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GLOBAL EXISTENCE AND UNIQUENESS RESULTS FOR SINGULAR SOLUTIONS OF THE CAPILLARITY EQUATION

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We study the singular solutions of the capillarity equation

$$\operatorname{div}\frac{Dv}{\sqrt{1+|Dv|^2}} = Kv \quad \text{in } \mathbf{R}^N,$$

with a $K < 0$. We prove the global existence of a rotationally symmetric solution. We prove the uniqueness of a symmetric solution negative and concave near the origin.

Introduction. In this paper we study the existence and uniqueness of a singular solution of the capillarity equation in $\mathbf{R}^N$:

$$\operatorname{div}\left(Dv/\left(\sqrt{1+|Dv|^2}\right)\right) = Kv, \tag{1}$$

with a $K < 0$. The situation is quite different from the case $K \geq 0$, where every isolated singularity is removable [4]. We restrict our attention to the symmetric case where v depends only on the distance r from the origin.

Let

$$u(r) = \sqrt{-\frac{K}{N-1}}\, v\left(\sqrt{-\frac{N-1}{K}}\, r\right).$$

Then the equation is equivalent to

$$\left(\frac{r^{N-1}u'}{\sqrt{1+u'^2}}\right)'(r) = -(N-1)r^{N-1}u(r). \tag{2}$$

In [1], P. Concus and R. Finn conjectured the global existence and uniqueness of a singular solution of (2). They proved the local existence of a function u of the form

$$u(r) = -\frac{1}{r} + \frac{N+3}{2(N-1)}r^3 + r^3\varepsilon(r), \tag{3}$$

where $\varepsilon(r) = o(r)$ when r goes to 0. Up to the change of u into $-u$, they got local uniqueness in a particular class: functions such that $\varepsilon(r)/r^p$ ($p < 4$) and $r\varepsilon'(r)$ are bounded. The solution has an asymptotic development in powers of r but the formal Taylor series is divergent.

In §1 we write the equation in terms of $z(r) = u'(r)/\sqrt{1 + u'^2(r)}$, which leads us to a second order nonlinear equation:

$$\Delta z(r) = (N-1)z(r)\left(\frac{1}{r^2} - \frac{1}{\sqrt{1 - z^2(r)}}\right), \tag{4}$$

with limit conditions $\lim_{r\to 0} z(r) = 1$, $\lim_{r\to 0} z'(r) = 0$.

We give an a priori energy estimate for z and u in §2.

Then, in §3 we improve the results of local existence and uniqueness: we try to draw the maximum profit from the fixed point method introduced in [**1**], adapted to the function z. We get the local existence and uniqueness of functions z such that $(z(r) - 1 + (r^4/2))/r^6$ is not too large, and then of functions u such that $(u'(r) - 1/r^2)$ is not too large. This result is an essential tool for uniqueness results of §5.

In §4, from the energy estimate for z, we get global existence in $[0, +\infty[$ for z, then for u. We study the behavior of u, z for large r. They are oscillatory and go to zero when r goes to infinity.

In §5, we prove the uniqueness of a solution z nonincreasing near 0, then the uniqueness of a solution u concave near 0. As the maximum principle fails, we use local comparison methods to obtain some accurate estimates near the origin, and prove that such functions z, u are in the classes of uniqueness defined in §3.

1. New formulation of the problem. Up to the change of u into $-u$, we shall deal with the existence and uniqueness of a singular symmetric solution of (2), *negative near the origin*. Let us recall the estimates given in [**2**]: every singular solution u satisfies near the origin

$$-\left(\frac{\pi + \sqrt{2}}{\sqrt{N-1}} r + o(r)\right) \le u(r) + \frac{1}{r} \le \frac{\sqrt{2}}{\sqrt{N-1}} r + o(r), \tag{5}$$

$$\frac{u'(r)}{\sqrt{1 + u'^2(r)}} \ge 1 - \left(\frac{(\pi + \sqrt{2})^2}{2} r^4 + o(r^4)\right). \tag{6}$$

Now we make a change of unknown function.

PROPOSITION 1. *The existence and uniqueness of a C^2 function u, singular symmetric solution of* (1), *is equivalent to the existence and uniqueness of a C^2 function z solution of the second order semilinear elliptic equation*:

$$z''(r) + (N-1)\frac{z'(r)}{r} = (N-1)\left(\frac{z(r)}{r^2} - \frac{z(r)}{\sqrt{1 - z^2(r)}}\right), \tag{7}$$

with limit conditions

$$\lim_{r\to 0} z(r) = 1; \quad \lim_{r\to 0} z'(r) = 0. \tag{8}$$

Functions u and z are linked by the relations

$$z(r) = \frac{u'(r)}{\sqrt{1+u'^2(r)}} = \sin\psi(r), \tag{9}$$

$$z'(r) + (N-1)\frac{z(r)}{r} = -(N-1)u(r), \tag{10}$$

where ψ *is the angle between the tangent at* $(r, u(r))$ *and the r axis.*

Proof. Let u be a singular solution of (2) and z be defined by (9). Then equation (2) takes the form (10), also equivalent to

$$z(r) = -\frac{N-1}{r^{N-1}}\int_0^r \rho^{N-1}u(\rho)\,d\rho, \tag{11}$$

since, from (5), (6), $r^{N-1}u(r) = O(1)$, $r^{N-1}z(r) = o(1)$, when r goes to 0. Now (9) is obviously equivalent to

$$u'(r) = z(r)/\sqrt{1-z^2(r)}\,; \tag{12}$$

then we derive (10) and get (7); then (8) using (5), (6). Conversely let z be a solution of (7), (8) and define u by (10); then u satisfies (12), (9), then (2), and $u(r) \sim_{r\to 0} -1/r$, so that u is singular.

