of convergence. Of course, these notions are identical: a function is holomorphic if and only if it is analytic. The equivalence holds for functions of n variables: any function holomorphic (in the Cauchy-Riemann sense) in the open set $U \subseteq \mathbb{C}^n$ is locally the sum of its Taylor series, and conversely.

It is somewhat surprising that the natural generalizations of "holomorphic" and "analytic" to noncommuting variables give rise to two different theories. For simplicity, we shall deal again with "entire" functions. We can consider functions that send n-tuples of $i \times i$ matrices to $i \times i$ matrices ($n \geq 2$ is fixed, i ranges over the positive integers). If we demand that these functions satisfy suitable intertwining conditions, we obtain the algebra $\mathcal{D}(\Omega)$ of "free entire holomorphic functions". Even simpler is the definition of $\mathcal{F}(\infty)$, the algebra of "free entire analytic functions": we take all free power series with infinite radius of convergence. J. L. Taylor, who introduced $\mathcal{D}(\Omega)$ and $\mathcal{F}(\infty)$ in [14], showed that $\mathcal{F}(\infty)$ embeds in $\mathcal{D}(\Omega)$. The inclusion is proper: take free indeterminates X_1 and X_2, and consider

$$ f(X_1, X_2) = \sum_{k=2}^{\infty} S_k \left(X_1, X_1 X_2, \ldots, X_1 X_2^{k-1} \right) $$

where S_k is the standard alternating free polynomial in k variables. It is easily shown that $f \in \mathcal{D}(\Omega)$, for the value of f on a couple of matrices is always given by a finite sum (Amitsur-Levitzki theorem, see [10]); but $f \notin \mathcal{F}(\infty)$: it cannot be written as a convergent power series (2.9).

Let us go back to the functional calculus problem. Since we can always substitute Banach algebra elements in a free power series, the map $F_n \to A$ extends to $\mathcal{F}(\infty)$. However, we are looking for a functional calculus, and $\mathcal{F}(\infty)$ is not really a function algebra. Thus, we now ask whether the map $F_n \to A$ extends to $\mathcal{D}(\Omega)$ (which is a function algebra)

$$
\begin{array}{ccc}
F_n & \to & A \\
\downarrow & \nearrow & \vdots \\
\mathcal{F}(\infty) & \to & \mathcal{D}(\Omega)
\end{array}
\qquad \text{(Taylor's question, [14])}
$$

In order for the dotted map to exist, it is necessary that the subalgebra of A generated by $a_1, \ldots, a_n$ be a PI-algebra, i.e., some free polynomial vanishes identically when evaluated on elements of the algebra (2.7).

The converse question is more delicate. We conjecture that $\mathscr{D}(\Omega)$ gives a functional calculus for all PI-algebras and any (finite) number of variables. We prove this

(a) when the algebra is a separated quotient of a closed subalgebra of a Banach Azumaya algebra;

(b) when the number of variables is two, and the algebra satisfies all polynomial identities of 2×2 matrices.

It is not known if every finitely generated Banach PI-algebra enjoys property (a). Even if this is not the case, (b) indicates that (a) might be superfluous.

To attack this question, we shall assume henceforth that the unital Banach algebra A satisfies all identities of $i \times i$ matrices. Then the map $F_n \to A$ vanishes on the ideal $T_{n,i}$ of those elements in F_n that are polynomial identities for $i \times i$ matrices, and yields a map $F_n/T_{n,i} = R_{n,i} \to A$, where $R_{n,i}$ is the algebra of n generic $i \times i$ matrices. $R_{n,i}$ can also be described as an algebra of polynomial functions $M_i^n \to M_i$ (M_i is the algebra of complex $i \times i$ matrices; likewise M_{ij} is the space of complex $i \times j$ matrices, GL_i the complex general linear group, ...). We can consider $\mathscr{R}$, the closure of $R_{n,i}$ in $\mathscr{O}(M_i^n, M_i)$. An intrinsic definition of $\mathscr{R}$ can be given for $n = i = 2$, but fails when $i > 2$ or $n > 2$. $\mathscr{R}$ is the algebra of "entire holomorphic functions of n $i \times i$ matrices". Does $R_{n,i} \to A$ extend to $\mathscr{R}$?

$$R_{n,i} \quad \to \quad A$$
$$\searrow \qquad \nearrow$$
$$\mathscr{R}$$

This question is equivalent to the following: can every function in $\mathscr{R}$ be represented by a power series, i.e., do "holomorphic" and "analytic" coincide for entire functions of n $i \times i$ matrices (where now both n and i are fixed). Still another way to phrase the question is: do $\mathscr{D}(\Omega)$ and $\mathscr{F}(\infty)$ agree after factoring out the functions that vanish on $i \times i$ matrices? If the answer is affirmative, we will have shown that $\mathscr{D}(\Omega) \to A$.

When $n = i = 2$, we can prove it, because we have precise structure results on $R_{2,2}$, the algebra of two generic 2×2 matrices [7]. Our approach does not seem to generalize to $n > 2$ or $i > 2$, when the structure of the generic matrix algebra is less well understood. The basic difficulty with our algebras $\mathscr{R}$ is that they are not finite modules over

their centers, and the centers are not finitely generated algebras. Nevertheless, we can try a more complicated construction. Instead of considering entire free power series, we take free power series with a finite polyradius of convergence. Also, we add q extra indeterminates to the n we started with, and impose q relations, before factoring out the identities of $i \times i$ matrices. If the relations are chosen suitably, the quotient turns out to be an Azumaya algebra (hence finite over its center), and its center is finitely generated. Then we can show that the quotient is isomorphic to the algebra of all "holomorphic functions" defined on an open set of M_i^n which is a natural generalization of a polynomial polyhedron. This implies the existence of $\mathscr{D}(\Omega) \to A$ when A is (a topological subquotient of) an Azumaya algebra (n, i arbitrary).

The contents of this paper can be summarized as follows. In Chapter One, we give basic facts about algebras with polynomial identities. The sole original result is a description of the algebra of two generic 2×2 matrices by intertwining conditions. Free algebras of analytic and holomorphic functions are introduced in Chapter Two, which is based on Taylor's [14]. We include a study of "domains of holomorphy", parallel to the domains of holomorphy in classical complex analysis. In Chapter Three, we discuss PI-algebras of holomorphic functions. Chapters Four and Five, where the commutative functional calculus plays an important role, constitute the heart of the paper. In Chapter Four, after developing an implicit function theorem due to Taylor, we show that holomorphic functions on n $i \times i$ matrices admits power series expressions in appropriate domains, for which the function algebras are Azumaya. Chapter Five concentrates on functions of two 2×2 matrices. Finally, Chapter Six describes the construction of a functional calculus, under the assumptions (a) or (b) stated on page 129. For closed subalgebras of Azumaya algebras, we obtain a functional calculus on polynomial polyhedra containing the spectrum.

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Chapter One. Polynomial Identities

Let A be a complex algebra with unit. A satisfies a polynomial identity if some noncommutative polynomial in n variables vanishes on all n-tuples of elements of A. We also say that A is a PI-algebra. We shall

write $A \leq M_i$ when A satisfies all the identities of $i \times i$ matrices (in particular the standard identity $S_{2i} = 0$).

Basic properties of PI-algebras can be found in [9], [12], [16]. Let us also mention this recent result.

1.1. PROPOSITION (*Braun*). *If A is a finitely generated PI-algebra, there exists $i \in \mathbf{N}$ such that $A \leq M_i$.*

Proof. See [4].

Azumaya algebras (also called central separable algebras) form an important class of PI-algebras. Here is a deep and useful fact.

1.2. PROPOSITION (*Artin-Procesi*). *A is an Azumaya algebra of rank i^2 over its center Z if and only if*

(a) $A \leq M_i$

(b) *For every maximal ideal I in A, A/I is a central simple algebra of dimension i^2 over its center.*

Proof. [16, II.3.5].

Recall that, for $A \leq M_i$, the Formanek center $F(A)$ of A is the subset of A obtained by evaluating all central polynomials (for $i \times i$ matrices) without constant term, in any number of indeterminates; see [9.VIII].

1.3. PROPOSITION

(a) *$F(A)$ is an ideal in $Z(A)$, the center of A.*

(b) *$F(A) = Z(A)$ if and only if A is an Azumaya algebra of rank i^2 over its center.*

Proof. [10, 5.1, 5.4].

Let $n \geq 2$, $i \geq 2$ be two integers. Let R be the algebra of n generic $i \times i$ matrices (see Introduction, or [11]).

Let B be the algebra of invariants (for n-tuples of $i \times i$ matrices), which we define by

$$B = \left\{ f \colon M_i^n \overset{\text{polyn.}}{\to} \mathbf{C} \colon f(mxm^{-1}) = f(x) \text{ for all } x \in M_i^n, \, m \in \mathrm{GL}_i \right\}.$$

Then $S = BR$ is the algebra of concomitants (also known as the trace algebra), and

1.4. PROPOSITION.

$$S = \left\{ f\colon M_i^n \overset{\text{polyn.}}{\to} M_i\colon f(mxm^{-1}) = mf(x)m^{-1} \right.$$

$$\left. \text{for all } x \in M_i^n, \, m \in \text{GL}_i \right\}.$$

Proof. [**11**, 2.1].

Is it possible to describe R by stronger intertwining conditions? The definition of Taylor's $\mathcal{D}(\Omega)$ algebra suggests that this might be possible, if we accept noninvertible intertwining matrices. Unfortunately, this works only for $n = i = 2$, where we have the

1.5. PROPOSITION. *Let*

$$R' = \left\{ f\colon M_2^2 \overset{\text{polyn.}}{\to} M_2, \, f(x)m = mf(y) \right.$$

$$\left. \text{whenever } x, y \in M_2^2, \, m \in M_2 \text{ and } xm = my \right\}.$$

Then $R' = R$, the algebra of 2 generic 2×2 matrices.

Proof. Basic structure results for R are given in [7]. By (1.4),

$$S = \left\{ f\colon M_2^2 \overset{\text{polyn.}}{\to} M_2, \, f(x)m = mf(y) \right.$$

$$\left. \text{when } x, y \in M_2^2, \, m \in \text{GL}_2 \text{ and } xm = my \right\}$$

and thus $R \subseteq R' \subseteq S$. Now R and S have the same commutator ideal K [7], so R' will also have K as its commutator ideal, and $R/K \subseteq R'/K \subseteq S/K$. Let $\overline{X}_1, \overline{X}_2, \overline{T}_1, \overline{T}_2$ be the classes mod K of $X_1, X_2, \text{tr } X_1, \text{tr } X_2 \in S$. Then $R/K \simeq \mathbf{C}[\overline{X}_1, \overline{X}_2]$ and $S/K \simeq \mathbf{C}[\overline{X}_1, \overline{X}_2, \overline{T}_1, \overline{T}_2]$. Take $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbf{C}$ and let

$$x_1 = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 - a_1 \end{pmatrix} \qquad x_2 = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 - a_2 \end{pmatrix}$$

$$y_1 = \begin{pmatrix} a_1 & 0 \\ 0 & c_1 - a_1 \end{pmatrix} \qquad y_2 = \begin{pmatrix} a_2 & 0 \\ 0 & c_2 - a_2 \end{pmatrix}.$$

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Note that $x\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)y$. Let $m = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Suppose $f \in S/K$. Write $f = \Sigma \lambda_{ijkl} \overline{X}_1^i \overline{X}_2^j \overline{T}_2^k \overline{T}_2^l$ (where $\lambda_{ijkl} \in \mathbf{C}$ and the sum is finite). If $f \in R'/K$, $f(x)m = mf(y)$, which forces $\lambda_{ijkl} = 0$ when $k \neq 0$ or $l \neq 0$. Thus $f \in R'/K$ implies that $f = \Sigma \lambda_{ij00} \overline{X}_1^i \overline{X}_2^j \in R/K$, and from $R'/K = R/K$, it follows that $R' = R$ (the final part of this argument is due to E. Formanek).

