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EXPOSED POINTS OF LEFT INVARIANT MEANS

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If S is a left amenable semigroup, let $ML(S)$ be the set of left invariant means on $m(S)$, the space of bounded real-valued functions on S. We prove in this paper that a left invariant mean on $m(S)$ is an exposed point of $ML(S)$ if and only if it is the arithmetic average on a minimal finite left ideal of S. In particular, $ML(S)$ has no exposed point when S is an infinite group. We also prove that if $ML(S)$ has an exposed point, then it is the w^*-closed convex hull of all its exposed points. This gives another proof of the Granirer-Klawe theorem on the dimension of $ML(S)$.

1. Introduction. For an arbitrary set X, let $m(X)$ be the Banach space of bounded real-valued functions on X with the supremum norm. An element $\mu \in m(X)^*$ is called a mean on $m(X)$ if μ is positive and $\|\mu\| = 1$. A finite mean on X is a positive element $\mu \in l^1(X)$ with $\|\mu\|_1 = 1$, and such that the support of μ, the set $\{x \in X \mid \mu(x) > 0\}$, is finite. Any finite mean, considered as an element of $m(X)^*$, is a mean. And the set of all finite means is w^*-dense in the set of all means on X (see Day [2]).

Let S be a semigroup. A mean μ on $m(S)$ is left invariant if $\mu(f) = \mu(l_s f)$ for all $f \in m(S)$ and $s \in S$, where $l_s f \in m(S)$ is defined by $(l_s f)(t) = f(st)$, $t \in S$. When $m(S)$ has a left invariant mean, we say S is left amenable, and denote by $ML(S)$ the set of all left invariant means on $m(S)$. $ML(S)$ is convex and w^*-compact in $m(S)^*$ (cf. [2]). If $s \in S$ and $A \subset S$, it is easy to see that $\mu(\chi_{sA}) \geq \mu(\chi_A)$ for any $\mu \in ML(S)$.

For a mean μ on $m(S)$ and $s \in S$, we define $s \cdot \mu \in m(S)^*$ by $(s \cdot \mu)f = \mu(l_s f)$, $f \in m(S)$. $s \cdot \mu$ is also a mean on $m(S)$, and $(st) \cdot \mu = s \cdot (t \cdot \mu)$ for $s, t \in S$. If $\{\mu_\lambda\}$ is a net of means on $m(S)$, we say that $\{\mu_\lambda\}$ is w^*-convergent to left invariance if the net $\{s \cdot \mu_\lambda - \mu_\lambda\}$ is w^*-convergent to 0 for each $s \in S$. Day [2] proved that S is left amenable if and only if there exists a net of finite means w^*-convergent to left invariance.

When S is left amenable, $ML(S)$, as a w^*-compact convex set, is the w^*-closed convex hull of all its extreme points. It is natural to ask how many exposed points (with respect to the w^*-topology) $ML(S)$ has. Chou

[1] proved that if G is a countable infinite amenable group, then $ML(G)$ has no exposed points. Later, Granirer [4] studied intensively the existence of exposed points of subsets of $ML(S)$ for a countable left amenable semigroup S. In particular, he proved [4, Cor. 4.1] that if S is a countable left amenable semigroup, then $ML(S)$ has exposed points if and only if S has finite left ideals. In this paper we characterize the exposed points of $ML(S)$ for an arbitrary left amenable semigroup S as the arithmetic averages on minimal finite left ideals. Thus we are abel to prove Chou and Granirer's results without the countability condition. We also prove that if $ML(S)$ has an exposed point, then it is the w^*-closed convex hull of all its exposed points. This gives another proof of the Granirer-Klawe Theorem on the dimension of $ML(S)$ (see [5]).

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2. Some lemmas. In this section we are going to prove some lemmas which are used to obtain the main results. They are also of independent interest.

For convenience we write $\mu(A)$ for $\mu(\chi_A)$ when μ is a mean and A is a subset of the underlying set or semigroup.

LEMMA 2.1. *Let X be an infinite set, $\{\mu_\lambda\}_{\lambda\in\Lambda}$ a net of finite means w^*-convergent to a mean μ. Let κ be an infinite cardinal. If for each subset A of X, $\mu(A) = 0$ whenever $|A| < \kappa$, then $|\Lambda| > \kappa$.*

Proof. Suppose $|\Lambda| = \kappa$. We are going to construct a function $f \in m(X)$ such that $\mu_\lambda(f)$ diverges.