2. A priori estimates. Now we get an estimate of the energy for z, which later on will be fundamental.

PROPOSITION 2. *Let z be a solution of* (7), (8), *defined on an interval* $[0, R[$. *Then*

$$g(r) = \frac{z'^2(r)}{2(N-1)} + \frac{1-z^2(r)}{2r^2} - \sqrt{1-z^2(r)} < 0, \tag{13}$$

and $g'(r) < 0$ *in* $]0, R[$. *Consequently*

$$o < \sqrt{1-z^2(r)} < 2r^2, \tag{14}$$

$$|z'(r)| < \sqrt{N-1}\,\min(r, \sqrt{2}), \quad in\ ]0, R[. \tag{15}$$

Proof. Multiplying (7) by $z'(r)$, we get

$$g'(r) = -\frac{z'^2(r)}{r} - \frac{1-z^2(r)}{r^3} < 0, \tag{16}$$

since $z^2(r) < 1$; multiplying (7) by $r^2 z'(r)$, we get also

$$(17)\qquad (r^2 g)'(r) = -\left(\frac{N-2}{N-1} z'^2(r) + 2r\sqrt{1 - z^2(r)}\right) < 0;$$

now from (8) we have $\lim_{r \to 0} r^2 g(r) = 0$, then $r^2 g(r) < 0$ in $]0, R[$; hence (13) and (14). Then (15) follows from the fact that

$$(18)\qquad 2g(r) = \frac{z'^2(r)}{N-1} + \left(\frac{\sqrt{1 - z^2(r)}}{r} - r\right)^2 - r^2.$$

Consequences.

(a) We obtain other estimates for z and u in $]0, R[$:

$$(19)\qquad 1 > z(r) > 1 - \frac{\sqrt{N-1}}{2} r^2,$$

from (8), (15), and

$$(20)\qquad -\frac{r}{\sqrt{N-1}} < u(r) + \frac{1}{r} < \frac{N+1}{2\sqrt{N-1}} r,$$

from (10), (19).

Now from (14), (19) and (20), we deduce

$$(21)\qquad r^2 \leq \max\left(\frac{1}{2}, \frac{2}{\sqrt{N-1}}\right) \Rightarrow z(r) > 0 \Rightarrow u'(r) > 0,$$

$$(22)\qquad r^2 < 2\frac{\sqrt{N-1}}{N+1} \Rightarrow u(r) < 0 \Rightarrow z(r) > 0 \Rightarrow u'(r) > 0.$$

(b) We can improve the local estimates (5), (6): from (10), (14) and (15) we get, near the origin,

$$(23)\qquad 1 > z(r) > 1 - (2r^4 + o(r^4)),$$

$$(24)\qquad -\frac{r}{\sqrt{N-1}} < u(r) + \frac{1}{r} < \frac{r}{\sqrt{N-1}} + O(r^3).$$

REMARK. Let us note an estimate of the energy for u, which has often been used in [**2**], [**3**]: let

$$(25)\qquad f(r) = \frac{u^2(r)}{2} - \frac{\sqrt{1 - z^2(r)}}{N-1};$$

then

$$(26)\qquad f'(r) = -\frac{z^2(r)}{r\sqrt{1 - z^2(r)}} < 0 \quad \text{in }]0, R[;$$

hence for any $r, s \in]0, +\infty[$ such that $r > s$,

$$\frac{u^2(r)}{2} - \frac{\sqrt{1 - z^2(r)}}{N-1} \leq \frac{u^2(s)}{2} - \frac{\sqrt{1 - z^2(s)}}{N-1}. \tag{27}$$

3. Local existence and uniqueness. From Proposition 1, and (3), (9), we still obtain the local existence of a solution Z of the problem (7), (8) of the form

$$Z(r) = 1 - r^4/2 + O(r^8),$$

near the origin. Now we prove a quite more accurate result, based on a fixed point method analogous to [**1**].

THEOREM 1. *Let $M < M_0 = (N+8)/3\sqrt{N-1}$. Then, for R_0 sufficiently small, the problem* (7), (8) *admits a unique C^2 solution Z in $]0, R_0]$ such that*

$$\begin{cases} Z(r) = 1 - r^4/2 + r^6 w(r), \\ |w(r)| \leq M \quad \text{in }]0, R_0]. \end{cases} \tag{28}$$

Proof. Let for any $y \in]-1, +1[$ and $r > 0$

$$\Phi(y, r) = (N-1)\left(\frac{y}{r^2} - \frac{y}{\sqrt{1-y^2}}\right). \tag{29}$$

Let $M < M_0$, $R > 0$, and denote

$$B_{M,R} = \left\{v \in C^0([0, R]) \mid \|v\| = \max_{r \in [0,R]} |v(r)| \leq M\right\}.$$