From this description of $R' = R$, we obtain

$$Z(R) = Z(R') = \left\{ f \colon M_2^2 \overset{\text{polyn.}}{\to} \mathbf{C}, f(x) = f(y) \right.$$

$$\left. \text{whenever } x, y \in M_2^2, \, m \neq 0, \text{ and } xm = my \right\}$$

from which we can get

1.6. COROLLARY.

$$Z(R) = \mathbf{C} + B(X_1 X_2 - X_2 X_1)^2.$$

This appears in [7] with a different proof.

However, results analogous to (1.5) fail for $n > 2$ or $i > 2$.

1.7. EXAMPLE (Formanek).

Let R be the algebra of 3 generic 2×2 matrices. Consider $f \in S$, $f(X_1, X_2, X_3) = \operatorname{tr} X_1(X_2 X_3 - X_3 X_2)$; then f satisfies the strong intertwining condition, but $f \notin R$. Similar examples can be given for pairs of $i \times i$ matrices $(i > 2)$, and of course for n-tuples of $i \times i$ matrices $(i, n > 2)$.

We now go back to the algebra of invariants B (n, i arbitrary). B is a finitely generated algebra; more precisely

1.8. PROPOSITION. *B is generated by* $\{ \operatorname{tr}(X_{q_1} \cdot \, \cdots \, \cdot X_{q_r}) \colon r \leq 2^i - 1 \}$.

Proof. [11, 1.3].

It follows that B is Noetherian.

Fix a set of generators $z_1, \ldots, z_p$ for B. This defines a map $\Pi \colon M_i^n \to \mathbf{C}^p$.

1.9. DEFINITION.

Let $\hat{M}_i^n = \{(x_1, \ldots, x_n) \in M_i^n \colon x_1, \ldots, x_n \text{ generate } M_i\}$. Let $\check{M}_i^n = M_i^n - \hat{M}_i^n$.

Note that $\hat{M}_i^n$ is open and dense in M_i^n.
Two important facts are contained in

1.10. PROPOSITION. (a) Π *maps* M_i^n *onto an affine algebraic subvariety* Δ *in* $\mathbf{C}^p$, $\dim \Delta = (n-1)i^2 + 1$.
   (b) Π *restricted to* $\hat{M}_i^n$ *maps onto an open, smooth subset* $\hat{\Delta}$ *of* Δ; *in fact,* $\hat{M}_i^n$ *is the total space of a principal bundle with fiber* PGL_i *and base* $\hat{\Delta}$.

Proof. [**10**, 4.1.5.10]

We now attack the spectral study of Banach PI-algebras. The fundamental structure result is

1.11. PROPOSITION. *Let* A *be a primitive Banach algebra satisfying a polynomial identity of degree* d.
   Then $A \simeq M_i$ *for some* $i \le [d/2]$

Proof. By a theorem of Kaplansky [**9**, II.1], A is a central simple algebra of dimension i^2 over its center Z, with $2i \le d$. By the Gelfand-Mazur theorem, $Z \simeq \mathbf{C}$. Since $\mathbf{C}$ is algebraically closed, we conclude that $A \simeq M_i$.

1.12. COROLLARY. *Assume* A *is a unital Banach algebra that satisfies an identity of degree* d, *and let* I *be an ideal of* A. *Then* (a), (b), (c) *are equivalent and imply* (d):
   (a) I *is maximal*
   (b) I *is primitive*
   (c) $A/I \simeq M_i$ $(2i \le d)$
   (d) I *is closed.*

Let A be a unital Banach PI-algebra, and take $a = (a_1, \ldots, a_n) \in A^n$. All irreducible representations of A are finite-dimensional, and this will allow us to define the joint spectrum of $(a_1, \ldots, a_n)$ as a subset of $\Omega = \bigcup_{i=1}^{\infty} M_i^n$.

1.13. DEFINITION. Let

$$\mathrm{sp}_i(a) = \{(\varphi(a_1), \ldots, \varphi(a_n)): \varphi \text{ irreducible representation } A \to M_i\}$$

and $\mathrm{sp}(a) = \bigcup_{i=1}^{\infty} \mathrm{sp}_i(a)$. (Actually the union is always finite.)
   We will need the notion of full spectrum.

1.14. DEFINITION. Let

$$f \operatorname{sp}_i(a) = \{(\psi(a_1), \ldots, \psi(a_n)): \psi \text{ nondegenerate}$$

$$\text{representation } A \to M_i\}$$

and $f \operatorname{sp}(a) = \bigcup_{i=1}^{\infty} f \operatorname{sp}_i(a)$.

Also interesting is the polynomial spectrum.

1.15. DEFINITION. Let

$$p \operatorname{sp}_i(a) = \{x = (x_1, \ldots, x_n) \in M_i^n: \rho p(x) \le \rho p(a) \text{ for all } p \in F_n\}$$

$$(\rho = \text{spectral radius}),$$

and $p \operatorname{sp}(a) = \bigcup_{i=1}^{\infty} p \operatorname{sp}(a)$.

Clearly $\varnothing \subset \operatorname{sp}(a) \subset f \operatorname{sp}(a) \subseteq p \operatorname{sp}(a)$.

We will not consider the problem of the Gelfand transform (can we represent any semiprimitive Banach PI-algebra as an algebra of continuous matrix-valued functions?) in full generality here. Let us just mention the

1.16. PROPOSITION (*Fell*). *Let A be a unital C^*-algebra such that $A/I \simeq M_i$ for all maximal ideals I of A (i is fixed). Let Z be the center of A: then $Z \simeq C(X)$ for some compact Hausdorff space X. There is a bundle $\mathcal{M}$ with fiber M_i and base X such that $A \simeq \Gamma(X, \mathcal{M})$, the algebra of all continuous sections of over X.*

Proof [**6**, Th. 3.2].

Thus Azumaya C^*-algebras can be represented as section algebras (since the bundle $\mathcal{M}$ need not be trivial, we cannot always write $A = C(X, M_i)$). A similar result will be given for some Azumaya algebras of holomorphic functions (4.19).

Chapter Two. Free Algebras of Functions

In his papers [**14**] and [**15**], Taylor studied "localizations" of the free algebra. We fix $n \ge 1$. An embedding $F_n \to \mathcal{A}$ (where $\mathcal{A}$ is a Fréchet algebra, i.e., a complete, metrizable, locally m-convex algebra) is said to be a localization if there exist n linear maps $\Delta_1, \ldots, \Delta_n: \mathcal{A} \to \mathcal{A} \hat{\otimes} \mathcal{A}$ ($\hat{\otimes}$ is the completed inductive tensor product) that have properties similar to partial differential quotients [**15**, p. 6]. If, furthermore, $\Delta_1, \ldots, \Delta_n$ satisfy

conditions analogous to the Cauchy estimates, $\mathscr{A}$ will be called a free analytic algebra [15, p. 13]. When the map $F_n \to \mathscr{A}$ is understood, we shall just say that $\mathscr{A}$ is a localization. Taylor gives many examples of localizations of F_1, the polynomials algebra in one indeterminate. He shows that $\mathscr{A}$ is a free analytic algebra (in one indeterminate) if and only if $\mathscr{A} = \mathcal{O}(U)$ for U open in $\mathbf{C}$. Since the algebras $\mathcal{O}(U)$ give a one-variable functional calculus, we can hope that free analytic algebras (in n indeterminates) will lead to a functional calculus for n noncommuting variables in a Banach algebra.

We shall also need the notion of "lmc completion". The lmc completion $\tilde{\mathscr{A}}$ of a topological algebra $\mathscr{A}$ is the Hausdorff completion of $\mathscr{A}$ with respect to the family of all continuous submultiplicative seminorms on $\mathscr{A}$. It has the universal property:

$$\mathscr{A} \; \to \; B$$
$$\searrow \qquad \nearrow$$
$$\tilde{\mathscr{A}}$$

viz. every continuous homomorphism $\mathscr{A} \to B$ (B Banach algebra) factors through $\tilde{\mathscr{A}}$; see [13, p. 178] for an equivalent definition.

We assume henceforth $n \geq 2$. Let us investigate some "free algebras of functions".

Every free polynomial p of degree d in F_n can be written as $p = \sum_{|\sigma|=0}^{d} \lambda_\sigma X^\sigma$ where σ is a noncommutative multi-index, $|\sigma|$ its length, and λ_σ is a complex coefficient. Let $I =]0\infty]$. On I^n, we have the partial ordering $a = (a_1, \ldots, a_n) < b = (b_1, \ldots, b_n)$ if and only if $a_1 < b_1, \ldots, a_n < b_n$. Let $r \in I^n$. We introduce the algebra of power series [14], [15]:

2.1. DEFINITION.

$$\mathscr{F}(r) = \left\{ \sum_{|\sigma|=0}^{\infty} \lambda_\sigma X^\sigma : \sum_{|\sigma|=0}^{\infty} |\lambda_\sigma| t^\sigma < \infty \text{ for all } t \in I^n, \, t < r \right\}.$$

Endowed with the obvious family of (semi)norms, $\mathscr{F}(r)$ is a Fréchet algebra, and a free analytic algebra [15, p. 17]. Let $\mathscr{F}(\infty) = \mathscr{F}(r)$ where $r = (\infty, \ldots, \infty)$. $\mathscr{F}(\infty)$ is the lmc completion of F_n, and can be regarded as the algebra of "free entire analytic functions" (although free power series do not correspond to functions in the sense of mappings). Taylor claimed [14, p. 235] that $\mathscr{F}(r)$ was never nuclear (as a Fréchet space). However, we have the

2.2. PROPOSITION. *Let $r = (r_1, \ldots, r_n) \in I^n$. $\mathscr{F}(r)$ is nuclear if and only if at most one of the $r_1, \ldots, r_n$ is finite.*

Proof. We follow [14]. $\mathscr{F}(r)$ is nuclear provided that, for every $\rho' \in I^n$, $\rho' < r$, there is $\rho \in I^n$, $\rho' < \rho < r$, for which the series

$$\sum_{k_1,\ldots,k_n} \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!} \left(\frac{\rho'_1}{\rho_1}\right)^{k_1} \cdots \left(\frac{\rho'_n}{\rho_n}\right)^{k_n}$$

is convergent. This series can be rewritten as

$$\sum_{k=0}^{\infty} \left(\frac{\rho'_1}{\rho_1} + \cdots + \frac{\rho'_n}{\rho_n}\right)^k \qquad (k = k_1 + \cdots + k_n)$$

which converges if

$$\frac{\rho'_1}{\rho_1} + \cdots + \frac{\rho'_n}{\rho_n} < 1.$$

If (and only if) at most one of the $r_1, \ldots, r_n$ is finite, then for every choice of $\rho' < r$ we will be able to find ρ, $\rho' < \rho < r$, such that the above inequality is satisfied, and $\mathscr{F}(r)$ will be nuclear.