Well order Λ as $\{\lambda_\alpha\}_{\alpha<\kappa}$. We define f by transfinite induction.

Let $\alpha < \kappa$ be an ordinal. Suppose we have defined for each $\beta < \alpha$ a function f_β with range $\{0, 1\}$ on a subset A_β of X, satisfying

(1) If β is finite, then A_β is finite. If β is infinite, then $|A_\beta| \le |\beta|$.

(2) $\beta_1 < \beta_2 < \alpha \Rightarrow A_{\beta_1} \subset A_{\beta_2}$ and $f_{\beta_2} \restriction A_{\beta_1} = f_{\beta_1}$.

(3) If $\beta < \alpha$, then there exists $\lambda', \lambda'' > \lambda_\beta$ in Λ, such that the supports of $\mu_{\lambda'}$ and $\mu_{\lambda''}$ are contained in A_β, and $\mu_{\lambda'}(f_\beta) < 1/4$, $\mu_{\lambda''}(f_\beta) > 3/4$.

If α is finite, then $\bigcup_{\beta<\alpha} A_\beta$ is finite. If α is infinite, then $|\bigcup_{\beta<\alpha} A_\beta| \le |\alpha|^2 = |\alpha|$. In both cases $\mu(\bigcup_{\beta<\alpha} A_\beta) = 0$. $\mu_\lambda \xrightarrow{w^*} \mu$ implies that there exists

$\lambda' > \lambda_\alpha$ in Λ, such that $\mu_{\lambda'}(X\setminus\bigcup_{\beta<\alpha}A_\beta) > 3/4$. Also since $|\bigcup_{\beta<\alpha}A_\beta \cup \operatorname{supp}\mu_{\lambda'}| < \kappa$, there exists $\lambda'' > \lambda_\alpha$ in Λ such that

$$\mu_{\lambda''}\left(X\setminus\left(\bigcup_{\beta<\alpha}A_\beta \cup \operatorname{supp}\mu_{\lambda'}\right)\right) > 3/4.$$

Let $A_\alpha = \bigcup_{\beta<\alpha}A_\beta \cup \operatorname{supp}\mu_{\lambda'} \cup \operatorname{supp}\mu_{\lambda''}$, and define

$$f_\alpha(s) = \begin{cases} f_\beta(s), & \text{if } s \in A_\beta \text{ for some } \beta < \alpha, \\ 0, & \text{if } s \in \operatorname{supp}\mu_{\lambda'} \setminus \bigcup_{\beta<\alpha} A_\beta, \\ 1, & \text{if } s \in A_\alpha \setminus \left(\bigcup_{\beta<\alpha} A_\beta \cup \operatorname{supp}\mu_{\lambda'}\right). \end{cases}$$

It is easy to see that A_α and f_α satisfy conditions (1)–(3).

Now let $f = f_\alpha$ on A_α, $\alpha < \kappa$, and $f = 0$ on $X\setminus\bigcup_{\beta<\alpha}A_\alpha$. Then $\mu_\lambda(f)$ diverges. In fact

$$\liminf \mu_\lambda(f) \le \tfrac{1}{4} < \tfrac{3}{4} \le \limsup \mu_\lambda(f).$$

COROLLARY 2.2. *Let S be an infinite left amenable right cancellative semigroup, $\{\mu_\lambda\}_{\lambda\in\Lambda}$ a net of finite means w^*-convergent to a left invariant mean μ. Then $|\Lambda| > |S|$.*

Proof. Let $A \subset S$ be such that $|A| < |S|$. Then it is not difficult to see that $\mu(A) = 0$. A proof can be found in [**5**, Prop. 2.5].

LEMMA 2.3. *Let S be an infinite left amenable semigroup, μ an extreme point of $ML(S)$. Define the cardinal function $\kappa(\mu) = \min\{|A| \mid A \subset S$ and $\mu(A) = 1\}$. If $\kappa(\mu)$ is infinite, then for each subset B of S, $|B| < \kappa(\mu)$ implies $\mu(B) = 0$.*

Proof. Suppose the contrary that there is a set $B \subset S$ such that $|B| < \kappa(\mu)$ and $\mu(B) > 0$.

