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COUNTING FUNCTIONS AND MAJORIZATION FOR JENSEN MEASURES

CHARLES S. STANTON

We establish a generalization for uniform algebras of the classical identities of Hardy and Stein. We use this and an estimate based on the isoperimetric inequality to give a proof of H. Alexander's spectral area theorem. We use similar methods to prove a theorem of Axler and Shapiro about VMOA of the unit ball in $\mathbf{C}^n$.

1. Introduction. Given a Jensen measure on the maximal ideal space of A, we introduce a "counting function" analogous to the classical counting function $N(r, w)$ of Nevanlinna's value distribution theory. In particular, this counting function is non-negative, supported on the spectrum of f, and a subharmonic function of w on the complex plane except for a logarithmic pole. We next establish an identity for integral means of f in terms of this counting function. This generalizes Theorems 2 and 9 of [**6**]. Classical identities of Cartan and of Hardy and Stein occur as special cases.

As an application, we give a proof (for Jensen measures) of H. Alexander's spectral area estimate:

THEOREM A [**1,2**]. *Let A be a uniform algebra, $\varphi \in M_A$, and σ a Jensen measure for φ. Then*

$$(1)\qquad \int_{M_A} |f|^2\, d\sigma \leq \frac{1}{\pi}\text{area}(\text{spec} f) + |f(\varphi)|^2.$$

Finally, we apply these counting function techniques to prove a slight generalization of the following result of Axler and Shapiro about analytic functions of vanishing mean oscillation (VMOA) of the unit ball in $\mathbf{C}^n$.

THEOREM B [3]. *Suppose $f \in H^\infty(\mathbf{B}^n)$ and for each $\zeta \in S$*

$$\text{area}(\text{cl}(f, \zeta)) = 0.$$

Then $f \in$ VMOA.

2. Uniform algebras and Jensen measures. We first recall some basic facts about uniform algebras and Jensen measures (for more details see [7]). We then introduce the counting function and establish its subharmonicity.

Let X be a compact Hausdorff space and A a uniform algebra on X, i.e. a closed subalgebra of $C(X)$ which contains the constants and separates points of X. Let M_A denote the maximal ideal space of A. The spectrum of $f \in A$, denoted $\operatorname{spec} f$, is the set $\{w \in \mathbf{C}\colon f - w \text{ is not invertible in } A\}$.

Let $\varphi \in M_A$. A probability measure σ on M_A is a *Jensen measure* for φ if and only if

$$\log|f(\varphi)| \leq \int_{M_A} \log|f|\,d\sigma, \tag{2}$$

for every $f \in A$. Since σ is a Jensen measure it is also an *Arens-Singer measure* for φ:

$$\log|f(\varphi)| = \int_{M_A} \log|f|\,d\sigma, \tag{3}$$

for each invertible f belonging to A. It follows that σ is also a *representing measure* for φ:

$$f(\varphi) = \int_{M_A} f\,d\sigma \tag{4}$$

for each $f \in A$.

DEFINITION. Suppose $f \in A$, $\varphi \in M_A$, and that σ is a Jensen measure for φ. Then, for each $w \in \mathbf{C} \setminus \{f(\varphi)\}$, we define

$$N(w; f, \sigma) = \int_{M_A} \log|f - w|\,d\sigma - \log|f(\varphi) - w|. \tag{5}$$

REMARK. If $\sigma = \alpha\tau + (1-\alpha)\delta_\varphi$ with $0 < \alpha < 1$ then τ is also a Jensen measure for φ and $N(w; f, \sigma) = \alpha N(w; f, \tau)$. We shall assume that $\sigma(\varphi) = 0$ in the following. We shall also denote $N(w; f, \sigma)$ by $N(w)$ when f and σ have been fixed.