2.3. COROLLARY. $\mathscr{F}(\infty)$ *is nuclear.*

Here is the relationship between free power series and the functional calculus problem.

2.4. PROPOSITION. *Let A be a unitial Banach algebra, and take $a_1, \ldots, a_n \in A$. Choose $r_1 > \|a_1\|, \ldots, r_n > \|a_n\|$. There is a continuous algebra homomorphism $\mathscr{F}(r) \to A$ that maps $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$.*

Proof. Elementary.

But, as we observed earlier, $\mathscr{F}(r)$ can hardly be regarded as an algebra of functions. Therefore, we shall now turn our attention to a quite different kind of algebra, based on the idea of function rather than power series.

The following definitions are borrowed from [14, p. 238]. Let $\Omega = \bigcup_{i=1}^{\infty} M_i^n$ (topological disjoint union). If $x = (x_1, \ldots, x_n) \in M_i^n$, $y = (y_1, \ldots, y_n) \in M_j^n$, $x \oplus y$ will denote

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \left(\begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_n & 0 \\ 0 & y_n \end{pmatrix}\right) \in M_{i+j}^n.$$

Let U be an open subset of Ω, let $U_i = U \cap M_i^n$. Then U_i is open in M_i.

We denote the Fréchet algebra of holomorphic functions in U_t with values in M_t by $\mathcal{O}(U_i, M_i)$.

Let $\mathcal{B}(U) = \prod_{i=1}^{\infty} \mathcal{O}(U_i, M_i)$ be the topological direct product. Thus an element $f \in \mathcal{B}(U)$ is a function on U which, when restricted to U_i, is a holomorphic function with values in M_t.

Note that $\mathcal{B}(U)$ is a nuclear Fréchet algebra. We shall say that U open in Ω is normal if $x \in U_i$, $y \in U_j$ implies $x \oplus y \in U_{i+j}$.

2.5. DEFINITION. Suppose U is a nonempty open normal subset of Ω. Then

$$\mathcal{D}(U) = \{ f \in \mathcal{B}(U): f(x)m = mf(y)$$

$$\text{whenever } x \in U_i, \ y \in U_j, \ m \in M_{ij} \text{ and } xm = my \}.$$

$\mathcal{D}(U)$ is a closed subalgebra of $\mathcal{B}(U)$ and hence is also a nuclear Fréchet algebra. We have a natural map $F_n \to \mathcal{D}(U)$. As no (nonzero) free polynomial vanishes in all matrix algebras, this map is injective. Here is a crucial property.

2.6. PROPOSITION. *If U is a nonempty open normal subset of Ω, $\mathcal{D}(U)$ is a projective limit of Banach PI-algebra.*

Proof. $\mathcal{D}(U)$ is a closed subalgebra of $\mathcal{B}(U)$. Any continuous semi-norm ν on $\mathcal{D}(U)$ is essentially given by $\nu(f) = \max\{ \|f(x)\|: x \in K \}$ for some K compact in U, so $K \subset \bigcup_{j=1}^{i} M_j^n$. Thus $\nu^{-1}(0)$ contain all polynomial identities of $i \times i$ matrices, and the normed algebra $\mathcal{D}(U)/\nu^{-1}(0) \le M_i$. Taking an increasing sequence of compact sets and completing, we can realize $\mathcal{D}(U)$ as a projective limit of Banach PI-algebras.

2.7. COROLLARY. *Let A be a Banach algebra, and $\varphi: \mathcal{D}(U) \to A$ a continuous homomorphism. There exists $i \in \mathbf{N}$ such that $\varphi\mathcal{D}(U) \le M_i$ (hence $\overline{\varphi\mathcal{D}(U)} \le M_i$).*

Thus, if A is a Banach algebra with unit, and $a_1, \ldots, a_n \in A$, we can hope to get a functional calculus map $\mathcal{D}(U) \to A$, $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$ only when the subalgebra of A generated by $a_1, \ldots, a_n$ satisfies some identity, hence all identities of $i \times i$ matrices for some i (1.1).

Since $\mathcal{D}(U)$ is lmc and complete, the embedding $F_n \to \mathcal{D}(U)$ yields an injection $\mathcal{F}(\infty) \to \mathcal{D}(U)$. In particular, $\mathcal{F}(\infty) \to \mathcal{D}(\Omega)$. We know that $\mathcal{F}(\infty)$ is an algebra of "free entire power series", while $\mathcal{D}(\Omega)$ is an algebra of "entire matrix-valued functions".

2.8. PROPOSITION. *The map $\mathscr{F}(\infty) \to \mathscr{D}(\Omega)$ is not surjective.*

Proof. If it were, $\mathscr{F}(\infty)$ and $\mathscr{D}(\Omega)$ would be isomorphic Fréchet spaces; but on $\mathscr{D}(\Omega)$ there is no continuous norm (2.7) and $\mathscr{F}(\infty)$ admits many continuous norms.

A more concrete explanation is the following (see Introduction).

2.9. EXAMPLE. Let $n = 2$, consider

$$f(X_1, X_2) = \sum_{k=2}^{\infty} S_k\left(X_1, X_1 X_2, \ldots, X_1 X_2^{k-1}\right).$$

Then $f \notin \mathscr{F}(\infty)$: write f as a power series $\sum \lambda_\sigma X^\sigma$, and observe that for all $d \in \mathbf{N}$, there is a multi-index σ, $|\sigma| \geq d$, $\lambda_\sigma = 1$; thus $\sum \lambda_\sigma X^\sigma$ cannot have infinite radius of convergence.

But $f \in \mathscr{D}(\Omega)$: if $x = (x_1, x_2) \in M_i^2$, $f(x_1, x_2)$ is given by the finite sum $\sum_{k=2}^{2i} S_k(x_1, x_1 x_2, \ldots, x_1 x_2^{k-1})$, and $f \in \mathscr{B}(\Omega)$; it is easily checked that $f \in \mathscr{D}(\Omega)$.

2.10. PROPOSITION. *Let $f \in \mathscr{D}(\Omega)$, $x \in U_i$, $y \in U_j$. There is a linear map $\Delta\colon M_{ij}^n \to M_{ij}$, $z \mapsto f(x, y, z)$ such that*

$$f\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} = \begin{pmatrix} f(x) & \Delta f(x, y, z) \\ 0 & f(y) \end{pmatrix}$$

whenever $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in U_{i+j}$.

Proof. See [**14**, p. 239]. Note that $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in U_{i+j}$, and as U is open, $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in U_{i+j}$ for all z in a neighborhood of 0 in M_{ij}^n.

2.11. PROPOSITION. *Every $f \in \mathscr{D}(\Omega)$ has a formal free power series in $X_1, \ldots, X_n$.*

Proof. For a given i, f will have a power series as a function $M_i^n = \mathbf{C}^{ni^2} \to \mathbf{C}^{i^2} = M_i$. This Taylor series can be written as a power series in $X_1, \ldots, X_n$, say $\sum_{|\sigma|=0}^{\infty} \lambda_\sigma^{(i)} X^\sigma$ [**15**, p. 20]. As M_i does not satisfy any polynomial identity of degree $< 2i$, the $\lambda_\sigma^{(i)}$ are uniquely determined for $|\sigma| < 2i$. We define $\lambda_\sigma = \lambda_\sigma^{(i)}$ for any $i > |\sigma|/2$, and $\sum \lambda_\sigma X^\sigma$ is the formal power series for f. Example (2.9) shows that we lose control on the growth of the coefficients.

Taylor proves [**14**, p. 242] that $\mathscr{D}(U)$ is a localization. However, it is not a free analytic algebra. Let us only show

2.12. PROPOSITION. $\mathscr{D}(\Omega)$ *is not a free analytic algebra.*

Proof. Let E be an infinite-dimensional Banach space. If $\mathscr{D}(\Omega)$ were a free analytic algebra, the set

$$V = \left\{ (x_1, \ldots, x_n) \in (\mathscr{L}(E))^n \colon \text{there is a continuous homomorphism} \right.$$

$$\left. \mathscr{D}(\Omega) \to \mathscr{L}(E), 1 \mapsto 1, X_1 \mapsto x_1, \ldots, X_n \mapsto x_n \right\}$$

would be open in the norm topology of $(\mathscr{L}(E))^n$ [**15**, p. 14]. But (2.7)

$$V \subseteq W = \left\{ (x_1, \ldots, x_n) \in (\mathscr{L}(E))^n \colon x_1, \ldots, x_n \right.$$

$$\left. \text{generate a PI-subalgebra of } \mathscr{L}(E) \right\}.$$

Since $0 = (0, \ldots, 0) \in W$ and W is not a neighborhood of 0, V cannot be open. A similar argument shows that $\mathscr{D}(U)$ is not a free analytic algebra.

In the setting of matrix-valued functions, we can study envelopes and domains of holomorphy.

As usual, let U be a nonempty, open normal subset of Ω. By a representation $\mathscr{D}(U) \to M_i$, we mean a continuous unital algebra homomorphism. Since $\mathscr{D}(U)$ is a localization of F_n, we have the

2.13. PROPOSITION. *Let* $(x_1, \ldots, x_n) \in M_i^n$. *There is at most one representation* $\mathscr{D}(U) \to M_i$, $X_1 \mapsto x_1, \ldots, X_n \mapsto x_n$.

Proof. [**15**, p. 7].

Of course, if $x = (x_1, \ldots, x_n) \in U_i$, evaluation at x will give a representation. A natural question is: are there (finite-dimensional) representations which are not evaluations?

2.14. DEFINITION. For $i = 1, 2, 3, \ldots$, we let

$$\tilde{U}_i = \left\{ x \in M_i^n \colon \text{there is one representation } \mathscr{D}(U) \to M_i, \right.$$

$$\left. X_1 \mapsto x_1, \ldots, X_n \mapsto x_n \right\}.$$

Then $\tilde{U} = \bigcup_{i=1}^{\infty} \tilde{U}_i$ is the envelope of holomorphy of U. Since $U_i \subseteq \tilde{U}_i \subseteq M_i^n$, we have $U \subseteq \tilde{U} \subseteq \Omega$; thus we do not have to consider "Riemann domains".

2.15. *Conjecture.* We believe that $\tilde{U}$ is always open in Ω. If it were true that $\mathscr{D}(U)$ is a free analytic algebra, this would follow from [15]. Note that U is normal; if U is open, we have a natural isomorphism $\mathscr{D}(\tilde{U}) \to \mathscr{D}(U)$.

2.16. DEFINITION. U is a domain of holomorphy if $\tilde{U} = U$.

Although we do not have a complete characterization for domains of holomorphy, we have necessary conditions contained in the

2.17. PROPOSITION. *Let U be a domain of holomorphy. Then*:

(a) *For all i, U_i is holomorphically convex (in the classical sense, viz. as a subset of $\mathbf{C}^{ni^2}$).*

(b) *If $x \in U_i$ and $m \in \mathrm{GL}_i$, then $mxm^{-1} \in U_i$*

(c) *If $x \in U_i$ and $y \in U_j$, then*

$$\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in U_{i+j} \quad \text{for all } z \in M^n_{ij}.$$

(d) *If $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in U_{i+j}$, then $x \in U_i$ and $y \in U_j$.*

Proof. (a) Assume U_i is not a domain of holomorphy in $\mathbf{C}^{ni^2}$. Let $\hat{U}_i$ be its classical envelope of holomorphy, which is a manifold spread over $\mathbf{C}^{ni^2}$. Every holomorphic function $g: U_i \to M_i$ extends to $\hat{g}: \hat{U}_i \to M_i$. Take $x \in \hat{U}_i$, let $\psi: \mathscr{D}(U) \to M_i$, $f \mapsto \hat{g}(x)$ (where $f|_{U_i} = g$); this defines a representation of $\mathscr{D}(U)$. If x does not lie over U_i, ψ cannot be given by any point of U_i, so U is not a domain of holomorphy.