Then one can see as in [**1**] that the problem is equivalent to a fixed point problem: find a function $w \in B_{M,R}$ such that

$$w = T(w), \tag{30}$$

where

$$T(w)(r) = \frac{r^{-(N+8)/2}}{\sqrt{N-1}} \int_0^r \tau^{(N+2)/2} F(w(\tau), \tau) \sin \frac{\sqrt{N-1}}{2}\left(\frac{1}{\tau^2} - \frac{1}{r^2}\right) d\tau, \tag{31}$$

$$F(w, r) = 2(N+2)r^2 + \frac{N(N-4)}{4} r^4 w + (N-1)w + \Phi\left(1 - \frac{r^4}{2} + r^6 w, r\right). \tag{32}$$

Let $w \in B_{M,R}$. Then there exists $\theta(r) \in [0,1]$ such that

$$\Phi\left(1-\frac{r^4}{2}+r^6w(r),r\right)=\Phi\left(1-\frac{r^4}{2},r\right)+r^6w(r)\frac{\partial\Phi}{\partial y}\left(1-\frac{r^4}{2},r\right)$$
$$+r^{12}\frac{w^2(r)}{2}\frac{\partial^2\Phi}{\partial y^2}\left(1-\frac{r^4}{2}+r^6\theta(r)w(r),r\right).$$

Now

$$\frac{\partial\Phi}{\partial y}(y,r)=(N-1)\left(\frac{1}{r^2}-(1-y^2)^{-3/2}\right),$$

$$\frac{\partial^2\Phi}{\partial y^2}(y,r)=-3(N-1)y(1-y^2)^{-5/2},$$

hence for sufficiently small r,

$$\Phi\left(-1+\frac{r^4}{2}+r^6w(r),r\right)$$
$$=(N-1)\left(-\frac{r^2}{8}+O(r^6)\right.$$
$$\left.+\frac{5}{8}r^4w(r)-w(r)-\frac{3}{2}r^2w^2(r)(1+O(r^2))\right),$$

$$r^{(N+2)/2}F(w(r),r)=\frac{15N+33}{8}r^{(N+6)/2}$$
$$+\frac{(N+1)(2N-5)}{8}r^{(N+10)/2}w(r)$$
$$-\frac{3(N-1)}{2}r^{(N+6)/2}w^2(r)(1+O(r^2)).$$

Then we integrate by parts the first term, cf. [**2**], and get

$$T(w)(r)=\frac{15N+33}{8(N-1)}r^2+O(r^4)+O(r^2)+R(r)=O(r^2)+R(r),$$

with

$$|R(r)|\leq\frac{r^{-(N+8)/2}}{\sqrt{N-1}}\frac{3(N-1)}{2}\|w\|^2\int_0^r\tau^{(N+6)/2}d\tau=\frac{\|w\|^2}{M_0}\leq\frac{M}{M_0}.$$

As $M<M_0$, we deduce that there exists $R_1=R_1(M)>0$ such that T maps $B_{M,R}$ into itself for $R\leq R_1$.

Moreover, let $w, \hat{w} \in B_{M,R_1}$; then there exists $\eta, \xi \in B_{M,R_1}$ such that $\eta(r) \in [w(r), \hat{w}(r)]$ and

$$\begin{aligned}
&\Phi\left(1 - \frac{r^4}{2} + r^6\hat{w}(r), r\right) - \Phi\left(1 - \frac{r^4}{2} + r^6 w(r), r\right) \\
&= r^6(\hat{w}(r) - w(r))\frac{\partial\Phi}{\partial y}\left(1 - \frac{r^4}{2} + r^6\eta(r), r\right) \\
&= (\hat{w}(r) - w(r))\left(r^6\frac{\partial\Phi}{\partial y}\left(1 - \frac{r^4}{2}, r\right) + r^{12}\eta(r)\frac{\partial^2\Phi}{\partial y^2}\left(1 - \frac{r^4}{2} + r^6\xi(r), r\right)\right) \\
&= (N-1)\left(-1 + \frac{5}{8}r^4 + O(r^6) - 3r^2\eta(r)(1 + O(r^2))\right)(\hat{w}(r) - w(r)) \\
&= (N-1)(-1 - 3r^2\eta(r) + O(r^4))(\hat{w}(r) - w(r)),
\end{aligned}$$

hence

$$\begin{aligned}
&r^{(N+2)/2}(F(\hat{w}(r), r) - F(w(r), r)) \\
&\qquad = \left(-3(N-1)r^{(N+6)/2}\eta(r) + O(r^{(N+10)/2})\right)(\hat{w}(r) - w(r)),
\end{aligned}$$

$$|T(\hat{w})(r) - T(w)(r)| \le \left(\frac{2}{M_0}\max(\|w\|, \|\hat{w}\|) + O(r^2)\right)\|\hat{w} - w\|.$$