The properties of $N(w)$ are summarized by

PROPOSITION 1. *Suppose $f \in A$, $\varphi \in M_A$, and that σ is a Jensen measure for φ. Then $N(w)$ is a non-negative function supported on $\operatorname{spec} f$. Furthermore, $N(w)$ is subharmonic on $\mathbf{C} \setminus \{f(\varphi)\}$, and $N(w) + \log|f(\varphi) - w|$ is subharmonic on $\mathbf{C}$.*

Proof. The non-negativity is a consequence of the definition (2) of a Jensen measure; since a Jensen measure is also an Arens-Singer measure the support of $N(w)$ is contained in spec f by (3). To prove the subharmonicity we introduce the Borel probability measure $f^*(d\sigma)$ supported on spec f defined by

$$\int_{M_A} (k \circ f)\, d\sigma = \int_{\mathbf{C}} h(\zeta) f^*\, d\sigma(\zeta)$$

for every $h \in L^1(d\sigma)$. Thus

$$\int_{M_A} \log |f - w|\, d\sigma = \int_{\mathbf{C}} \log |\zeta - w| f^*(d\sigma).$$

This establishes $N(w) + \log|f(\varphi) - w|$ as the potential of the measure $f^*(d\sigma)$ and hence a subharmonic function on $\mathbf{C}$. Since $\log|f(\varphi) - w|$ is harmonic on $\mathbf{C} \setminus \{f(\varphi)\}$, we set that $N(w)$ is subharmonic on $\mathbf{C} \setminus \{f(\varphi)\}$.

When A is the disc algebra the following result is known as Lehto's principle of majorization (see **[10]**):

THEOREM 1. *Let* Ω *be an open set which contains* spec f, *and* $G_\Omega(w; f(\varphi))$ *be the Green function for* Ω *with pole at* $f(\varphi)$. *Then*

$$N(w) \le G_\Omega(w; f(\varphi)). \tag{6}$$

REMARK. We shall always extend a Green function G_Ω to all of $\mathbf{C}$ by defining it to be identically zero outside of Ω.

Proof. The theorem follows immediately from the maximum principle since $G_\Omega(w; f(\varphi)) + \log|f(\varphi) - w|$ is harmonic on Ω while $N(w) + \log|f(\varphi) - w|$ is subharmonic on Ω and $N(w) \le G_\Omega(w; f(\varphi))$ on the boundary of Ω.

3. Identities for integral means.

Our next result expresses integral means of f as an integral of $N(w)$ weighted by an appropriate measure.

THEOREM 2. *Suppose* Ψ *is subharmonic on a disc* $\Delta_R = \{z: |z| < R\}$ *which contains* spec f. *Let* $d\mu$ *be the Riesz measure for* Ψ. *Assume* $\mu(f(\varphi)) = 0$. *Then*

$$\int_{M_A} \Psi(f)\, d\sigma = \int_{\mathbf{C}} N(w)\, d\mu + \Psi(f(\varphi)). \tag{7}$$

Proof. By the Riesz decomposition theorem for subharmonic functions

$$\Psi(\zeta) = \int_{\operatorname{spec} f} \log|w - \zeta|\, d\mu(w) + h(\zeta). \tag{8}$$

Here the Riesz measure $d\mu = (1/2\pi)\Delta\Psi$ in the sense of distributions and h is harmonic in the interior of Δ_R. Thus $h = \operatorname{Re} H$ for some function H holomorphic on $\{w\colon |w| < R\}$. It follows from the "functional calculus" that $H \circ f \in A$. Since σ is a representing measure for φ we have by (4)

$$\int_{M_A} h \circ f \, d\sigma = h(f(\varphi)).$$

We now calculate, using the Riesz decomposition (8) and the definition of $N(w)$:

$$\begin{aligned}\int_{M_A} \Psi \circ f \, d\sigma &= \int_{M_A} \left\{ \int_{\Delta_R} \log|w - f| \, d\mu(w) + h \circ f \right\} d\sigma \\ &= \int_{\Delta_R} \int_{M_A} \log|w - f| \, d\sigma \, d\mu(w) + h(f(\varphi)) \\ &= \int_{\Delta_R} N(w) + \log|f(\varphi) - w| \, d\mu(w) + h(f(\varphi)) \\ &= \int_{\Delta_R} N(w) \, d\mu(w) + \Psi(f(\varphi)).\end{aligned}$$

Since $N(w)$ is supported on $\operatorname{spec} f$ we may extend the last integral to be taken over the entire plane to obtain (7).