(b) $\psi(f) = mf(x)^{-1}$ defines a representation, which must be given by evaluation at mxm^{-1}.

(c) As U is normal, $x \in U_i$ and $y \in U_j$ imply that

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in U_{i+j}.$$

As U is open, for $w \in M^n_{ij}$, w small enough,

$$\begin{pmatrix} x & w \\ 0 & y \end{pmatrix} \in U_{i+j}.$$

Now, by (b),

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & w \\ 0 & y \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & tw \\ 0 & y \end{pmatrix} \in U_{i+j},$$

for all $t \in \mathbf{C}^*$. Since tw is arbitrary, we see that

$$\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in U_{i+j} \quad \text{for all } z \in M^n_{ij} \text{ (see also (2.10))}.$$

(d) This follows from [**15**, p. 7].

We do not know if these four conditions are sufficient, i.e., if every normal, open U that satisfies (a), (b), (c), (d) is a domain of holomorphy.

We also have a class of examples, the holomorphic polyhedra. Let V be a domain of holomorphy (e.g., Ω), take $f_1, \ldots, f_q \in \mathscr{D}(V)$, and consider

$$U = \left\{ x \in V : \rho f_1(x) < 1, \ldots, \rho f_q(x) < 1 \right\}.$$

U is open and normal, moreover

2.18. PROPOSITION. *U is a domain of holomorphy.*

Proof. First observe that $U \subseteq V \Rightarrow U \subseteq \tilde{U} \subseteq \tilde{V} = V$. Suppose $x \in \tilde{U} - U$. We can assume, without loss of generality, that $\rho f_1(x) \geq 1$, so there is $\eta \in \mathbf{C}$, $|\eta| \geq 1$, such that $f_1(x) - \eta$ is not invertible in M_i. But $f_1 - \eta$ is invertible in $\mathscr{D}(U)$, and a unital representation should map invertibles onto invertibles.

In classical complex analysis, domains of holomorphy are holomorphically convex, and conversely (Cartan-Thullen). Here, the $\mathscr{D}(U)$-convex hull of a compact subset of U will almost never be compact, even if U is a domain of holomorphy. Still, we think that domains of holomorphy should be "limits" of holomorphic polyhedra.

Chapter Three. PI-Algebras of Functions

Let A be a unital Banach algebra, and $a_1, \ldots, a_n \in A$. We have shown that a functional calculus map $\mathscr{D}(U) \to A$ could exist only if the subalgebra of A generated by $a_1, \ldots, a_n$ satisfied a polynomial identity, hence all identities of $i \times i$ matrices for some i (2.7).

Without loss of generality, we shall assume that A itself satisfies the identities of $i \times i$ matrices, i.e., $A \leq M_i$.

Suppose there exists $\varphi: \mathscr{D}(\Omega) \to A$, $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$. φ will then vanish on the ideal of polynomial identities of $i \times i$ matrices, hence on its closure $I_i = \{ f \in \mathscr{D}(\Omega): f = 0 \text{ on } M_i^n \}$ (actually $f = 0$ on M_i^n implies $f = 0$ on M_j^n, $j \leq i$). Thus φ will yield a map $\mathscr{D}(\Omega)/I_i \to A$. The quotient $\mathscr{D}(\Omega)/I_i$ can be expressed as an algebra of functions taking n-tuples of $j \times j$ matrices ($1 \leq j \leq i$) to $j \times j$ matrices, and $\mathscr{D}(\Omega)/I_i \leq M_i$.

In order to attack the question of the existence of φ, we need a satisfactory theory of such quotient algebras, more generally of quotients $\mathscr{D}(U)/\{f \in \mathscr{D}(U): f$ vanishes on U_j for all $j \leq i\}$, where U is open and normal in Ω. Therefore, we are led to investigate PI-algebras of functions, similar to the $\mathscr{D}(U)$, but now $U \subseteq \bigcup_{j=1}^{i} M_j^n$ has only finitely many "levels".

We shall make a simplifying assumption, viz. consider just $U \subseteq M_i^n$ (U has only one level). In the two cases we will study later, this is no loss of generality:

— If $i = 2$, we can always embed (n-tuples of) scalars as (n-tuples of) scalar matrices (Chapter Five)

— When our PI-algebra of functions turns out to be Azumaya of rank i^2, U must be contained in M_i^n (Chapter Four)

Another argument for restricting ourselves to "one-level" algebras is that we can embed all M_j, $1 \leq j \leq i$, in $M_{i!}$ (see also 6.6)

Let us thus take U open in M_i^n.

3.1. DEFINITION.

$$\mathscr{H}(U) = \left\{ f: U \xrightarrow{\text{holom.}} M_i, \, f(x)m = mf(y) \text{ whenever} \right.$$

$$\left. x, y \in U, \, m \in M_i, \, xm = my \right\}.$$

3.2. DEFINITION. $\mathscr{R}(U) =$ the closure in $\mathcal{O}(U, M_i)$ of the algebra R of n generic $i \times i$ matrices.

Clearly $\mathscr{R}(U) \subseteq \mathscr{H}(U)$. In some sense, $\mathscr{R}(U)$ is the smallest algebra we can define, if we demand that it be closed in $\mathcal{O}(U, M_i)$ and contain the generic matrices. We also expect every function of n $i \times i$ matrices to satisfy the intertwining condition in (3.1), so $\mathscr{H}(U)$ is the largest algebra we can reasonably work with.

Neither of these two definitions is totally satisfying, (3.1) in view of example (1.7), (3.2) because we do not expect all domains U to be Runge. But both will prove to be useful in the sequel. In the next two chapters, we shall show, under suitable assumptions on U, that $\mathscr{R}(U) = \mathscr{H}(U)$. Let us already mention

3.3. PROPOSITION. $\mathscr{R}(M_2^2) = \mathscr{H}(M_2^2)$.

Proof. Let $f \in \mathscr{H}(M_2^2)$; f is the sum of its Taylor series (as f: $\mathbf{C}^8 \to M_2$), i.e., $f = \sum_{|\mu|=0}^{\infty} c_\mu t^\mu$ where $c_\mu \in M_2$, μ is a commutative multi-

index, and t is a 8-tuple of complex numbers (the 8 entries of the matrices X_1 and X_2). Since the intertwining condition (3.1) is linear, $f_k = \sum_{|\mu|=0}^{k} c_\mu t^\mu \in R$ (the algebra of 2 generic 2×2 matrices) by (1.5), and $f_k \xrightarrow{k \to \infty} f$, so $f \in \mathscr{R}(M_2^2)$. The reverse inclusion is obvious.

This result will be generalized later (4.17). Note that the analog of (3.3) fails when $n > 2$ or $i > 2$, because of (1.7).

Chapter Four. The Implicit Function Theorem

Algebras of holomorphic functions on polynomially convex sets in $\mathbf{C}^n$ were studied by H. Cartan and K. Oka in the 1940's. They yield one road (Waelbroeck) to the commutative functional calculus [8], while another approach (Shilov, Arens-Calderon) uses the Weil integral formula. We do not see any generalization of the Weil formula to functions of several noncommuting variables. On the other hand, Taylor [15, p. 23] proved an "implicit function theorem" that will play the role of the Oka-Cartan results.

4.1. PROPOSITION. *Let $\mathscr{F} = \mathscr{F}(r_1, \ldots, r_n, s_1, \ldots, s_q)$ be a power series algebra (2.1) in the indeterminates $X_1, \ldots, X_n, Y_1, \ldots, Y_q$. Let $p_1, \ldots, p_q$ be free polynomials in $X_1, \ldots, X_n$ and let I be the closed ideal generated in $\mathscr{F}$ by $p_1 - Y_1, \ldots, p_q - Y_q$.*

Then $\mathscr{E} = \mathscr{F}/I$ is a free analytic algebra in the indeterminates $X_1, \ldots, X_n$, provided $I \neq \mathscr{F}$ (we do not distinguish between X_k in $\mathscr{F}$ and its class $\mod I$, $k = 1, \ldots, n$).

Proof. [**15**, p. 26 (Example 3)].

Recall that algebras of power series do map into Banach algebras (2.4), but not all functions in $\mathscr{D}(\Omega)$ have convergent power series expressions (2.9). However, this very example (2.9) indicates that power series algebras and matricial function algebras might coincide after factoring out the identities of $i \times i$ matrices. Our goal is to prove this, under suitable hypotheses. The idea is to map quotients of power series algebras into PI-algebra of functions (Chapter Three) by a "Gelfand transformation" and show that the homomorphism must be an isomorphism.

We want to map $\mathscr{E} = \mathscr{F}/I$ (4.1) into an algebra $\mathscr{H}(U)$ (3.1). Let us study $\mathscr{E}$ by looking at its finite-dimensional representations (a representation is always assumed to be continuous and nondegenerate).

4.2. DEFINITION. For $i = 1, 2, 3, \ldots$, we let

$$U_i = \left\{ x = (x_1, \ldots, x_n) \in M_i^n : \text{there is a (unique, 2.13)} \right.$$

$$\left. \text{representation } \mathscr{E} \to M_i, \; X_1 \mapsto x_1, \ldots, X_n \mapsto x_n \right\}.$$

Note that each U_i is open in M_i^n [15, p. 14] and invariant under conjugation. What does U_i look like? Can it be defined by analytic inequalities?

4.3. PROPOSITION. *Let* $\| \; \|$ *be any algebra norm on* M_i. *If* $x = (x_1, \ldots, x_n) \in M_i^n$, *and* $\|x_1\| < r_1, \ldots, \|x_n\| < r_n$, $\|p_1(x)\| < s_1, \ldots, \|p_q(x)\| < s_q$, *then* $x \in U_i$.

Proof. This follows from (2.4)
We have a converse result

4.4. PROPOSITION. *If* $x = (x_1, \ldots, x_n) \in U_i$, *then* $\rho(x_1) < r_1, \ldots, \rho(x_n) < r_n$, $\rho p_1(x) < s_1, \ldots, \rho p_q(x) < s_q$.

Proof. Suppose $x \in U_i$. Let α be the corresponding representation $\mathscr{E} \to M_i$, and $\bar{\alpha}$ its lift $\mathscr{F} \to M_i$, so $\bar{\alpha}(X_1) = x_1$. Now $f = \sum_{k=0}^{\infty} c_k X_1^k \in \mathscr{F}$ if $\sum_{k=0}^{\infty} c_k t_1^k$ converges for all $t_1 \in \mathbf{R}_+$, $t_1 < r_1$, and then $\bar{\alpha}(f) = \sum_{k=0}^{\infty} c_k x_1^k$ converges only if $\rho(x_1) \leq r_1$. As U_i is open we must have $\rho(x_1) < r_1$. Similarly, we can consider power series of a single variable, and show $\rho(x_2) < r_2, \ldots, \rho(x_n) < r_n$, $\rho p_1(x) < s_1, \ldots, \rho p_q(x) < s_q$.

