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PLANE CURVES AND REMOVABLE SETS

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Various function spaces defined outside a curve Γ are introduced, along with their subspaces of holomorphic functions. The removability of Γ depends on the modulus of continuity; the results obtained are quite precise, as shown by examples based on careful estimation of Fourier coefficients. It is most surprising that the results are nearly the same for the holomorphic functions, and even for functions conformal off Γ.

Let Γ be the graph of a continuous real function $y = y(x)$, $0 \le x \le 1$. Then $C^r(\Gamma)$ $(r = 1, 2, 3, \ldots)$ denotes the class of complex functions, continuous on R^2 and of class $C^r(R^2 \setminus \Gamma)$, whose partial derivatives up to order r admit continuous extensions to all of R^2. Again, $A^r(\Gamma)$ contains the elements of $C^r(\Gamma)$, holomorphic in $R^2 \setminus \Gamma$. We say that Γ is removable C^r, abbreviated N_r, if the functions in $C^r(\Gamma)$ are necessarily of class $C^1(R^2)$, and N_r^a is defined with $A^r(\Gamma)$ in place of $C^r(\Gamma)$. (Remark 1, explaining the definition of N_r, is placed after Theorem 2.)

Our conclusions on class N_r and N_r^a can be summarized as follows: a close connection exists between the modulus of continuity of y and removability properties of Γ, and this connection is about the same for N_r and the (ostensibly larger) class N_r^a. We do not know how to prove that $N_r^a \neq N_r$; an explanation for this anomaly appears as Remark 2. In finding curves Γ, not of class N_r^a, we are led to find elements of $A^r(\Gamma)$ with even stronger properties, so that one obtains a larger class that might coincide with N_r^a.

The modulus of continuity of y is

$$\omega(h) = \omega(y; h) \equiv \sup\{|y(x_1) - y(x_2)|, |x_1 - x_2| \le h\}.$$

THEOREM 1. *If* $\limsup_{h \to 0} \omega(h) h^{-1/r+1} < +\infty$, *then* Γ *is of class* N_r.

THEOREM 2. *Let* $\psi(t)$ *be a positive, increasing function on* $0 < t < 1$ *and* $\limsup \psi(t) t^{-1/r+1} = +\infty$; *then there is a curve* Γ: $y = y(x)$, $0 \le x \le 1$, *such that*

$$\limsup \omega(h)/\psi(h) = 0,$$

and an element φ *of* $A^r(\Gamma)$, *which is not entire, defining a homeomorphism of* R^2 *onto itself, whose inverse is a contraction mapping of* R^2.

For the method used, compare [2].

1. *Proof of Theorem* 1. Let M be a square $|x| < c$, $|y| < c$ containing Γ and $\varphi \in C^r(\Gamma)$. When $(x, y_1) \in M$, $(x, y_2) \in M$, then plainly $|\varphi(x, y_1) - \varphi(x, y_2)| \le c|y_1 - y_2|$, for each fixed function φ in $C^r(\Gamma) \subseteq C^1(\Gamma)$. Suppose next that $(x_1, y_1) \in M$, $(x_2, y_2) \in M$, and $y_1 > y(x_1)$, $y_2 > y(x_2)$. We join (x_1, y_1) to (x_2, y_2) by a path γ, entirely in $M \setminus \Gamma$; clearly the length l of γ can be made at most $|x_1 - x_2| + |y_1 - y_2| + 2\omega(|x_1 - x_2|)$. Let $p(x, y)$ be the Taylor polynomial, of order r, of φ at z_1. We assert that $\varphi(x_2, y_2) - p(x_2, y_2) = o(1) \cdot l^r$, where $o(1)$ refers to small distances $|z_1 - z_2|$. More exactly $\varphi(x, y) - p(x, y) = o(1) \cdot l(x, y)^r$, where $l(x, y)$ is the distance between (x_1, y_1) and (x, y) along γ. This is true if $r = 0$ (no derivatives!) and then follows by induction. Combining this with out estimate for $\varphi(x, y_1) - \varphi(x, y_2)$, we see that for z_1, $z_2 \in M$

$$\begin{aligned}|\varphi(z_1) - \varphi(z_2)| &= O(|z_1 - z_2|) + o(1)\omega(|z_1 - z_2|)^r \\ &\equiv \Omega(|z_1 - z_2|),\end{aligned}$$

say.