Two important special cases of (7) occur when we take $\Psi(\zeta) = \log^+|\zeta|$ and $\Psi(\zeta) = |\zeta|^p$. In the first case Theorem 2 implies

$$\int_{M_A} \log^+|f| \, d\sigma = \frac{1}{2\pi} \int_0^{2\pi} N(e^{i\vartheta}) \, d\vartheta + \log^+|f(\vartheta)|. \tag{9}$$

If A is the disc algebra and $d\sigma$ is Lebesgue measure on the unit circle this is known as Cartan's formula [**9**, p. 8]. In the second case we obtain, for $p > 0$,

$$\int_{M_A} |f|^p \, d\sigma = \frac{p^2}{2\pi} \int_{\mathbf{C}} N(w)|w|^{p-2} \, du \, dv + |f(\varphi)|^p. \tag{10}$$

which is a version of the Hardy-Stein identity [**13**]. For applications of other choices of Ψ see [**6**].

4. Alexander's spectral area theorem. The key estimate we will need is the following consequence of the isoperimetric inequality:

PROPOSITION 3 ([**11**, p. 115], [**4**, p. 60]). *Let Ω be a plane domain of finite area. Let $G_\Omega(w, w_0)$ be the Green function for Ω with pole at w_0. Then*

$$\int_\Omega G_\Omega(w, w_0) \, du \, dv \le \frac{1}{2} \operatorname{area}(\Omega). \tag{11}$$

Proof of Theorem A. Let Ω be a region containing $\operatorname{spec} f$ such that $\operatorname{area}\Omega \leq \operatorname{area}(\operatorname{spec} f) + \varepsilon$. By the Hardy-Stein identity (10), Lehto's principle of majorization (6), and the proposition above we have

$$\begin{aligned}\int_{M_A} |f|^2\,d\sigma &= \frac{2}{\pi}\int_{\mathbf{C}} N(w)\,du\,dv + |f(\varphi)|^2\\ &\leq \frac{2}{\pi}\int_{\mathbf{C}} G_\Omega(w; f, \varphi)\,du\,dv + |f(\varphi)|^2\\ &\leq \frac{2}{\pi}\frac{1}{2}\operatorname{area}(\Omega) + |f(\varphi)|^2\\ &\leq \frac{1}{\pi}(\operatorname{area}(\operatorname{spec} f) + \varepsilon) + |f(\varphi)|^2.\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain (1).

5. Counting functions on $\mathbf{B}^n$. Let $\mathbf{B}^n$ denote the open unit ball in $\mathbf{C}^n$ with normalized measure $d\sigma$ on $\partial\mathbf{B}^n$. Suppose $\alpha \in \mathbf{B}^n$. The *Poisson-Szegö measure* for α is

$$d\nu_\alpha(\zeta) = \left\{\frac{1 - |\alpha|^2}{|1 - \langle\alpha, \zeta\rangle|^2}\right\}^n d\sigma(\zeta).$$

The Möbius transformation φ_α is defined by

$$\varphi_\alpha(z) = \frac{\alpha - P_\alpha z - (1 - |\alpha|^2)^{1/2} Q_\alpha z}{1 - \langle z, \alpha\rangle},$$

where P_α is the orthogonal projection of $\mathbf{C}^n$ onto the subspace generated by α, and Q_α is the orthogonal complement to P_α. The properties of φ_α are summarized by

PROPOSITION 4. *Let $\alpha \in \mathbf{B}^n$, $z \in \overline{\mathbf{B}^n}$, $h \in \mathbf{C}^n$, and $g \in L^1(\partial\mathbf{B}^n)$. Then*

(12.a) φ_α *is a biholomorphism of $\mathbf{B}^n$ onto $\mathbf{B}^n$,*

(12.b) $$\varphi_\alpha^{-1} = \varphi_\alpha,$$

(12.c) $$1 - |\varphi_\alpha(z)|^2 = \frac{(1 - |\alpha|^2)(1 - |z|^2)}{|1 - \langle z, \alpha\rangle|^2},$$