Then for any $\varepsilon > 0$ there exists $R_0 = R_0(\varepsilon, M) < R_1$ such that if $R \le R_0$,

$$\|T(\hat{w}) - T(w)\| \le \left(\frac{2}{M_0}\max(\|w\|, \|\hat{w}\|) + \varepsilon\right)\|\hat{w} - w\|$$

and

$$\|T(w)\| \le \varepsilon M_0 + \frac{\|w\|^2}{M_0}.$$

Then $\|T^n(w)\| \le v_n$ where $v_n = \varepsilon M_0 + (v_{n-1}^2/M_0)$, $v_0 = M$. Now take $\varepsilon < \min((M/M_0^2)(M_0 - M), 1/6)$; then $v_n \searrow \lambda$ where $\lambda = (M_0/2)(1 - \sqrt{1 - 4\varepsilon}) < 2\varepsilon M_0 < M_0/3$. Then

$$\|T^n(\hat{w}) - T^n(w)\| \le a_n\|\hat{w} - w\|,$$

where

$$a_n = \prod_{k=0}^{n}\left(\frac{2v_n}{M_0} + \varepsilon\right); \qquad \lim_{n\to+\infty}\frac{a_{n+1}}{a_n} = \frac{2\lambda}{M_0} + \varepsilon < 1,$$

then $\lim_{n\to+\infty} a_n = 0$; hence for large n, T^n is a strict contraction. Then T has a unique fixed point in B_{M,R_0}.

REMARK. As in [**1**], we can prove that the function Z has an asymptotic development near 0 in powers of r^4 whose first terms are

$$Z(r) = 1 - \frac{r^4}{2} + \frac{15N+33}{8(N-1)}r^8 + o(r^8). \tag{33}$$

Now from (7)

$$(r^{N-1}Z')'(r) = r^{N-1}\Phi(Z(r), r),$$

hence Z', then Z'' and all the derivatives of Z have an asymptotic development near 0, obtained by successive differentiations of the development of Z, and Z is in $C^\infty([0, R_0])$. Indeed, by recursion the derivatives cannot have a development with negative powers of r. Then with equation (7) we obtain by recursion all the terms of the development and deduce the divergence of the Taylor series. Now observe that

$$Z'(r) = -2r^3 + o(r^3), \quad Z''(r) = -6r^2 + o(r^3),$$

so that $Z'(r)$ and $Z''(r)$ are negative near the origin.

Theorem 1 is still an improvement of the results in [**1**]. Let us apply it to the function u.

COROLLARY 1. *Let $\tilde{M} < M_0$. Then for $\tilde{R}_0$ sufficiently small, the problem* (2) *admits a unique C^2 solution U in $]0, \tilde{R}_0]$, singular, such that*

$$\begin{cases} U'(r) = \dfrac{1}{r^2} + \omega(r), \\ |\omega(r)| \le \tilde{M} \quad \textit{in }]0, \tilde{R}_0]. \end{cases} \tag{34}$$

Proof. Let $\tilde{M} < M_0$ and $M < \tilde{M}$, and U be the singular solution of (2) associated with the solution Z defined by (28). Then by calculation

$$U'(r) = \frac{Z(r)}{\sqrt{1 - Z^2(r)}} = \frac{1}{r^2} + w(r) + O(r^2),$$

hence for $\tilde{R}_0$ sufficiently small U satisfies (34). Let u be another singular solution satisfying (34) in $]0, \tilde{R}_0]$ and z be the solution of (7) (8) associated to u. Then by calculation

$$z(r) = \frac{u'(r)}{\sqrt{1 + u'^2(r)}} = 1 - \frac{r^4}{2} + r^6\omega(r) + o(r^6).$$

Then for R sufficiently small

$$\begin{cases} z(r) = 1 - \dfrac{r^4}{2} + r^6 w(r), \\ |w(r)| \leq \dfrac{\tilde{M} + M_0}{2} \quad \text{in }]0, R]; \end{cases}$$

hence $z(r) = Z(r)$ near the origin, hence in $]0, \tilde{R}_0]$.

4. Global existence and asymptotic properties. Here we prove the existence of global solutions.

THEOREM 2. *Each solution z of* (7)(8), *or equivalently each singular solution u of* (2), *admits a unique extension defined on the whole interval* $]0, +\infty[$.

Proof. From Proposition 1 we have only to consider z. Let z be a solution of (7) (8) defined on an interval $[0, R)$. Let $x = (z, z')$, then equation (7) takes the form

$$x'(r) = G(r, x(r)), \tag{35}$$

where G is a C^1 function on the open set $W =]0, +\infty[\times]-1, +1[\times\mathbb{R}$. Then z admits a unique maximal extension, still called z, defined on an interval $[0, R_m)$.

Suppose $R_m < +\infty$. From (15), z' is bounded; hence $z(r)$ has a limit z_m when $r \nearrow R_m$. From Proposition 2, the energy function g, decreasing and bounded below by -1, has a limit $\gamma < 0$. By contradiction this implies $z_m \neq \pm 1$. Then, from (7), z'' is bounded near R_m, hence $z'(r)$ has a limit z'_m. Then $(R_m, z_m, z'_m) \in W$, hence z admits an extension to an interval $[0, R_m + \varepsilon)$, which is impossible.