Let $g \in C^1(R^2)$, $\iint g(x, y)\,dx\,dy = 1$, $g \ge 0$, $g(z) = 0$ when $|z| > 1$, and then define

$$\varphi_h(z) = h^{-2} \iint \varphi(x - y, y - v) g(h^{-1}u, h^{-1}v)\,dx\,dy.$$

We shall show that $\nabla\varphi_h$ converges in $L^1(M)$ as $h \to 0+$ along a subsequence h_ν, and this clearly proves that $\varphi \in C^1(R^2)$. On the part of M defined by $d(z, \Gamma) > h$, $\nabla\varphi_h = O(1)$, and on the remaining part of M a change of variables yields the estimate $\nabla\varphi_h = O(h^{-1})\Omega(h)$.

Now $\liminf h^{-1}\Omega(h)\omega(h) = 0$, so it will be enough to prove that $m\{z\colon d(z, \Gamma) < h\} = O(\omega(h)) + O(h)$. To do so we observe that when $d(z, \Gamma) < h$ and $(k - 1) < x < kh$, then $|y(x) - y((k - \frac{1}{2})h)| \le h + \omega(h)$, so the measure is $O(h^{-1}) \cdot O(h) \cdot O(h + \omega(h))$.

In the definition of $C^r(\Gamma)$, the continuity of the rth order derivatives can be weakened to boundedness, provided $\liminf \omega(h)h^{-1/r+1} = 0$. It is unclear whether both hypotheses can be retained in the weaker form, to conclude that $\nabla\varphi$ is locally in L^∞.

2. *Proof of Theorem* 2. Because of the complications of the construction, we present an outline first. The operator J is defined by

$$(Jf)(\zeta) \equiv \iint f(z)(z - \zeta)^{-1}\,dx\,dy.$$

When f is smooth and integrable, then Jf is smooth, and $\bar{\partial}(Jf) \equiv \frac{1}{2}(\partial/\partial x + i\partial/\partial y)(Jf) = -\pi f$ (a formula that will be useful later). The proof begins with a smooth function $f_0 \geq 0$, vanishing off $0 \leq x \leq 1/2$, $0 \leq y \leq 1/2$, and constructs $f_1, \ldots, f_k, \ldots$ so that $J(f_0 \cdots f_k), \ldots, \partial^r J(f_0 \cdots f_k)/\partial y^r$ converge uniformly as $k \to \infty$. The singular set of the limit w will be the graph of a function defined over a set $E \subseteq [0,1]$, and linear interpolation will yield the graph Γ. Since $\partial^r w/\partial y^r$ is continuous on all of R^2, $w \in A^r(\Gamma)$ by the Cauchy-Riemann equations. Using the operator $\bar{\partial}$, we then find that $\varphi = +w - Az$ with a constant $A > 0$, has the properties claimed.

LEMMA 1. *Let $g \in C^1(R^2)$, $g = 0$ when $x^2 + y^2 \geq 1$. Then*

$$\iint \exp 2\pi i(ux + vy) g(x, y)(z - \zeta)^{-1} dx\, dy = O(1) \cdot (1 + |u| + |v|)^{-1}.$$

Proof. It will be convenient to abbreviate $e(t) \equiv \exp 2\pi i t$. Clearly nothing is lost in assuming $v = 0$, $u > 0$. Moreover, only the case $|\zeta| < 2$ is interesting and $\zeta = 0$ is typical. Let $H(x, y)$ be a radial function, of class C^1 and vanishing for $x^2 + y^2 > 1$, and let $H(0,0) = 1$. Then

$$\iint e(ux) H(x, y) z^{-1} dx\, dy = \int_0^{2\pi} \int_0^{\infty} e(ur\cos\theta) H(r) e^{-i\theta}\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^{\infty} e(ur\cos\theta) H(r) \cos\theta\, dr\, d\theta.$$

Now $\int_0^t e(ur\cos\theta)\cos\theta\, dr = O(|u|^{-1})$ uniformly for $u > 0$, $t > 0$, so the integral is $O(|u|^{-1})$, since $\int_0^\infty |H'(r)|\, dr < \infty$.