4.5. REMARK. Combining (4.3) and (4.4), we see that, for $x \in M_i^n$,

$$\left. \begin{array}{l} \|x_1\| < r_1, \ldots, \|x_n\| < r_n \\ \|p_1(x)\| < s_1, \ldots, \|p_q(x)\| < s_q \end{array} \right\} \Rightarrow x \in U_i \Rightarrow \left\{ \begin{array}{l} \rho(x_1) < r_1, \ldots, \rho(x_n) < r \\ \rho p_1(x) < r_1, \ldots, \rho p_q(x) < s_q \end{array} \right.$$

which does not completely determine U_i (except when $i = 1$), but does give some control on it. Later, projective limit arguments and the Gelfand-Beurling formula will allow us to obtain functional calculi on domains defined by analytic inequalities (6.4).

4.6. REMARK. If (for example) p_1 is a central polynomial for $i \times i$ matrices, then $p_1(x)$ is a scalar matrix for all $x \in M_i^n$, $\|p_1(x)\| = \rho p_1(x)$, and the condition for p_1 is completely known.

4.7. REMARK. With results like (2.18) in mind, one could believe that actually (4.4) determines U_i, i.e.,

$$U_i = \left\{ x \in M_i^n \colon \rho(x_1) < r_1, \ldots, \rho(x_n) < r_n, \right.$$

$$\left. \rho p_1(x) < s_1, \ldots, \rho p_q(x) < s_q \right\}.$$

This is not the case; let $n = 2$, $q = 0$ (so $\mathscr{E} = \mathscr{F}$), $r_1 = r_2 = 1$, $i = 2$. Then

$$U_2 \subset \left\{ (x_1, x_2) \colon \rho(x_1) < 1, \, \rho(x_2) < 1 \right\}$$

and inclusion is strict: let $f(X_1, X_2) = \sum_{k=0}^{\infty} (X_1 X_2)^k$, $f \in \mathscr{F}$, and $(x_1, x_2) \in U_2$ implies $\rho(x_1 x_2) < 1$, which does not follow from $\rho(x_1) < 1$, $\rho(x_2) < 1$.

Now let i be chosen such that $U_i \neq \varnothing$, but $U_j = \varnothing$ for $j < i$. Note that all we are going to do is trivial for $i = 1$, so the reader may assume $i \geq 2$.

Our strategy is to study a quotient algebra of $\mathscr{E}$ which is Azumaya of constant rank i^2. All irreducible representations of such an algebra will be i-dimensional. Thus, if $U_j = \varnothing$ for $j < i$, but $U_i \neq \varnothing$, we will factor out the identities of $i \times i$ matrices, in order to get rid of all the irreducible representations in dimension greater than i.

Let $U = U_i$. Also note that $U_j = \varnothing$ for $j < i$ implies $U \subseteq \hat{M}_i^n$ (1.9); if $x \in U$, $x \in \check{M}_i^n$, then x corresponds to a reducible i-dimensional representation, which yields a subrepresentation and a quotient representation in dimension less than i, contradicting the emptiness of U_j, $j < i$. We have the "Gelfand transformation": for $f \in \mathscr{E}$, $x \in U$, let $\hat{f}(x) = \varphi(f)$, where φ is the unique representation $\mathscr{E} \to M_i$, $X_1 \mapsto x_1, \ldots, X_n \mapsto x_n$.

4.8. PROPOSITION. *The map $f \mapsto \hat{f}$ defines a continuous homomorphism $\gamma \colon \mathscr{E} \to \mathscr{H}(U) = \mathscr{H}$.*

Proof. [**15**, p. 15].

γ is certainly not injective; $\ker \gamma$ contains J, the closure in $\mathscr{E}$ of the polynomial identities of $i \times i$ matrices.

4.9. DEFINITION. Let $\mathscr{G} = \mathscr{E}/J$.

We obtain a homomorphism $\mathscr{G} \to \mathscr{H}$. We want to prove that $\mathscr{G} \to \mathscr{H}$ is in fact a (topological, as both $\mathscr{G}$ and $\mathscr{H}$ are Fréchet) isomorphism.

We shall prove (4.12, 4.13) that $\mathscr{G}$ and $\mathscr{H}$ are Azumaya of rank i^2. Of course $\mathscr{G} \leq M_i$ (we factored out J which contains the identities of $i \times i$ matrices) and $\mathscr{H} \leq M_i$ ($\mathscr{H}$ is an algebra of M_i-valued functions). All closed maximal ideals of $\mathscr{G}$ and $\mathscr{H}$ correspond to points in $U \subseteq \hat{M}_i^n$, and they have codimension i^2. but we cannot apply Artin-Procesi (1.2) here, since $\mathscr{G}$ and $\mathscr{H}$ have maximal ideals that are not closed. We need the

4.10. PROPOSITION. *Let A be a Fréchet algebra which is Azumaya over its center Z. Then the trace map* [5, III §2] *$T \colon A \to Z$ is continuous.*

Proof. A is a finite Z-module. Take $e_1, \ldots, e_k$ a generating set for A over Z; if $a \in A$, $a = \sum_{j=1}^{k} a_j e_j$ for some $a_1, \ldots, a_k \in Z$. Since T is Z-linear, $T(a) = \sum a_j T(e_j)$. Consider the map $\psi \colon Z^k \to A, (a_1, \ldots, a_k) \mapsto \sum a_j e_j \cdot \psi$ is continuous and surjective, hence open, and $Z^k / \ker \psi \simeq A$ (as Fréchet spaces). Let $T' \colon Z^k \to Z, (a_1, \ldots, a_k) \mapsto T(\sum a_j e_j) = \sum a_j T(e_j)$. T' is continuous and vanishes on $\ker \psi$, yielding a continuous map $T'' \colon Z^k / \ker \psi \to Z$. Now T is just T'' provided we identify A with $Z^k / \ker \psi$. (I am indebted to R. Brooks for this argument.)

Recall that we defined a map $\Pi \colon M_i^n \to \Delta \subseteq \mathbf{C}^p$ which sends $\hat{M}_i^n$ onto $\hat{\Delta}$, $\hat{\Delta}$ smooth (1.10). Since $U \subseteq \hat{M}_i^n$, we have $V = \Pi U \subseteq \hat{\Delta}$, V is a smooth locally closed submanifold of $\mathbf{C}^p$, and

$$Z(\mathscr{H}) = \left\{ f \colon U \overset{\text{holom.}}{\to} \mathbf{C}, \, f(mxm^{-1}) = f(x) \text{ whenever} \right.$$

$$\left. x \in U, \, m \in \mathrm{GL}_i \right\} = \mathcal{O}(V).$$

4.11. PROPOSITION. *V is polynomially convex in $\mathbf{C}^p$.*

Proof. Write $\mathscr{F} = \varprojlim \mathscr{F}_k$, where the $\mathscr{F}_k$ are Banach algebras, and $\mathscr{G} = \varprojlim \mathscr{G}_k$. Then each $\mathscr{G}_k$ is a Banach Azumaya algebra, as shown by (1.2): all maximal ideals of $\mathscr{G}_k$ are closed and have codimension i^2. Now R (the algebra of n generic $i \times i$ matrices) is dense in $\mathscr{G}$, for F_{n+q} was dense in $\mathscr{F}$ (4.1) and density is preserved under quotients, and R is dense in each $\mathscr{G}_k$. The trace map $T \colon \mathscr{G}_k \to Z(\mathscr{G}_k)$ is just (up to a coefficient which we may assume is 1) the usual trace, so T maps R into B, $B \subseteq Z(\mathscr{G}_k)$; since $R \subseteq \mathscr{G}_k$, we see that $BR = S \subseteq \mathscr{G}_k$. As R is dense in $\mathscr{G}_k$, so is S. Furthermore, B is dense in $Z(\mathscr{G}_k)$: if $f \in Z(\mathscr{G}_k)$, $f = Tf$, S dense in $\mathscr{G}_k \Rightarrow f = \lim f_j$ $(f_j \in S)$, T continuous (4.10) $\Rightarrow f = Tf = \lim Tf_j$, and each $Tf_j \in Z(S) = B$. Since B is dense in $Z(\mathscr{G}_k)$, $Z(\mathscr{G}_k)$ is a

finitely generated commutative Banach algebra. Its maximal ideal space V_k can be identified with a compact, polynomially convex subset of $\hat{\Delta}$. Now $V_k \subseteq V$, in fact $V = \bigcup_k V_k$ is an increasing union of compact polynomially convex subsets of $\hat{\Delta}$, and each compact subset of V is contained in some V_k. Therefore V is polynomially convex.

We can now prove

4.12. PROPOSITION. $\mathscr{H}$ is an Azumaya algebra.

Proof. In view of (1.3), it is sufficient to prove that $F(\mathscr{H}) = Z(\mathscr{H})$. We already know that $Z(\mathscr{H}) = \mathcal{O}(V)$. Let $\mathscr{C}$ be the set of central polynomials without constant term. Then $\check{M}_i^n = \bigcap\{ f^{-1}(0)\}$ ($f \in \mathscr{C}$). As $Z(R) \subset B$ and B is Noetherian, we can suppose $\check{M}_i^n = \{ x \in M_i^n : f_1(x) = \cdots = f_r(x) = 0\}$, with $f_1, \ldots, f_r \in B$. These functions $f_1, \ldots, f_r$ can be seen as functions on $\prod M_i^n = \Delta$, they have no common zero on $\hat{\Delta}$, hence no common zero on V. As V is polynomially convex, we can find $g_1, \ldots, g_r \in \mathcal{O}(V)$, with $f_1 g_1 + \cdots + f_r g_r = 1$. This shows $1 \in F(\mathscr{H})$, and $\mathscr{H}$ is Azumaya.

Similarly, we have

4.13. PROPOSITION. $\mathscr{G}$ is an Azumaya algebra.

Proof. The space of closed maximal ideals of $Z(\mathscr{G})$ is V (same proof as in (4.11)). Take $f_1, \ldots, f_r$ as in (4.12). They have no common zero in V, and by a result of Arens [2] we can find $g_1, \ldots, g_r \in Z(\mathscr{G})$ with $f_1 g_1 + \cdots + f_r g_r = 1$. Thus $1 \in F(\mathscr{G})$ and $\mathscr{G}$ is Azumaya.

4.14. REMARK. We have a map $\mathscr{G} \to \mathscr{H}$ between two Azumaya algebras of rank i^2. In order to show that this map is an isomorphism, it is enough to establish the isomorphism on the centers. Injectivity of $Z(\mathscr{G}) \to Z(\mathscr{H})$ implies injectivity of $\mathscr{G} \to \mathscr{H}$, because we have a natural correspondence between ideals of $\mathscr{G}$ and ideals of $Z(\mathscr{G})$. If $Z(\mathscr{G}) \to Z(\mathscr{H})$ is also surjective, then $\mathscr{G} \to \mathscr{H}$ is bijective [12, 1.8.49].