Now we make precise the behavior near infinity of any solution:

THEOREM 3. *Each solution z of* (7), (8) *admits a countable number of zeros, asymptotically separated by a distance of $\pi/\sqrt{N-1}$, and*

$$\frac{z^2(r) + z'^2(r)}{r} \in L^1(]a, +\infty[), \quad \textit{for any } a > 0, \tag{36}$$

$$\lim_{r \to +\infty} z(r) = \lim_{r \to +\infty} z'(r) = \lim_{r \to +\infty} u(r) = \lim_{r \to +\infty} u'(r) = 0, \tag{37}$$

$$\frac{u^2(r) + u'^2(r)}{r} \in L^1(]a, +\infty[), \quad \textit{for any } a > 0. \tag{38}$$

Proof. Let z be a solution of (7), (8) on $[0, +\infty[$. From Proposition 2, the energy function g has a limit $\gamma < 0$ when r goes to $+\infty$. By contradiction, this implies that $\liminf_{r\to+\infty}\sqrt{1-z^2(r)} > 0$. Then there exists $\alpha > 0$ such that $\sqrt{1-z^2(r)} > \alpha$ for large r.

Let us make the substition $z = r^{-(N-1)/2}y$ in equation (7); this equation becomes

$$y''(r) + p(r)y(r) = 0, \tag{39}$$

where

$$p(r) = (N-1)\left(\frac{1}{\sqrt{1-z^2(r)}} - \frac{N+1}{4r^2}\right); \tag{40}$$

for large r, we have $(N-1)/2 < p(r) < (N-1)/\alpha$; hence, from the Sturm comparison theorem, z is oscillatory; moreover, let

$$0 < r_1 < r_2 < \cdots < r_n < r_{n+1} < \cdots$$

be the zeros of z, simple because of the local uniqueness in (35); then the distance $d_n = r_{n+1} - r_n$ between two consecutive zeros satisfies

$$\sqrt{\alpha}\,\frac{\pi}{\sqrt{N-1}} < d_n < \sqrt{2}\,\frac{\pi}{\sqrt{N-1}}. \tag{41}$$

Moreover for any r_n such that $r_n \geq 1$ — if $N = 2$ there is no condition since $r_1 > 1$ from (21) — there exists a unique point $s \in]r_n, r_{n+1}[$ where $z'(s_n) = 0$: indeed, if not, there would exist an $r \in]r_n, r_{n+1}[$ such that

$$0 = (r^{N-1}z')'(r) = r^{N-1}z(r)\left(\frac{1}{r^2} - \frac{1}{\sqrt{1-z^2(r)}}\right),$$

and then $z(r) = 0$, which is impossible.

On the other hand, from (16) we deduce that, for any $a > 0$, $z'^2(r)/r \in L^1(]a, +\infty[)$. In the same way, the function f defined in (25) is decreasing and bounded below by $-1/(N-1)$; hence it has a limit when r goes to $+\infty$; then from (26) we deduce that

$$\frac{z^2(r)}{r\sqrt{1-z^2(r)}} \in L^1(]a, +\infty[),$$

hence (36).

Now let us prove (37), (38). Suppose first that $\gamma = -1$; then

$$\lim_{r\to+\infty}\left(\frac{z'^2(r)}{2(N-1)} + \frac{1-z^2(r)}{2r^2} + 1 - \sqrt{1-z^2(r)}\right) = 0,$$

then $\lim_{r\to+\infty} z'(r) = \lim_{r\to+\infty} z(r) = 0$. From (11) and (13) we get (37) and (38).

Suppose now that $\gamma > -1$; we will obtain a contradiction. For the extremal points s_n of z on $[r_n, r_{n+1}]$ we have $\lim_{n\to+\infty}\sqrt{1-z(s_n)^2} = -\gamma \in]0,1[$, then $\lim_{n\to+\infty}|z(s_n)| = k \in]0,1[$. Let σ_n be the unique point of $]r_n, s_n[$ where $z(\sigma_n) = z(s_n)/2$. Then, from (15),

$$\left|\frac{z(s_n)}{2}\right| = |z(\sigma_n) - z(r_n)| \le \sqrt{2(N-1)}(\sigma_n - r_n).$$

Hence with (41) we get for large n

$$\frac{k}{4\sqrt{N-1}} < \sigma_n - r_n < \frac{\sqrt{2}\pi}{\sqrt{N-1}}. \tag{42}$$

Now for any $r \in [r_n, \sigma_n]$, $\sqrt{1-z^2(r)} \ge \sqrt{1-z^2(s_n)/4}$, then from the expression of g,

$$\frac{z'^2(r)}{2(N-1)} \ge g(r) - \frac{1-z^2(r)}{2r^2} + \sqrt{1-\frac{z^2(s_n)}{4}};$$

let $\mu = \gamma + \sqrt{\gamma^2+3}/2 > 0$; hence for large n

$$z'^2(r) \ge 2(N-1)\mu. \tag{43}$$

From (42), (43) we deduce that for n_0 sufficiently large,

$$\int_{n_0}^{+\infty} \frac{z'^2(r)}{r}\,dr \ge \sum_{n\ge n_0}\int_{r_n}^{\sigma_n} \frac{z'^2(r)}{r}\,dr \ge \frac{\sqrt{N-1}}{2}k\mu \sum_{n\ge n_0}\frac{1}{\sigma_n}.$$

Now from (41), (42), $\sigma_n = O(r_n) = O(n)$. This is impossible, since $z'^2(r)/r$ is integrable on $]n_0, +\infty[$.