If B is finite, then there is an $s \in S$ with $\mu(\{s\}) > 0$. For any $t \in S$, $\mu(\{ts\}) \ge \mu(\{s\})$. So the left ideal $I = Ss$ of S is finite, and $0 < \mu(I) < 1$ since $\kappa(\mu)$ is infinite. For any $t \in S$, $\mu(tI) \ge \mu(I)$ and $tI \subset I$ give that $\mu(tI) = \mu(I)$ and $\mu(I\Delta tI) = 0$.

Suppose B is infinite. Let $x = \sup\{\mu(A) \mid A \subset S,\ |A| \le |B|\}$. By taking countable union, we can get a subset I of S such that $|I| = |B|$ and $\mu(I) = x$. For any $t \in S$, $\mu(I) \le \mu(tI) \le \mu(tI \cup I) \le x = \mu(I)$ since $|I \cup tI| = |I| = |B|$. So the equalities hold everywhere. Thus we also have

$0 < \mu(I) < 1$, $\mu(I) = \mu(tI)$ and $\mu(I \Delta tI) = 0$. Denote by $t^{-1}(tI)$ the set $\{s \in S \mid ts \in tI\}$. It is easy to see that $\mu(I \Delta t^{-1}(tI)) = 0$, since $t^{-1}(tI) \supset I$ and $\mu(t^{-1}(tI)) = \mu(tI) = \mu(I)$.

Let $\mu_1 \in m(S)^*$ be defined by

$$\mu_1(f) = \frac{\mu(f \cdot \chi_I)}{\mu(I)}, \qquad f \in m(S).$$

Then μ_1 is positive, $\|\mu_1\| = 1$, and is left invariant:

$$\mu_1(l_t f) = \frac{\mu((l_t f) \cdot \chi_I)}{\mu(I)} = \frac{\mu((l_t f) \cdot \chi_{t^{-1}(tI)})}{\mu(I)}$$

$$= \frac{\mu(l_t(f \cdot \chi_{tI}))}{\mu(I)} = \frac{\mu(f \cdot \chi_{tI})}{\mu(I)} = \frac{\mu(f \cdot \chi_I)}{\mu(I)} = \mu_1(f),$$

since $\mu(I \Delta t^{-1}(tI)) = 0$ and $\mu(I \Delta tI) = 0$.

Let $\mu_2 = (\mu - \mu(I) \cdot \mu_1)/(1 - \mu(I))$. Then for $f \in m(S)$,

$$\mu_2(f) = \frac{\mu(f) - \mu(f \cdot \chi_I)}{1 - \mu(I)} = \frac{\mu(f \cdot \chi_{S \setminus I})}{\mu(S \setminus I)}.$$

So μ_2 is also in $ML(S)$, and

$$\mu = \mu(I)\mu_1 + (1 - \mu(I))\mu_2$$

is not an extreme point.

LEMMA 2.4. *Let S be a left amenable semigroup, $\{\mu_\alpha\}_{\alpha \in \Gamma}$ a net of finite means w^*-convergent to left invariance. Then for any $\alpha \in \Gamma$, any $\varepsilon > 0$, and any $s_1, \ldots, s_n \in S$, there exists a finite mean μ'_α which is a convex combination of elements μ_β, $\beta > \alpha$, such that*

$$\|s_i \cdot \mu'_\alpha - \mu'_\alpha\| < \varepsilon, \qquad i = 1, \ldots, n.$$

Proof. This was proved by Day [**2**, p. 524].

3. Main results. We are now ready to prove our main results concerning the exposed points of $ML(S)$. In all cases we shall consider only the w^*-topology on $ML(S)$.

THEOREM 3.1. *Let S be a left amenable semigroup, and μ an exposed point of $ML(S)$ (if any). Then μ is a finite mean.*

Proof. Let μ be an extreme point of $ML(S)$ and define $\kappa(\mu)$ as in Lemma 2.3. Suppose μ is not a finite mean. Then $\kappa(\mu)$ is infinite. Take $A \subset S$ so that $|A| = \kappa(\mu)$ and $\mu(A) = 1$. Then for any $t \in S$, $\mu(tA \cap A) = 1$ since $\mu(tA) = 1$. In particular $tA \cap A \neq \varnothing$; i.e., there exist $a, b \in A$ with $ta = b$. For fixed $a, b \in A$, let $S_{(a,b)} = \{t \in S \mid ta = b\}$. Then $\bigcup\{S_{(a,b)} \mid a, b \in A\} = S$.