4.15. PROPOSITION. *The map* $Z(\mathscr{G}) \to Z(\mathscr{H})$ *is an isomorphism.*

Proof. Again, write $\mathscr{G} = \varprojlim \mathscr{G}_k$ and $Z(\mathscr{G}) = \varprojlim Z(\mathscr{G}_k)$. The maximal ideal space V_k of $Z(\mathscr{G}_k)$ is a compact polynomially convex subset of V. By a form of the (commutative) holomorphic functional calculus due to Allan [1], there is a continuous homomorphism $\mathcal{O}(V) \to Z(\mathscr{G}_k)$ which is

the identity on B. Passing to the projective limit, we obtain a map $\mathcal{O}(V) \to Z(\mathcal{G})$. We also have a map (Gelfand) $Z(\mathcal{G}) \to Z(\mathcal{H}) = \mathcal{O}(V)$, and the composition $Z(\mathcal{G}) \to Z(\mathcal{H}) \to Z(\mathcal{G})$ is the identity on B, which is dense in $Z(\mathcal{G})$. Hence $Z(\mathcal{G}) \simeq Z(\mathcal{H})$.

From this, we obtain, in view of (4.14)

4.16. THEOREM. *The map $\mathcal{G} \to \mathcal{H}$ is an isomorphism.*

This theorem has several consequences

4.17. COROLLARY. $\mathcal{R}(U) = \mathcal{H}(U)$ *(recall (3.1, 3.2)) (for R is dense in $\mathcal{H}(U) = \mathcal{G}$).*

4.18. COROLLARY. *The ideal J (see 4.9) is equal to $\ker \gamma$ (for $\mathcal{H}(U)$ is separated by its i-dimensional representations).*

4.19. PROPOSITION. *There is a holomorphic bundle $\mathcal{M}$ over V with fiber M_i such that $\mathcal{H}$ is isomorphic with $\Gamma(V, \mathcal{M})$ the algebra of holomorphic sections of $\mathcal{M}$ over V.*

Proof. Cover U by open sets U_α such that $U_\alpha \simeq \Pi(U_\alpha) \times \mathrm{PGL}_i$ ($\simeq$ as complex manifolds). Let $V_\alpha = \Pi U_\alpha$. On U_α, Π has a holomorphic section $\sigma_\alpha \colon V_\alpha \to U_\alpha$, $\Pi(\sigma_\alpha(z)) = z$ for all z in V_α. For $f \in \mathcal{H}(U)$, define $f_\alpha \colon V_\alpha \to M_i$ by $f_\alpha(z) = f(\sigma_\alpha(z))$ ($z \in V_\alpha$). We get homomorphisms $\mathcal{H}(U) \to \mathcal{O}(V_\alpha, M_i)$, $f \mapsto f_\alpha$. Pasting these, we obtain a bundle $\mathcal{M}$ over V and a map $\mathcal{H}(U) \to \Gamma(V, \mathcal{M})$. This map induces an isomorphism on the centers, and both algebras are Azumaya. Thus $\mathcal{H}(U) \simeq \Gamma(V, \mathcal{M})$.

Chapter Five. Two Generic 2 × 2 Matrices

Let R be the algebra of generic 2×2 matrices generated by X_1 and X_2. The structure of R is described in [7]. As generators for the algebra of invariants B, we take $z_1 = \mathrm{tr}\, X_1$, $z_2 = \det X_1$, $z_3 = \mathrm{tr}\, X_2$, $z_4 = \det X_2$, $z_5 = \mathrm{tr}\, X_1 X_2$, so $B \simeq \mathbb{C}[z_1, z_2, z_3, z_4, z_5]$. Note that

$$\det X_1 = \tfrac{1}{2}\left((\mathrm{tr}\, X_1)^2 - \mathrm{tr}\, X_1^2\right), \ldots,$$

so all invariants could be expressed in terms of traces (1.8).

The algebra of concomitants S is a free B-module of dimension 4: $S = B + BX_1 + BX_2 + BX_1 X_2$. We shall also need the commutator ideal of R (cf. 1.5)), $[R, R] = [S, S] = S(X_1 X_2 - X_2 X_1)$.

We let $\tau = X_1 X_2 - X_2 X_1$, $\sigma = \tau^2 = (X_1 X_2 - X_2 X_1)^2$, and $Z(R) = \mathbf{C} + B\sigma$ (see 1.6).

We now find the lmc completion of R. As $\mathscr{R} = \mathscr{R}(M_2^2)$ is lmc and complete, we have an injective map $\tilde{R} \to \mathscr{R}$, and we shall show that it is a (topological, as both $\tilde{R}$ and $\mathscr{R}$ are Fréchet) isomorphism. Note that $\tilde{R}$ and $\mathscr{R}$ are not Azumaya (they have irreducible representations in dimension 1 and 2), so the method of Chapter Four (reduction to the center) will not do. Instead we are going to use a Five lemma argument.

We have a diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \to & [\tilde{R}, \tilde{R}] & \to & \tilde{R} & \to & \tilde{R}/[\tilde{R}, \tilde{R}] & \to & 0 \\
& & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\
0 & \to & [\mathscr{R}, \mathscr{R}] & \to & \mathscr{R} & \to & \mathscr{R}/[\mathscr{R}, \mathscr{R}] & \to & 0
\end{array}
$$

where $[\tilde{R}, \tilde{R}]$ and $[\mathscr{R}, \mathscr{R}]$ are the closed commutator ideals of $\tilde{R}$ and $\mathscr{R}$. If we can prove that both α and γ are isomorphisms, it will follow that β is bijective too.

5.1. PROPOSITION. γ *is an isomorphism.*

Proof. Both $R/[\tilde{R}, \tilde{R}]$ and $\mathscr{R}/[\mathscr{R}, \mathscr{R}]$ are commutative Fréchet algebras generated by (the classes of) X_1 and X_2, and we have

$$
\begin{array}{ccc}
 & \tilde{R}/[\tilde{R}, \tilde{R}] & \\
\mathscr{O}(\mathbf{C}^2) \nearrow & & \searrow \mathscr{O}(\mathbf{C}^2) \\
\searrow & & \nearrow \\
 & \mathscr{R}/[\mathscr{R}, \mathscr{R}] &
\end{array}
$$

where the maps on the left are holomorphic functional calculi, and the maps on the right are Gelfand transforms. It follows that $\mathscr{O}(\mathbf{C}^2) \simeq \tilde{R}/[\tilde{R}, \tilde{R}] \simeq \mathscr{R}/[\mathscr{R}, \mathscr{R}]$.

Now we have to show that α is an isomorphism. We let $\mathscr{B} = \tilde{B} \simeq \mathscr{O}(\mathbf{C}^5)$, and define $\mathscr{S} = \mathscr{B} + \mathscr{B}X_1 + \mathscr{B}X_2 + \mathscr{B}X_1X_2$.

5.2. PROPOSITION. $\mathscr{S}$ *is a free $\mathscr{B}$-module.*

Proof. If $a + bX_1 + cX_2 + dX_1X_2 = 0$, with $a, b, c, d \in \mathscr{B}$, write four scalar equatins (corresponding to the four matrix entries) and observe that the determinant of the system is $\sigma = (X_1X_2 - X_2X_1)^2$. Since σ is not a

zero-divisor, $a = b = c = d = 0$ is the only solution.

Next let

$$\mathcal{T} = \left\{ f \colon M_2^2 \overset{\text{hol.}}{\to} M_2, \; f(mxm^{-1}) = mf(x)m^{-1} \right.$$

$$\left. \text{for all } x \in M_2^2, \, m \in \mathrm{GL}_2 \right\}.$$

5.3. PROPOSITION. $\mathcal{S} = \mathcal{T}$.

Proof. Clearly $\mathcal{S} \subseteq \mathcal{T}$, and $Z(\mathcal{S}) = Z(\mathcal{T}) = \mathcal{B}$. Let $f \in \mathcal{T}$. An argument similar to (3.3) allows us to write f as the sum of a power series, all partial sums of which belong to $S = B + BX_1 + BX_2 + BX_1X_2$. Thus we can write $f = a + bX_1 + cX_2 + dX_1X_2$, where a, b, c, d are formal power series in z_1, z_2, z_3, z_4, z_5. On the other hand, a, b, c, d can also be computed by solving a 4×4 system with entries in $\mathcal{B}$ and determinant σ, so by Cramer's rule a, b, c, d will be meromorphic functions: $a = A/\sigma$, ($A \in \mathcal{B}$), etc. But the fact that a has a formal power series forces a to be holomorphic on $\mathbf{C}^5$, i.e. $a \in \mathcal{B}$. Similarly, $b, c, d \in \mathcal{B}$ and $f \in \mathcal{S}$.

We let $\mathcal{K} = \mathcal{S}\tau$, and claim that $\mathcal{K} = [\tilde{R}, \tilde{R}] = [\mathcal{R}, \mathcal{R}]$. Actually, since $[\tilde{R}, \tilde{R}] \subseteq [\mathcal{R}, \mathcal{R}]$ we need only show that $\mathcal{K} \subseteq [\tilde{R}, \tilde{R}]$ and $[\mathcal{R}, \mathcal{R}] \subseteq \mathcal{K}$.

5.4. PROPOSITION. $\mathcal{K} \subseteq [\tilde{R}, \tilde{R}]$.

Proof. It is enough to show that $\mathcal{B}\tau \subseteq [\tilde{R}, \tilde{R}]$.

Let ν be any submultiplicative seminorm on R. Without loss of generality, we can suppose ν is a norm. Let E be the completion of $B\tau$ with respect to ν. E is a Banach space, and $\mathcal{L}(E)$ is a Banach algebra. We have an embedding $\psi \colon B \to \mathcal{L}(E)$, given by ($f \in B$):

$$\psi(z_1)(f\tau) = X_1 f\tau + f\tau X_1$$

$$\psi(z_2)(f\tau) = X_1 f\tau X_1$$

$$\psi(z_3)(f\tau) = X_2 f\tau + f\tau X_2$$

$$\psi(z_4)(f\tau) = X_2 f\tau X_2$$

$$\psi(z_5)(f\tau) = X_1 X_2 f\tau + f\tau X_1 X_2 \qquad (\text{see } [7]).$$

These relations show that $\psi(z_1)$, $\psi(z_2)$, $\psi(z_3)$, $\psi(z_4)$, $\psi(z_5)$ are ν-continuous ($B\tau \to B\tau$), hence extend to continuous linear maps $E \to E$. Let A be the closure of $\psi(B)$ in $\mathcal{L}(E)$. A is a finitely generated

commutative Banach algebra. At this point, we can employ the (commutative) holomorphic functional calculus: there is a continuous homomorphism $\mathcal{O}(\mathbf{C}^5) = \mathscr{B} \to A$, $1 \mapsto 1$, $z_1 \mapsto \psi(z_1)$, $z_2 \mapsto \psi(z_2)$, $z_3 \mapsto \psi(z_3)$, $z_4 \mapsto \psi(z_4)$, $z_5 \mapsto \psi(z_5)$. Hence, if $f \in \mathscr{B}$, the linear map $\psi(f)$ will be in A, it will send τ onto $f\tau \in E$. Thus $\mathscr{B}\tau \subseteq E$. This holds for any choice of ν, and $\mathscr{B}\tau \subseteq \overline{B\tau}$, the closure of $B\tau$ in $[\tilde{R}, \tilde{R}]$ (i.e., in $\tilde{R}$). We can multiply by X_1, X_2 and $X_1 X_2$, proving that $\mathscr{K} \subseteq [\tilde{R}, \tilde{R}]$.