Moreover, $[g(x, y) - g(0,0)H(x, y)](x + iy)^{-1}$ has a gradient in $L^1(R^2)$, whence

$$\iint e(ux)[g(x, y) - g(0,0)H(x, y)](x + iy)^{-1} dx\, dy = O(|u|^{-1}).$$

A sequence of functions. We choose and fix a function $a \in C^\infty(R)$, such that $a \geq 0$, $\int_{-1}^1 a(t)\, dt = 1$, and $a(t) = 0$ for $|t| \geq 1$. We then define $A_T(x)$ (or $A(T; x)$ for typographical reasons) for $T \geq 2$ as $Ta(Tx)$ for $|x| \leq 1/2$, and extend A_T to be 1-periodic on R. For the Fourier expansion

$$A_T(x) = 1 + \sum{}' a_n^T e(nx)$$

we have

$$|a_n^T| \leq 1, \quad |a_n^T| \leq c_p\, T^p |n|^{-p},$$

for $p = 1, 2, 3, \ldots$.

Let now $G(x, y)$ be a function of class C^∞ and compact support, let $S > 2$ and $T > 2$ be parameters, and

$$F(S, T; \zeta) = \iint G(x, y) A(T; y - Sx)(z - \zeta)^{-1}\, dx\, dy.$$

LEMMA 2. *$F(S, T)$ and $\partial^p F(S, T)/\partial\eta^p = O(T^r S^{-1})$, $1 \le p \le r$.*

Proof. Writing the Fourier expansion

$$A(T; y - Sx) - 1 = \Sigma' a_n^T e(ny - nSx),$$

we substitute this in $F(S, T; \zeta)$. The partial derivatives $\partial/\partial\eta$, $\partial^2/\partial\eta^2, \ldots$ can be effected by applying $\partial/\partial y$, $\partial^2/\partial y^2, \ldots$ to the cofactor of $(z - \zeta)^{-1}$. After using Leibniz' rule in these derivatives, and applying Lemma 1 to the resulting integrals we get, for $\partial^r/\partial y^r$ (taking $p = r + 1$) $c(G)\Sigma_1^\infty n^r \cdot \min(1, n^{-r-1}T^{r+1})n^{-1}S^{-1} \le c(G)T^r S^{-1}$.

To prove Theorem 2, we choose a smooth function $G_0 \ge 0$, vanishing off $0 < x < 1$, $0 < y < 1/2$, and choose $S_k > T_k > 4 + k$ so that the sequences defined by

$$w_k(\zeta) \equiv \iint G_0(x, y) \prod_1^j A(T_j, y - S_j x)(z - \zeta)^{-1}\, dx\, dy,$$

and $\partial^p w_k/\partial\eta^p$ $(1 \le p \le r)$ converge uniformly; of course we require that $w = \lim w_k \not\equiv 0$. At each step, this can be accomplished by choosing $S_k = c_{k-1}T_k^r$, with a constant c_{k-1} depending only on G_0 and $S_1, T_1, \ldots, S_{k-1}, T_{k-1}$. Furthermore, since $\limsup \psi(t)t^{-1/r+1} = +\infty$, we can choose T_k so that $T_k^{-1} < k^{-1}\psi(c_{k-1}^{-1}T_k^{-r-1})$.

Clearly w is holomorphic off the set $0 \le x \le 1$, $0 \le y \le \frac{1}{2}$ $|y - S_k x| \le T_k^{-1}$ (modulo 1). On this set y is a single-valued function of x, $y = y(x)$; let E be the closed set in $[0, 1]$ over which y is defined, and let $\bar{y}$ be obtained from y by linear interpolation on the intervals contiguous to E in its convex hull.