Pick $f \in m(S)$ with $\|f\| = 1$, and choose a net $\{\mu_\alpha\}_{\alpha \in \Gamma}$ of finite means w^*-convergent to μ. Then $\{\mu_\alpha\}_{\alpha\in\Gamma}$ is w^*-convergent to left invariance and $\mu_\alpha(f) \to \mu(f)$.

Let Λ be the set of all finite nonempty subsets of $A \times A$, directed by inclusion. Then Λ is a directed set with $|\Lambda| = |A| = \kappa(\mu)$. Take $F = \{(a_i, b_i) \mid i = 1, \ldots, n\} \in \Lambda$. There exists $\alpha \in \Gamma$ such that for any $\beta > \alpha$, $|\mu_\beta(f) - \mu(f)| < 1/2n$. By the finite intersection property on right ideals (see [3]), $\bigcap_{i=1}^n a_i S \neq \varnothing$. Choose $a \in \bigcap_{i=1}^n a_i S$ (a is not necessarily in A), say $a = a_i s_i$, $i = 1, \ldots, n$. By Lemma 2.4, there exists a finite mean μ'_α which is a convex combination of elements μ_β, $\beta > \alpha$, such that

$$\|a \cdot \mu'_\alpha - \mu'_\alpha\| < \frac{1}{2n}$$

and

$$\|(b_i s_i) \cdot \mu'_\alpha - \mu'_\alpha\| < \frac{1}{2n}, \qquad i = 1, \ldots, n.$$

For $t \in S_{(a_i, b_i)}$, we have

$$\|t \cdot (a \cdot \mu'_\alpha) - a \cdot \mu'_\alpha\| = \|(b_i s_i) \cdot \mu'_\alpha - a \cdot \mu'_\alpha\| < \frac{1}{n}.$$

Also

$$|(a \cdot \mu'_\alpha)(f) - \mu(f)| < \frac{1}{n}.$$

Define $\mu_F = a \cdot \mu'_\alpha$. Then the net $\{\mu_F\}_{F\in\Lambda}$ w^*-converges to left invariance and $\lim \mu_F(f) = \mu(f)$. Since μ is an extreme point of $ML(S)$, by Lemma 2.3, for any $B \subset S$, $|B| < \kappa(\mu)$ implies $\mu(B) = 0$. By Lemma 2.1, $\{\mu_F\}_{F\in\Lambda}$ does not converge to μ since $|\Lambda| = |A| = \kappa(\mu)$. So it has a w^*-cluster point μ_1 different from μ. Since $\mu_1 \in ML(S)$ and $\mu_1(f) = \mu(f)$, μ is not an exposed point of $ML(S)$.

For a finite nonempty set $I \subset S$, the arithmetic average on I is the finite mean μ such that for each $a \in I$, $\mu(\{a\}) = 1/|I|$.

THEOREM 3.2. *Let S be a left amenable semigroup. Then μ is an exposed point of $ML(S)$ if and only if it is the arithmetic average on a minimal finite left ideal of S.*

Proof. Let I be a minimal finite left ideal of S. By a result of Mitchell ([**6**, pp. 256–257]), there exists $\mu \in ML(S)$ with $\mu(I) = 1$. Since $Ia = I$ for any $a \in I$, I is right cancellative. Also $\mu(aI) = \mu(I)$ implies that $aI = I$ for any $a \in S$. Thus I is left cancellative and in fact a finite group. μ as the unique invariant mean on I is the arithmetic average on I. Let f be the characteristic function of I. Then $\mu(f) = 1$. For any $\mu_1 \in ML(S)$, if $\mu_1(f) = \mu_1(I) = 1$, then by the above argument, $\mu = \mu_1$. Thus μ is an exposed point of $ML(S)$. (Remark. Part of the proof is adopted from [**3**, Thm. 4.1].)