(12.d) $$\varphi_\alpha(z)h = \frac{(1 - |\alpha|^2)^{1/2}}{(1 - \langle z, \alpha\rangle)^2} \times\left\{-(1 - |\alpha|^2)^{1/2} P_\alpha h - Q_\alpha h - z\langle h, \alpha\rangle + h\langle z, \alpha\rangle\right\}$$

(12.e) $$\int_{\partial \mathbf{B}^n} g \circ \varphi_\alpha \, d\sigma = \int_{\partial \mathbf{B}^n} g \, d\nu_\alpha,$$

Proof. Assertions (12.a–12.c) are contained in Theorem 2.2.2 of [**12**], and (12.d) may be obtained in the same manner as part (ii) of that Theorem. Part (12.e) is Theorem 3.3.8 of [**12**].

The Green function with pole at 0 for $\mathbf{B}^n$ is

(13) $$G_{\mathbf{B}}(z,0) = \begin{cases} \log \dfrac{1}{|z|} & (\text{if } n = 1) \\ \dfrac{1}{2n-2}\left(|z|^{2-2n} - 1\right), & (\text{if } n > 1) \end{cases}$$

We introduce the form

$$\beta = \frac{i}{2\pi} \sum_j dz_j \wedge d\bar{z}_j.$$

Then β^n is Lebesgue measure on $\mathbf{C}^n$, normalized so that $\int_{\mathbf{B}} \beta^n = 1$. We recall Wirtinger's Theorem [**8**, p. 5]: If V is a k-dimensional variety then β^k is the induced volume form.

It follows from Jensen's formula [**14**, p. 248] that $d\sigma$ is a Jensen measure for 0 and that the counting function $N(0; f, \sigma)$ defined by (5) is the usual counting function of value distribution theory in $\mathbf{C}^n$. In particular, if $\mu(z)$ is the multiplicity of the zero at z then [**14**, p. 248]

$$N(0; f, \sigma) = \int_{f^{-1}(0)} \mu(z) G_{\mathbf{B}}(z,0) \beta^{n-1}.$$

It follows from (5) and (12.e) that

$$N(w; f, \nu_\alpha) = N(0; f \circ \varphi_\alpha - w, \sigma)$$

and hence

(14) $$N(w; f, \nu_\alpha) = \int_{\varphi_\alpha(f^{-1}(w))} \mu(z) G_{\mathbf{B}}(z,0) \beta^{n-1}$$

where μ is the multiplicity of the zero for $f \circ \varphi_\alpha - w$ at z. Since the integrand on the right is non-negative it follows that ν_α is a Jensen measure for α.

The counting function $N(w; f, \nu_\alpha)$ may be extended to f in the Nevanlinna class (see [**5**, §4] for more details) by setting

$$N(w; f, \nu_\alpha) = \limsup_{\zeta \to w} \left(\lim_{r \to 1} \left(N(w; f_r, \nu_\alpha) \right) \right)$$

where $f_r(z) = f(rz)$ for $r < 1$. Theorem 1 remains true in this context. If Ψ is a positive subharmonic function then Theorem 2 may be extended to

$$\lim_{r\to 1} \int_{\partial \mathbf{B}^n} f_r\, d\nu_\alpha = \int_{\mathbf{C}} N(w; f, \nu_\alpha)\, d\mu + \Psi(f(\alpha)).$$

6. Functions of vanishing mean oscillation on $\mathbf{B}^n$. Definition. A function $f \in H^2(\mathbf{B}^n)$ is said to belong to BMOA if

$$\|f\|_*^2 = \sup_{\alpha\in\mathbf{B}} \int_{\partial\mathbf{B}^n} |f - f(\alpha)|^2\, d\nu_\alpha$$

is finite in which case $\|f\|_* + |f(0)|$ is a norm on the space BMOA.