5.5. PROPOSITION. $[\mathscr{R}, \mathscr{R}] \subseteq \mathscr{K}$.

Proof. $[\mathscr{R}, \mathscr{R}] = [\mathscr{H}, \mathscr{H}] \subseteq [\mathscr{T}, \mathscr{T}] = [\mathscr{S}, \mathscr{S}] = \mathscr{K}$.

Combining (5.4) and (5.5), we have

5.6. PROPOSITION. $[\tilde{R}, \tilde{R}] = [\mathscr{R}, \mathscr{R}]$, so α is an isomorphism.

Now, by the Five lemma, (5.1) and (5.6) yield

5.7. THEOREM. β *is an isomorphism, and* $\tilde{R} = \mathscr{R}(M_2^2) = \mathscr{H}(M_2^2)$.

5.8. REMARK. We have not been able to find the lmc completion of the algebra of n generic $i \times i$ matrices for $n > 2$ or $i > 2$. We believe that it should equal $\mathscr{R}(M_i^n)$, i.e., that $\mathcal{O}(M_i^n, M_i)$ induces the finest lmc topology on the algebra of n generic $i \times i$ matrices. But it is only for $n = i = 2$ that we have enough information on the generic matrix algebra (given in [7]) to prove it. The structure of the other generic matrix algebras is less well understood.

As an application of the above result, let us consider the following situation. A is a topological algebra and $Z(A)$ its center. We have maps

$$
\begin{array}{ccc}
Z(A) & \to & \overline{Z(A)} \\
\downarrow & & \uparrow \\
A & \to & \tilde{A}
\end{array} \searrow Z(\tilde{A}) \nwarrow
$$

and the homomorphism $\overline{Z(A)} \to Z(\tilde{A})$ need not be one-to-one or onto, as shown by (5.9, 5.11).

5.9. PROPOSITION. *If A is the algebra of two generic 2×2 matrices, the map $\overline{Z(A)} \to Z(\tilde{A})$ is not surjective.*

Proof. $\tilde{A} = \mathscr{H}(M_2^2) = \mathscr{R}(M_2^2)$ (5.7), and (cf. 1.6) $Z(\tilde{A}) = \mathbf{C} + \mathcal{O}(\mathbf{C}^5)\sigma$. Recall that $Z(A) = \mathbf{C} + B\sigma$. Endow $\mathbf{C}^5$ with a norm, and let

$T = \{ z \in \mathbf{C}^5 \colon |\sigma(z)| \leq \exp(-\|z\|) \}$. T is ia "rapidly decreasing" neighborhood of $\{ z \in \mathbf{C}^5 \colon \sigma(z) = 0 \}$. T is not compact. Nevertheless, every $f \in Z(A)$ is bounded on T: $f \in Z(A) \Rightarrow f = a + b\sigma$ ($a \in \mathbf{C}$, $b \in B = \mathbf{C}[z_1, z_2, z_3, z_4, z_5]$); $|f(z)| \leq |a| + |b(z)|\exp(-\|z\|)$ for $z \in T$. We can define a submultiplicative seminorm on $Z(A)$ by $\nu(f) = \sup_T |f(z)|$; ν does not extend to $Z(\tilde{A})$, as $g(z) = \sigma(z)\exp(z_1^2)$ is unbounded on T. So $\widetilde{Z(A)}$ is a proper subalgebra of $Z(\tilde{A})$.

5.10. REMARK. If it were true that $\widetilde{Z(A)} = \mathbf{C} + \mathcal{O}(\mathbf{C}^5)\sigma$, then it would be possible to show (using localization, in the ring theoretic sense) that $A = \mathcal{H}(M_2^2)$. This is the case, but for different reasons.

Here is a different phenomenon.

5.11. PROPOSITION. *If A is the enveloping algebra of* $\mathrm{sl}(2, \mathbf{C})$, *then the map* $\widetilde{Z(A)} \to Z(\tilde{A})$ *is not injective.*

Proof. By [**14**, p. 250], $\tilde{A}$ is a direct product of matrix algebras. We know that A has one (up to equivalence) irreducible representation in each dimension 1, so $\tilde{A} \simeq \prod_{i=1}^{\infty} M_i$, and $Z(\tilde{A}) \simeq \mathbf{C}^{\mathbf{N}}$ is the algebra of sequences of complex numbers. Now $Z(A) = \mathbf{C}[\Omega]$ (Ω is "the" Casimir element), and $\widetilde{Z(A)} \simeq \mathcal{O}(\mathbf{C})$, the entire functions of Ω. We have the map $\varphi \colon \widetilde{Z(A)} \to Z(\tilde{A})$. Looking at the infinitesimal character, we see that φ sends the function $\Omega \in \mathcal{O}(\mathbf{C})$ to the sequence $(0, 3/4, 2, \ldots, (n^2 - 1)/4, \ldots) \in \mathbf{C}^{\mathbf{N}}$. Hence

$$\varphi(f) = \big(f(0), f(3/4), \ldots, f((n^2 - 1)/4), \ldots \big).$$

By the Weierstrass theorem [**8**, 1.5.4], φ is a surjection $\mathcal{O}(\mathbf{C}) \to \mathbf{C}^{\mathbf{N}}$. φ is not injective, for its kernel contains $f(\Omega) = \sin 4\pi\Omega$.

Chapter Six. The Functional Calculus

The results of Chapter Four will now yield a functional calculus for topological subquotients of Banach Azumaya algebras, giving a positive answer to Taylor's question: if A is a separated quotient of a closed (unital) subalgebra of a Banach Azumaya algebra, if $a = (a_1, \ldots, a_n) \in A^n$, then there is a continuous algebra homomorphism $\mathcal{D}(\Omega) \to A$ that sends $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$. If we suppose that A is a closed subalgebra of a Banach Azumaya algebra, we can even get a functional calculus on a polynomial polyhedron. We will also give a result valid for two variables in any Banach algebra satisfying all identities of 2×2 matrices.

First let A be a Banach Azumaya algebra of rank i^2 over its center $Z(A)$. Take $(a_1, \ldots, a_n) = a \in A^n$. Our idea is the following: choosing suitable free polynomials $p_1, \ldots, p_q$, radii $r_1, \ldots, r_n$, $s_1, \ldots, s_q$, using the implicit function theorem of Chapter Four, we will get a homomorphism $\mathscr{G} \to \mathscr{H}(U)$ which, provided $\mathscr{G}$ and $\mathscr{H}$ are Azumaya, will be an isomorphism (4.16). Since $\mathscr{G}$ maps into A, we will obtain a continuous homomorphism $\mathscr{H} \to A$. But things are not that simple. Since all irreducible representations of A are i-dimensional, we have have (see 1.13) that $\mathrm{sp}_j(a) = \varnothing$ for $j < i$. This does not imply that $\mathrm{sp}_i(a) \subseteq \hat{M}_i^n$ (a trivial counterexample: let $a_1 = \cdots = a_n = 0$). Even if $\mathrm{sp}_i(a) \subseteq \hat{M}_i^n$, we do not know that $p\mathrm{sp}_i(a) \subseteq \hat{M}_i^n$ (see 1.15), and even if $p\mathrm{sp}_i(a) \subseteq \hat{M}_i^n$, can we find suitable $p_1, \ldots, p_q$ and corresponding radii such that $U_i = U \subseteq \hat{M}_i^n$? If this can be done, then the machinery of Chapter Four will apply. The difficulties can be overcome. As A is Azumaya, $Z(A) = F(A)$ (1.3); in particular, $1 \in F(A)$, there exists P_0, a free polynomial without constant term which is central for $i \times i$ matrices, and $b_1, \ldots, b_m \in A$, such that $P_0(b_1, \ldots, b_m) = P_0(b) = 1$. Let $p_0 = 1 - P_0$. Let us use the construction of Chapter Four, $p_1, \ldots, p_q$ being free polynomials in $X_1, \ldots, X_n$, p_0 a free polynomial in $X_1', \ldots, X_m'$. We choose $r_1 > \|a_1\|, \ldots, r_n > \|a_n\|$, $s_1 > \|p_1(a)\|, \ldots, s_q > \|p_q(a)\|$. We take $\mathscr{F}^+$ to be the free power series algebra in $X_1, \ldots, X_n, X_1', \ldots, X_n', Y_1, \ldots, Y_q, Y_0$, with the radii $r_1, \ldots, r_n$, $\infty, \ldots, \infty$, $s_1, \ldots, s_q$, 1. We have a continuous homomorphism (2.4) $\mathscr{F}^+ \to A$, $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$, $X_1' \mapsto b_1, \ldots, X_m' \mapsto b_m$, $Y_1 \mapsto p_1(a), \ldots, Y_q \mapsto p_q(a)$, $Y_0 \mapsto 0 = 1 - P_0(b) = p_0(b)$. This homomorphism vanishes on the closed ideal generated in $\mathscr{F}^+$ by $Y_0 - p_0$, $Y_1 - p_1, \ldots, Y_q - p_q$, and by all polynomial identities of $'$ Letting $\mathscr{G}^+$ be the quotient, we obtain a map $\mathscr{G}^+ \to A$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$, $X_1' \mapsto b_1, \ldots, X_m' \mapsto b_m$. Let

$$U^+ = \left\{ (x_1, \ldots, x_n, x_1', \ldots, x_m') \in M_i^{n+m} : \text{there is a representation} \right.$$

$$\left. \mathscr{G}^+ \to A, \ X_1 \mapsto x_1, \ldots, X_n \mapsto x_n, \ X_1' \mapsto x_1', \ldots, X_m' \mapsto x_m' \right\}.$$

Now $U^+ = U \times U'$, where $U \subseteq M_i^n$, $U' \subseteq M_i^m$; in fact (4.6) $U' = \{(x_1', \ldots, x_m') \in M_i^m : |p_0(x')| < 1\}$, and $U' \subseteq \hat{M}_i^m$ (for $P_0 = 1 - p_0$ vanishes on $\check{M}_i^m$), hence $U \subseteq \hat{M}_i^{n+m}$. By (4.16), the map $\mathscr{G}^+ \to \mathscr{H}^+ = \mathscr{H}(U^+)$ is an isomorphism. This shows

6.1. PROPOSITION. *There is a unique continuous homomorphism* $\mathscr{H}(U^+)$ $\to A$, $1 \mapsto 1$, $X_1 \mapsto a, \ldots, X_n \mapsto a_n$, $X_1' \mapsto b_1, \ldots, X_m' \mapsto b_m$. (*Uniqueness follows from polynomial density* (4.17)).

This is not satisfactory yet, because

(a) we do not know U^+ in terms of analytic inequalities;

(b) we are interested in functions of $a, \ldots, a_n$, and $b_1, \ldots, b_m$ have been thrown in.