Finally, we have $\lim_{r\to+\infty} p(r) = (N-1)$, since $\lim_{r\to+\infty} z(r) = 0$. From the Sturm comparison theorem we get

$$\lim_{n\to+\infty}\left(d_n - \frac{\pi}{\sqrt{N-1}}\right) = 0.$$

REMARKS.

(i) Obviously the function u admits a countable number of zeros ρ_n, such that, from (22):

$$0 < \rho_1 < r_1 < \rho_2 < r_2 < \cdots < \rho_n < r_n < \rho_{n+1} < r_{n+1} \cdots; \tag{44}$$

on $[\rho_n, \rho_{n+1}]$, u has a unique extremum in r_n. From (27) we get $|u(r_n)| > |u(r_{n+1})|$, that is to say $|z'(r_n)| > |z'(r_{n+1})|$, for any n.

Moreover $f(r_1) < f(\rho_1)$; this or (15) implies, cf. [5]:

$$0 < u(r_1) = -\frac{z'(r_1)}{N-1} < \sqrt{2/(N-1)}\,. \tag{45}$$

(ii) Consider for simplification the case $N = 2$. The function p defined by (40) satisfies $p(r) > (1 - \frac{3}{4}r^2)$. In the Bessel equation of order 1,

$$\zeta''(r) + \frac{\zeta'(r)}{r} = \frac{\zeta(r)}{r^2} - \zeta(r), \tag{46}$$

we make the substition $\zeta = r^{-1/2}\xi$; this equation becomes

$$\xi''(r) + \left(1 - \frac{3}{4r^2}\right)\xi(r) = 0. \tag{47}$$

From the Sturm comparison theorem, between two successive zeros in $]0, +\infty[$ of any Bessel function of order 1, there exists at least one zero of z; in fact exactly one for large r since the zeros of the Bessel functions are asymptotically separated by π. Likewise between 0 and the first zero $R_1 \neq 0$ of the function J_1, there exists at least one zero of z (if not, for any $\varepsilon \in]0, R[$, we would have, with $\xi = r^{1/2}J_1$,

$$[y\xi' - \xi y']_{\varepsilon}^{R_1} = \int_{\varepsilon}^{R_1} \left(p(r) - 1 + \frac{3}{4}r^2\right) y(r)\xi(r)\,dr > 0;$$

now $\xi(\varepsilon) = O(\varepsilon^{3/2})$, $y'(\varepsilon) = O(\varepsilon^{-1/2})$, hence $\lim_{\varepsilon\to 0}\xi(\varepsilon)y'(\varepsilon) = 0$;

$$\lim_{\varepsilon\to 0} y(\varepsilon)\xi'(\varepsilon) = \lim_{\varepsilon\to 0} z(\varepsilon)\varepsilon^{1/2}\left(\varepsilon^{1/2}J_1'(\varepsilon) + \varepsilon^{-1/2}\frac{J_1(\varepsilon)}{2}\right) = 0,$$

hence $y(R_1)\xi'(R_1) > 0$, which is impossible since $y(R_1) > 0$, $\xi'(R_1) < 0$).

Using (22), we deduce the estimates

$$\sqrt{2/3} < \rho_1 < r_1 < R_1 \simeq 3.8; \quad \sqrt{2} < r_1; \tag{48}$$

notice that for the solutions Z and U we get numerically $\rho_1 \cong 1.5$, $r_1 \cong 2.8$.

It is an open question whether the extremal points of the function z satisfy $(z(s_n)) = O(1/\sqrt{s_n})$, as is the case for Bessel functions.

5. Uniqueness under growth conditions. We have seen in §4 that the solution Z defined in Theorem 1 is a decreasing function for small r. Differentiating (12), we get

$$u''(r) = z'(r)/(1 - z^2(r))^{3/2}, \tag{49}$$

so that the solution U is strictly concave for small r. We are going to prove reciprocally that any solution z nonincreasing for small r is equal to Z, any solution u concave for small r is equal to U:

THEOREM 4. *There is a unique solution z of* (7)(8) *in* $]0,+\infty[$ *such that z is nonincreasing near the origin. There is a unique singular solution u of* (2) *in* $]0,+\infty[$ *such that u is concave near the origin.*

Proof. Step 1. *An estimate for z.*

Let z be a solution such that $z'(r) \leq 0$ in an interval $]0, \alpha[$; in terms of u, that means from (49) that $u''(r) \leq 0$ in $]0, \alpha[$. Let $\rho \in]0, \alpha[$ be fixed. We are going to compare z to a function $\overline{w}$ of the form

$$\overline{w}(r) = ar^2 + br + cr^{1-N}, \tag{50}$$

such that

$$\overline{w}(\rho) = z(\rho), \quad \overline{w}'(\rho) = z'(\rho), \quad \overline{w}''(\rho) = z''(\rho). \tag{51}$$

We find

$$\begin{cases} a = -\dfrac{N-1}{N+1}\dfrac{z(\rho)}{\sqrt{1-z^2(\rho)}}, \\ b = \dfrac{1}{N}\left((N-1)\dfrac{z(\rho)}{\rho} + z'(\rho) + (N-1)\dfrac{\rho z(\rho)}{\sqrt{1-z^2(\rho)}}\right), \\ c = \dfrac{\rho^{N-1}}{N}\left(z(\rho) - \rho z'(\rho) - \dfrac{N-1}{N+1}\rho^2\dfrac{z(\rho)}{\sqrt{1-z^2(\rho)}}\right). \end{cases} \tag{52}$$