We first calculate the modulus of continuity $\omega(y; h)$ for certain values of $h > 0$. Let $(x_1, y_1), (x_2, y_2)$ belong to E, and $|x_1 - x_2| \le S_k^{-1}/4$. We write, for $j = 1, 2$

$$y_j = S_k x_j + \theta_j T_k^{-1} + N_j \qquad \left(N_j \in Z,\ |\theta_j| \le 1\right).$$

Since $T_k^{-1} < 1/4$ and $S_k|x_1 - x_2| < 1/4$, we have $|y_1 - y_2| \le 2T_k^{-1} + S_k|x_1 - x_2| < 1$ (modulo 1). Since $|y_1 - y_2| < 1/2$, then $N_1 = N_2$ and $|y_1 - y_2| \le 2T_k^{-1} + S_k|x_1 - x_2|$. It follows from this that $|y_1 - y_2| \le 2T_k^{-1} + 2S_k|x_1 - x_2|$ for *all* values of x_1, x_2 in E, that is $\omega(y, h) \le 2T_k^{-1} + 2S_k h$. From this it follows that $\omega(\bar{y}, h) \le 4T_k^{-1} + 4S_k h$, and in particular $\omega(\bar{y}, S_k^{-1}T_k^{-1}) \leq 8T_k^{-1}$. We recall that $S_k = c_{k-1}T_k^r$, so that

$$\omega\left(\bar{y}, S_k^{-1}T_k^{-1}\right) = \omega\left(\bar{y}, c_{k-1}^{-1}T_k^{-r-1}\right) \leq 8T_k^{-1} < 3k^{-1}\psi\left(c_{k-1}^{-1}T_k^{-r-1}\right),$$

or $\liminf \omega(\bar{y}, h)/\psi(h) = 0$.

We shall now select a constant $A > 0$ so that each mapping $w_k - Az = \varphi_k$ has positive determinant and $|d\varphi_k| \geq |dz|$. The determinant is $|\partial\varphi_k|^2 - |\bar{\partial}\varphi_k|^2$, and the minimum of $|d\varphi_k|/|dz|$ is $||\partial\varphi_k| - |\bar{\partial}\varphi_k||$. Thus we need to choose A so that $|\partial w_k - A| \geq 1 + |\bar{\partial} w_k|$, using the inequalities $|\partial w_k/\partial y| \leq c_1$, $\bar{\partial} w_k \leq 0$. This can be done with $A = 1 + 2c_1$. It is worthwhile to observe that $\varphi(\Gamma)$ must have positive measure, for otherwise φ^{-1} is entire and $w \equiv 0$. This completes the proof of Theorem 2.

Remark 1. The function w found in Theorem 2 can be represented

$$w(\zeta) = \iint (z - \zeta)^{-1}\mu(dz),$$

with a certain non-negative μ on Γ. Choosing Γ of class N_{r+1}^a, but not N_r^a (this will be true if $\psi(t) = t^\sigma$, $\sigma = 2(2r + 3)^{-1}$) we define

$$\Phi(\zeta) = \iint \log|z - \zeta|\mu(d\zeta), \qquad \zeta \notin \Gamma.$$

Then $\partial\Phi = -1/2w$ off G, and Φ has a gradient locally in L^1. Hence $\partial\Phi = -w/2$ in the classical sense, and since Φ is real, $\Phi \in C^{r+1}(\Gamma)$. Now $\Phi \in C^1(R^2)$ is clear, but $\Phi \notin C^2(R^2)$, because $\nabla^2\Phi = 2\pi\mu$, a singular measure.

Remark 2. It seems very difficult to prove that $N_1^a \neq N_1$. To explain this, we summarize a sufficient condition for a Cantor set S to be of class N_1^a, from [1]. To each $\varepsilon > 0$, there exist Jordan curves $\gamma_1, \dots, \gamma_m$ in $R^2 \setminus S$, of length $< \varepsilon$, surrounding S in the homology sense, and $\sum l(\gamma_j)^2 \leq C$. Unfortunately, this condition also forces $S \in N_1$. Our condition can be adapted to curves crossing Γ only in horizontal segments, and again forces $\Gamma \in N_1$.

References

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