Suppose μ is an exposed point of $ML(S)$. Then μ is a finite mean by Theorem 3.1. Let I be the support of μ. For $a \in I$ and $t \in S$, $\mu(\{ta\}) \geq \mu(\{a\}) > 0$, so $ta \in I$. Thus I is a left ideal and it contains a minimal left ideal I_1. If $I \neq I_1$, then as in the proof of Lemma 2.3 we have $0 < \mu(I_1) < 1$ and $\mu(I_1 \Delta t I_1) = 0$ for any $t \in S$. These give that μ is not an extreme point of $ML(S)$. So I must be a minimal finite left ideal. By the proof of the first part, μ is the arithmetic average on I.

COROLLARY 3.3. *For any left amenable semigroup S, $ML(S)$ has exposed points if and only if S has finite left ideals. The number of exposed points of $ML(S)$ is exactly the number of minimal finite left ideals of S.*

COROLLARY 3.4. *If S is a right cancellative, left amenable infinite semigroup, then $ML(S)$ has no exposed points.*

Proof. For any $s \in S$, $|Ss| = |S|$. So S does not have finite left ideals.

COROLLARY 3.5. *If* $\dim(ML(S)) < \infty$, *then S has finite left ideals.*

Proof. If $\dim(ML(S)) < \infty$, then $ML(S)$ is a compact convex subset of a Banach space. So it has exposed points.

COROLLARY 3.6. *Different exposed points of $ML(S)$ are linearly independent.*

Corollary 3.3 extends [**4**], Corollary 4.1. Corollary 3.5 is the main result of Klawe [**5**].

Suppose S is an infinite left amenable semigroup and K is an invariant subset of βS. Let $M(S, K)$ denote the set of all $\mu \in ML(S)$ with its support contained in K (see [**1**] for the definitions). Chou [**1**] proved that if G is a countably infinite amenable group, then $M(G, K)$

has no exposed points. He asked whether this holds for any infinite amenable group. Our Corollary 3.4 gives a partial answer to this problem with $K = \beta G$.

Motivated by Granirer [**3**, Thm. 3.1], we get the following generalization.

THEOREM 3.7. *If $ML(S)$ has exposed points, then it is the w^*-closed convex hull of all its exposed points.*

Proof. Suppose $ML(S)$ has exposed points. Then S has finite left ideals. Let $\{I_\alpha\}$ be the class of all its minimal finite left ideals and $A = \bigcup I_\alpha$. Then A is a right ideal of S since for any $s \in S$, $I_\alpha s$ is also a minimal left ideal. For any $\mu \in ML(S)$, $\mu(A) = 1$. Thus μ is the w^*-limit of a net $\{\mu_\lambda\}_{\lambda \in \Lambda}$ of finite means with supports in A. For each $\lambda \in \Lambda$, define

$$\mu'_\lambda = \sum_\alpha \mu_\lambda(I_\alpha)\varphi_\alpha,$$

where φ_α is the arithmetic average on I_α. Then μ'_λ is a convex combination of some φ_α. Take a minimal finite left ideal $I_0 = \{a_1, \dots, a_n\}$. For any I_α and any $a \in I_\alpha$, it is easy to see that $\sum_{i=1}^n a_i \cdot \mu_\lambda(a) = \mu_\lambda(I_\alpha)$. So $\mu'_\lambda = n^{-1}\sum_{i=1}^n a_i \cdot \mu_\lambda$. Since $\{\mu_\lambda\}$ converges to left invariance, we obtain that $\{\mu'_\lambda\}$ converges to μ in the w^*-topology.

COROLLARY 3.8. (*Granirer-Klawe Theorem. See* [**5**].) *For any left amenable semigroup S,* $\dim(ML(S)) = n$ *if and only if S contains exactly n minimal finite left ideals.*

Proof. If S has n minimal finite ideals, then $ML(S)$ has n exposed points. By Corollary 3.6, $\dim(ML(S)) \geq n$. By Theorem 3.7, $ML(S)$ is the convex hull of those exposed points. So $\dim(ML(S)) = n$.

On the other hand, if $\dim(ML(S)) = n$, by Corollary 3.5, S has finite left ideals. Again by Corollary 3.6, S has only finitely many minimal finite left ideals. By the proof of the first part, this number is n.

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