Then from equation (7) we get

$$\begin{aligned} &\left((\overline{w}-z)' + \frac{N-1}{r}(\overline{w}-z)\right)'(r) \\ &\quad = (N-1)\left(\frac{z(r)}{\sqrt{1-z^2(r)}} - \frac{z(\rho)}{\sqrt{1-z^2(\rho)}}\right) \\ &\quad = (N-1)\big(u'(r) - u'(\rho)\big). \end{aligned} \tag{53}$$

As u' is nonincreasing, we deduce from (51) that

$$\begin{aligned} &(\overline{w}-z)'(r) + \frac{N-1}{r}(\overline{w}-z)(r) \\ &\quad = r^{1-N}\big(r^{N-1}(\overline{w}-z)\big)'(r) \leq 0, \quad \text{in }]0, \alpha[, \end{aligned}$$

and then that

$$(\overline{w}-z)(r)(r-\rho)\le 0,\quad \text{in }]0,\alpha[.$$

As z is nonincreasing we deduce that

$$(\overline{w}(r)-z(\rho))(r-\rho)\le 0,\quad \text{in }]0,\alpha[.$$

Let $k=r/\rho$. Then

$$(k-1)(\overline{w}(k\rho)-z(\rho))\le 0\quad \text{in }]0,\alpha/\rho[. \tag{54}$$

From (50), (52), we obtain

$$\begin{aligned}\overline{w}(k\rho)-z(\rho)\\ &=\frac{k^{1-N}}{N}\Bigg[z(\rho)\big((N-1)k^N-Nk^{N-1}+1\big)+\rho z'(\rho)(k^N-1)\\ &\qquad -\frac{N-1}{N+1}\frac{\rho^2 z(\rho)}{\sqrt{1-z^2(\rho)}}\big(Nk^{N+1}-(N+1)k^N+1\big)\Bigg]\\ &=\frac{k^{1-N}}{N}(k-1)^2\left(z(\rho)P(k)+\frac{\rho z'(\rho)}{k-1}Q(k)-\frac{\rho^2 z(\rho)}{\sqrt{1-z^2(\rho)}}R(k)\right),\end{aligned}$$

where

$$\begin{cases}P(k)=(N-1)k^{N-2}+(N-2)k^{N-3}+\cdots+2k+1,\\ Q(k)=k^{N-1}+k^{N-2}+\cdots+1,\\ R(k)=\dfrac{N-1}{N+1}\big(Nk^{N-1}+(N-1)k^{N-2}+\cdots+2k+1\big).\end{cases} \tag{55}$$

As z is positive near 0, we obtain the inequalities, for sufficiently small ρ,

$$\begin{cases}\dfrac{\rho^2}{\sqrt{1-z^2(\rho)}}R(k)\ge P(k)+\dfrac{\rho z'(\rho)}{z(\rho)}\dfrac{Q(k)}{k-1}, & \text{if } k\in]1,\alpha/\rho[,\\[2ex] \dfrac{\rho^2}{\sqrt{1-z^2(\rho)}}R(k)\le P(k)+\rho\dfrac{z'(\rho)}{z(\rho)}\dfrac{Q(k)}{k-1}, & \text{if } k\in]0,1[.\end{cases} \tag{56}$$

Take first $k=1+\rho$, for sufficiently small ρ. From the majorization (16) we obtain

$$\begin{aligned}&\frac{\rho^2}{\sqrt{1-z^2(\rho)}}\frac{N(N-1)}{2}\left(1+\frac{2(N-1)}{3}\rho+o(\rho)\right)\\ &\quad\ge\frac{N(N-1)}{2}\left(1+\frac{2(N-2)}{3}\rho+o(\rho)\right)-N\sqrt{N-1}\,(\rho+o(\rho)),\end{aligned}$$

hence we get the estimate

$$\sqrt{1 - z^2(\rho)} \le \rho^2 + 2\left(\frac{1}{3} + \frac{1}{\sqrt{N-1}}\right)\rho^3 + o(\rho^3). \tag{57}$$

Now take $k = 1 - \rho$. Then we get in the same way the estimate

$$\sqrt{1 - z^2(\rho)} \ge \rho^2 - 2\left(\frac{1}{3} + \frac{1}{\sqrt{N-1}}\right)\rho^3 + o(\rho^3). \tag{58}$$

Hence

$$\sqrt{1 - z^2(\rho)} = \rho^2 + O(\rho^3), \tag{59}$$

so we still sharpen the estimate (14).