DEFINITION. A function in $H^2(\mathbf{B}^n)$ belongs to VMOA if for every $\zeta \in \partial\mathbf{B}^n$

$$\lim_{\alpha\to\zeta} \int_{\partial\mathbf{B}^n} |f - f(\alpha)|^2\, d\nu_\alpha = 0.$$

We note that

$$\int_{\partial\mathbf{B}^n} |f - f(\alpha)|^2\, d\nu_\alpha = \int_{\partial\mathbf{B}^n} |f|^2\, d\nu_\alpha - |f(\alpha)|^2 = \int_{\partial\mathbf{B}^n} |f\circ\varphi_\alpha|^2\, d\sigma - |f(\alpha)|^2. \tag{15}$$

Since area $(f \circ \varphi_\alpha(\mathbf{B}^n)) = \text{area}(f(\mathbf{B}^n))$ it follows from Theorem A that if area$(f(\mathbf{B}^n))$ is finite then $f \in$ BMOA and

$$\|f\|_*^2 \le \frac{1}{\pi}\,\text{area}(f(\mathbf{B}^n)).$$

For $\zeta \in \partial\mathbf{B}^n$ we define

$$D_{\rho,\zeta} = \{z \in \mathbf{B}^n\colon |1 - \langle z, \zeta\rangle| < \rho\}.$$

Our generalization of Theorem B is

THEOREM 3. *Suppose f is holomorphic in $\mathbf{B}^n$ and for every $\zeta \in \partial\mathbf{B}^n$*

$$\lim_{\rho\to 0} \text{area}\big(f(D_{\rho,\zeta})\big) = 0. \tag{16}$$

Then $f \in$ VMOA.

Before giving the proof of Theorem 3 we will show how Theorem B follows from it. Since the sets $D_{\rho,\zeta}$ form a basis for the topology at ζ the cluster set of f at ζ may be defined by

$$\text{cl}(f,\zeta) = \bigcap_{\rho>0} \overline{f(D_{\rho,\zeta})}.$$

We have

$$(17)\qquad \lim_{\rho\to 0}\operatorname{area}\big(f(D_{\rho,\zeta})\big) = \operatorname{area}\Big(\bigcap_{\rho>0} f(D_{\rho,\zeta})\Big) \le \operatorname{area}\Big(\bigcap_{\rho>0} \overline{f(D_{\rho,\zeta})}\Big) = \operatorname{area}(\operatorname{cl}(f,\zeta)).$$

By the hypothesis of Theorem B area $(\operatorname{cl}(f,\zeta)) = 0$ so the hypothesis (16) of Theorem 3 is satisfied and hence $f \in$ VMOA.

The equality in the first line of (17) follows from the dominated convergence theorem; we note that the hypothesis $f \in H^\infty$ could be replaced by the assumption area$(f(\mathbf{B}^n))$ is finite.

The following two lemmas will be used in the proof of Theorem 3.

LEMMA 1 (*see* [**12**, *Proposition* 5.1.2]). *If* $\alpha, z, \zeta \in \overline{\mathbf{B}^n}$ *then*

$$(18)\qquad |1-\langle z,\zeta\rangle|^{1/2} + |1-\langle z,\alpha\rangle|^{1/2} \ge |1-\langle \zeta,\alpha\rangle|^{1/2}.$$

LEMMA 2. *Suppose* $\alpha \in D_{\tau,\zeta}$ *with* $\tau < \rho/16$ *and* $w \notin f(D_{\rho,\zeta})$. *Then there is a constant* C *depending only on* ρ *and* n *such that*

$$(19)\qquad N(w; f, \nu_\alpha) < C(1-|\alpha|^2)^n N(w; f, \sigma).$$

Proof of Lemma 2. Suppose $\alpha \in D_{\tau,\zeta}$ and $z \in \mathbf{B}^n \setminus D_{\rho,\zeta}$. We deduce from (18) that $|1-\langle \alpha, z\rangle| > 9/16$. Hence, by (12.b),

$$1-|\varphi_\alpha(z)|^2 = \frac{(1-|\alpha|^2)(1-|z|^2)}{|1-\langle\alpha,z\rangle|^2} < \frac{2}{9}.$$

This implies that $|\varphi_\alpha(z)| > 7/9$, and thus

$$(20)\qquad G_{\mathbf{B}}(\varphi_\alpha(z),0) \le c\big(1-|\varphi_\alpha(z)|\big)^2$$

for some constant c depending only on the dimension n.