In order to get rid of the $b_1, \ldots, b_m$, observe that $\mathscr{H}(U^+) = \mathscr{R}(U^+)$. Restrict the map $\mathscr{H}(U^+) \mapsto A$ to the algebra of n generic matrices generated by $X_1, \ldots, X_n$, and extend it by continuity to the closure, viz. $\mathscr{R}(U)$ (recall that U^+ is a product domain). This implies

6.2. PROPOSITION. *There is a (unique) continuous homomorphism* $\mathscr{R}(U) \to A, 1 \mapsto 1, X_1 \mapsto a_1, \ldots, X_n \mapsto a_n.$

All we know about U is (4.5). We let

$$U^* = \left\{ x \in M_i^n : \rho(x_1) < r_1, \ldots, \rho(x_n) \leq r_n, \right.$$

$$\left. \rho p_1(x) < s_1, \ldots, \rho p_q(x) < s_q \right\}$$

Then $U^* \supseteq U$ (4.4), which gives a homomorphism $\mathscr{R}(U^*) \to \mathscr{R}(U)$, and (6.2) yields

6.3. PROPOSITION. *There is a unique continuous homomorphism* $\mathscr{R}(U^*) \to A, 1 \mapsto 1, X_1 \mapsto a_1, \ldots, X_n \mapsto a_n.$

The proposition is not quite optimal: we chose $r_1 > \|a_1\|, \ldots, r_n > \|a_n\|$, $s_1 > \|p_1(a)\|, \ldots, s_q > \|p_q(a)\|$; and we define U^* by spectral radii. Fortunately, we can express everything in terms of spectral radii. Take $\bar{r}_1 > \rho(a_1), \ldots, \bar{r}_n > \rho(a_n), \bar{s}_1 > \rho p_1(a), \ldots, \bar{s}_q > \rho p_q(a)$; then for k large enough, $\|a_1^k\| < \bar{r}_1^k, \ldots, \|a_n^k\| < \bar{r}_n^k, \|p_1^k(a)\| < \bar{s}_1^k, \ldots, \|p_q^k(a)\| < \bar{s}_q^k$. We just repeat the whole construction, but with $(n + q)$ extra polynomials $X_1^k, \ldots, X_n^k, p_1^k, \ldots, p_q^k$, and radii $\bar{r}_1^k, \ldots, \bar{r}_n^k, \bar{s}_1^k, \ldots, \bar{s}_q^k$. Since $\rho(x_1^k) = \rho(x_1)^k, \ldots, (6.3)$ becomes

6.4. THEOREM. *If*

$$W = \left\{ x \in M_i^n : \rho(x_1) < \bar{r}_1, \ldots, \rho(x_n) < \bar{r}_n, \right.$$

$$\left. \rho p_1(x) < \bar{s}_1, \ldots, \rho p_q(x) < \bar{s}_q \right\},$$

there is a unique continuous homomorphism $\mathscr{R}(W) \to A, X_1 \mapsto a_1, \ldots, X_n \mapsto a_n.$

This gives the functional calculus on polynomial polyhedra for Banach Azumaya algebras of constant rank. Now let A_0 be a closed unital subalgebra of the Banach algebra A, Azumaya of rank i^2. Let $a, \ldots, a_n \in A_0$. We have

6.5. PROPOSITION. *With W as in (6.4), there is a unique continuous homomorphism $\mathscr{R}(W) \to A_0$, $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$.*

Proof. This follows immediately from (6.4), since A_0 is closed in A.

6.6. PROPOSITION. *If A_0 is a Banach Azumaya algebra, it is a closed subalgebra of a Banach Azumaya algebra of constant rank.*

Proof. Let $A_0 = \oplus_{j=1}^i A_j$, A_j Azumaya of rank j^2. Let $k = i!$ We can embed A_j as a closed subalgebra of $M_{k/j}(A_j)$ which is Azumaya of rank k^2, and $A_0 \simeq \oplus_{j=1}^i M_{k/j}(A_j)$; the embedding has closed range, for the sum is direct.

6.7. COROLLARY. *Proposition 6.5 holds for closed subalgebras of Banach Azumaya algebras (of nonconstant rank).*

Now let A be a Banach Azumaya algebra, A_0 a closed subalgebra of A, and A_1 the quotient of A_0 by a closed ideal. Let $(a, \ldots, a_n) \in A_1^n$. Lift it to $(\bar{a}_1, \ldots, \bar{a}_n) \in A_0^n$.

6.8. PROPOSITION. *If $r_1 > \rho(\bar{a}_1), \ldots, r_n > \rho(\bar{a}_n)$, $s_1 > \rho p_1(\bar{a}), \ldots, s_q > \rho p_q(\bar{a})$, if*

$$W = \left\{ x \in M_i^n: \rho(x_1) < r_1, \ldots, \rho(x_n) < r_n, \right.$$

$$\left. \rho p_1(x) < s_1, \ldots, \rho p_q(x) < s_q \right\},$$

there is a unique continuous homomorphism $\mathscr{R}(W) \to A_1$, $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$.

Proof. Compose $\mathscr{R}(W) \to A_0$ (6.5) with the quotient map $A_0 \to A_1$.

This proposition is not fully satisfactory: we had to take $r_1 > \rho(\bar{a}_1), \ldots$. We do not know if it holds with $r_1 > \rho(a_1), \ldots$. Anyway, (6.8) gives us the

6.9. COROLLARY. *If A_1 is a topological subquotient of a Banach Azumaya algebra of rank i^2, and $a, \ldots, a_n \in A_1$, there is a unique continuous homomorphism $\mathscr{R}(M_i^n) \to A_1$, $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$.*

We are ready for a partial answer to Taylor's question.

6.10. PROPOSITION. *If A_1 is a topological subquotient of a Banach Azumaya algebra, if $a, \ldots, a_n \in A_1$, there is a unique continuous homomorphism $\mathscr{D}(\Omega) \to A_1$, $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \to a_n$.*

Proof. Compose $\mathscr{D}(\Omega) \to \mathscr{R}(M_i^n) \to A_1$. To see that $\mathscr{D}(\Omega) \to \mathscr{R}(M_i^n)$, combine (2.11) and the argument of (3.3).

6.11. REMARK. It is not known if every finitely generated Banach PI-algebra is a topological subquotient of a Banach Azumaya algebra. In a purely algebraic setting, every algebra with n generators that satisfies some identity is a quotient of the algebra R of n generic $i \times i$ matrices for some i (1.1), and R can be presented as a subalgebra of a matrix (hence Azumaya) algebra. In the Banach case, the best we can say is

6.12. PROPOSITION. *Let $A \leq M_i$ be a unital Banach algebra. Assume $a \in F(A)$ and a is not a topological divisor of zero. Then A can be embedded as a closed subalgebra of a Banach algebra B which is Azumaya of rank i^2.*

Proof. Since a is not a topological divisor of zero, there is a Banach algebra B that isometrically contains A and where a is invertible [3] (here A is not commutative, but a is central, and Aren's proof works in our case). Now $a \in F(B)$, a invertible in B imply $1 \in F(B)$, and B is Azumaya.

We now employ the results of Chapter Five.
Let A be a unital Banach algebra, $A \leq M_2$; take $(a_1, a_2) \in A^2$.

6.13. PROPOSITION. *There is a unique continuous homomorphism $\mathscr{R}(M_2^2) \to A$, $1 \mapsto 1$, $X_1 \mapsto a_1$, $X_2 \mapsto a_2$.*

Proof. This is just a restatement of (5.7).

6.14. COROLLARY. *There is a continuous homomorphism $\mathscr{D}(\Omega) \to A$, $1 \mapsto 1$, $X_1 \mapsto a_1$, $X_2 \mapsto a_2$.*

Let us summarize (2.7, 6.10, 6.14).

6.15. THEOREM. *Let A be a unital Banach algebra; $a, \ldots, a_n \in A$. In order for a continuous homomorphism $\mathscr{D}(\Omega) \to A$, $1 \mapsto 1$, $X_1 \to a_1, \ldots, X_n \mapsto a_n$ to exist, it is necessary that B, the closed subalgebra of A generated by 1, $a, \ldots, a_n$ satisfy a polynomial identity; it is sufficient that B be a topological subquotient of a Banach Azumaya algebra or that $n = 2$ and $B \leq M_2$. We conjecture that the necessary condition is also sufficient.*

In Chapter One, we defined various notions of spectrum, including the "polynomial spectrum" (1.15). We can restate (6.5) in terms of p sp. First we need a

6.16. PROPOSITION. *Let A be a unital Banach PI-algebra, and $b \in A$. Then $\rho(b) = \sup\{ \rho\varphi(b) : \varphi$ irreducible representation of $A\}$.*

Proof. It is obvious that $\rho(b) \geq \rho\varphi(b)$ for all φ. Now there is $s \in \mathbf{C}$, $|s| = \rho(b)$, such that $(b - s)$ is not invertible, hence not invertible on either side [9, II.4.3]. Thus $A(b - s)$ is a proper left ideal, which is contained in a maximal left ideal I. This gives an irreducible (finite-dimensional, 1.12) representation φ of A. As $\varphi(b - s) = \varphi(b) - s$ is not invertible, $\rho\varphi(b) \geq |s| = \varphi(b)$.

Take $p_1, \ldots, p_q$ free polynomials, $r_1, \ldots, r_n$, $s_1, \ldots, s_q > 0$. An open set of the form

$$W = \Big\{ x \in M_i^n : \rho(x_1) < r_1, \ldots, \rho(x_n) < r_n,$$

$$\rho p_1(x) < s_1, \ldots, \rho p_q(x) < s_q\Big\}$$

is called a polynomial polyhedron.

6.17. PROPOSITION. *Let A be a Banach algebra that is Azumaya of rank i^2 over its center. Let $(a, \ldots, a_n) = a \in A^n$. If $W \subseteq M_i^n$ is a polynomial polyhedron that contains $\mathrm{sp}_i(a) = f\,\mathrm{sp}_i(a)$, there is a continuous homomorphism $\mathscr{R}(W) \to A$, $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \to a_n$.*

Proof. Recall (6.4). All we have to show is that $W \supseteq \mathrm{sp}_i(a)$ implies $r_1 > \rho(a_1), \ldots, r_n > \rho(a_n), s_1 > \rho p_1(a), \ldots, s_q > \rho p_q(a)$. By (6.16), $\rho(a_1) = \sup\{\rho\varphi(a_1) : \varphi$ irreducible$\}$. So

$$\mathrm{sp}_i(a) \subseteq W \Rightarrow (\varphi(a_1), \ldots, \varphi(a_n)) \in W \quad (\text{all } \varphi)$$

$$\Rightarrow \rho\varphi(a_1) < r_1 \quad (\text{all } \varphi)$$

$$\Rightarrow \rho(a_1) < r_1, \ldots.$$

Here is the corresponding formulation of (6.5). Let A_0 be a closed subalgebra of A, a Banach Azumaya of rank i^2. In view of (6.6), we can assume that the dimension of all irreducible representations of A_0 divides i (this is automatic when $i = 2$). Take $a = (a, \ldots, a_n) \in A_0^n$.

6.18. PROPOSITION. *If W is a polynomial polyhedron that contains $f \mathrm{sp}_i(a)$, there is a continuous homomorphism $\mathscr{R}(W) \to A_0$, $1 \mapsto 1$, $X_1 \mapsto a_1, \ldots, X_n \mapsto a_n$.*

Proof. Similar to (6.17), as every irreducible representation yields a i-dimensional representation.

6.19. REMARK. In the commutative case, the Arens-Calderon trick [8, 3.2.3] allows us to obtain a functional calculus on (an open neighborhood of) the spectrum. For PI-algebras, it is not clear that we can pass from polynomial polyhedra to arbitrary open sets containing the full spectrum. Every commutative algebra with unit "contains" **C**, so the resolvent equation $\sum_{k=1}^n (a_k - s_k)x_k = 1$ makes sense, and is solvable precisely for $s \notin \mathrm{sp}(a)$. For $A \leq M_i$, there is in general no embedding $M_i \to A$, and the problem should be formulated differently.

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