Now suppose $w \notin f(D_{\rho,\zeta})$. Then by (20) and a change of variables

$$\begin{aligned} N(w; f, \nu_\alpha) &= \int_{\varphi_\alpha(f^{-1}(w))} G_{\mathbf{B}}(z,0)\beta^{n-1} \\ &= \int_{f^{-1}(w)} G_{\mathbf{B}}(\varphi_\alpha(z),0)\varphi_\alpha^*\beta^{n-1} \\ &\le c\int_{f^{-1}(w)} 1-|\varphi_\alpha(z)|^2\varphi_\alpha^*\beta^{n-1}, \end{aligned}$$

since $z \in \varphi_\alpha(f^{-1}(w))$ implies $z \notin D_{\rho,\zeta}$.

From (12.d) we deduce the estimate

$$\varphi_\alpha^*\beta^{n-1} \le \sup_{|I|=|J|=n-1}\left|\frac{\partial^I\varphi_\alpha}{\partial z^J}\right|^2\beta^{n-1} \le \frac{\left(1-|\alpha|^2\right)^{n-1}}{|1-\langle z,\alpha\rangle|^{4(n-1)}}\beta^{n-1}.$$

We conclude that

$$N(w;f,\nu_\alpha) < C\left(1-|\alpha|^2\right)^n\int_{f^{-1}(w)}(1-|z|^2)\beta^{n-1}$$

$$< C\left(1-|\alpha|^2\right)^n N(w;f,\sigma)$$

with C depending on n and ρ.

Proof of Theorem 3. Under the hypotheses of the Theorem it suffices (recalling (15)) to show that for each fixed $\zeta \in \partial\mathbf{B}^n$ and $\rho > 0$ that

$$\lim_{\alpha\to\zeta}\int_{\partial\mathbf{B}^n}|f|^2\,d\nu_\alpha - |f(\alpha)|^2 \le \frac{1}{\pi}\operatorname{area}\left(f(D_{\rho,\zeta})\right).$$

By the Hardy-Stein identity (10) this is equivalent to

$$\lim_{\alpha\to\zeta}\int_\Omega N(w;f,\nu_\alpha)\,du\,dv \le \tfrac{1}{2}\operatorname{area}\left(f(D_{\rho,\zeta})\right). \tag{21}$$

Let ζ and ρ be fixed and define $\Omega = f(\mathbf{B}^n)$ and $\Omega_\rho = f(D_{\rho,\zeta})$. Let C be the constant in (19) of Lemma 2 and define a function h by

$$h(w,\alpha) = C\left(1-|\alpha|^2\right)^n G_\Omega(w;f(0)) + G_{\Omega_\rho}(w;f(\alpha)).$$

The Green function G_Ω is harmonic on $\Omega\setminus\{f(0)\}$, while G_{Ω_ρ} is harmonic on $\Omega_\rho\setminus\{f(\alpha)\}$ and 0 on $\Omega\setminus\overline{\Omega}_\rho$. It follows from Lemma 2 and the majorization principle (6) that $N(w;f,\nu_\alpha)\le h(w)$ on $\Omega\setminus\overline{\Omega}_\rho$. Since h has a logarithmic pole at $f(\alpha)$ it now follows from the maximum principle that $N(w;f,\nu_\alpha)\le h(w)$ on $\overline{\Omega}_\rho$, and hence $N(w;f,\nu_\alpha)\le h(w)$ on all of Ω.

We now have

$$\limsup_{\alpha\to\zeta}\int_\Omega N(w;f,\nu_\alpha)\,du\,dv \le \limsup_{\alpha\to\zeta}\int_\Omega h(w,\alpha)\,du\,dv$$

$$\le \limsup_{\alpha\to\zeta}\int_{\Omega_\rho} G_{\Omega_\rho}(w,f(\alpha))\,du\,dv$$

$$\le \tfrac{1}{2}\operatorname{area}(\Omega_\rho).$$

This proves (21) as desired.

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UNIVERSITY OF CALIFORNIA, RIVERSIDE
RIVERSIDE, CA 92521