Step 2. *Improvement of the estimates.*

Consider first a point ρ where $z'(\rho) \ge -C(\rho^3 + o(\rho^3))$ for a $C > 0$. Take $k = 1 + q\rho^2$, where q is a parameter. Then from (56) we get

$$\frac{\rho^2}{\sqrt{1 - z^2(\rho)}} \frac{N(N-1)}{2}\left(1 + \frac{2(N-1)}{3}q\rho^2 + o(\rho^2)\right)$$
$$\ge \frac{N(N-1)}{2}\left(1 + \frac{2(N-2)}{3}q\rho^2 + o(\rho^2)\right) - N\frac{C}{q}\left(\rho^2 + o(\rho^2)\right),$$

hence, taking $q = \sqrt{3C/(N-1)}$ for the better estimate, we get

$$\sqrt{1 - z^2(\rho)} \le \rho^2 + 4\sqrt{\frac{3C}{N-1}}\,\rho^4 + o(\rho^4), \tag{60}$$

and, in the same way, with $k = 1 - q\rho^2$,

$$\sqrt{1 - z^2(\rho)} \ge \rho^2 - 4\sqrt{\frac{3C}{N-1}}\,\rho^4 + o(\rho^4). \tag{61}$$

Now consider the function $\varphi = \psi^2$, where,

$$\psi(r) = \frac{r^2 - \sqrt{1 - z^2(r)}}{r^4}; \tag{62}$$

then

$$\varphi'(r) = 2\psi(r)\psi'(r)$$
$$= 2\psi(r)\frac{r^{-5}}{\sqrt{1 - z^2(r)}}\left(rz(r)z'(r) - 2r^2\sqrt{1 - z^2(r)} + 4\left(1 - z^2(r)\right)\right).$$

Observe that there exists no neighborhood of 0 where $\psi(r) \le 0$: suppose $\psi(r) \le 0$ in $]0, \beta]$; from (7) we have

$$r^{1-N}(r^{N-1}z')'(r) = (N-1)z(r)\frac{\sqrt{1 - z^2(r)} - r^2}{r^2\sqrt{1 - z^2(r)}}, \tag{63}$$

hence, from (8), $r^{N-1}z'$ would be nondecreasing near 0, then z would be nondecreasing near 0; hence $z(r) = 1$, $\psi(r) = r^{-2}$ near 0, which is impossible.

Now consider three cases:

First case. There exists $\alpha > 0$ such that $\varphi'(r) \neq 0$, $\forall r \in]0, \alpha]$. Then $\psi(r) \neq 0$, hence $\psi(r) > 0$, $\forall r \in]0, \alpha]$. Moreover we have $\varphi'(r) > 0$, $\forall r \in]0, \alpha]$: if not, we would have $\varphi(r) > \varphi(\alpha) > 0$, hence $r^2 - \sqrt{1 - z^2(r)} > \psi(\alpha) r^4$, then from (8), (59) and (63)

$$(r^{N-1}z')'(r) < -\frac{N-1}{2}\psi(\alpha) r^{N-1},$$

near the origin; and integrating twice

$$z(r) \leq 1 - \frac{N-1}{4N}\psi(\alpha) r^2,$$

which is in contradiction with (23).

Now take ρ sufficiently small; since $\psi'(\rho) > 0$, we have

$$\begin{aligned} z'(\rho) &> \frac{1}{z(\rho)}\left(2\rho\sqrt{1 - z^2(\rho)} - 4\frac{1 - z^2(\rho)}{\rho}\right) \\ &\geq -2\rho^3(1 + O(\rho)); \end{aligned}$$

then from (60) (61) we get the estimate

$$\left|\sqrt{1 - z^2(\rho)} - \rho^2\right| \leq 4\sqrt{\frac{2}{3(N-1)}}\,\rho^4 + o(\rho^4). \tag{64}$$

Second case. For any $\alpha > 0$ there exists $r < \alpha$ such that $\psi(r) = 0$. Then there exists $r_1 < 1$ such that $\psi(r_1) = 0$. There exists $r_2 < r_1$ such that $\psi(r_2) > 0$. Consider a small $\rho < r_2$; then there exists $r_3 < \rho$ such that $\psi(r_3) = 0$. The function φ has a maximum on $[r_3, r_1]$ in a point $\bar{\rho}$ such that $\varphi(\bar{\rho}) > \varphi(r_2) > 0$. Then $\psi'(\bar{\rho}) = 0$, hence

$$z'(\bar{\rho}) = -2\bar{\rho}^3(1 + O(\bar{\rho})),$$

so that we have the estimate (64) at point $\bar{\rho}$, that is to say

$$|\psi(\bar{\rho})| \leq 4\sqrt{\frac{2}{3(N-1)}} + o(1);$$

then $|\psi(\rho)| \leq \psi(\bar{\rho})$, hence we get the estimate (64) at the point ρ.

Third case. There exists $\alpha_0 > 0$ such that $\psi(r) > 0$ in $]0, \alpha_0]$, and for any $\alpha > 0$, there exists $r < \alpha$ such that $\varphi'(r) = 0$. Then there exists

$r_1 < \alpha_0$ such that $\varphi'(r_1) = 0$. Consider a small $\rho < r_1$; there exists $r_2 < \rho$ such that $\varphi'(r_2) = 0$. The function φ has a maximum in $[r_2, r_1]$ in a point $\bar{\rho}$ such that $\varphi'(\bar{\rho}) = 0$, hence $\psi'(\bar{\rho}) = 0$. Hence we have again (64) at $\bar{\rho}$, then at ρ.

Step 3. *Conclusion.*

Consequently in any case we have the estimate (64). We deduce easily that, near the origin:

$$z(\rho) = 1 - \frac{\rho^4}{2} + \rho^6 w(\rho), \tag{65}$$

with

$$|w(\rho)| \leq 4\sqrt{\frac{2}{3(N-1)}} + o(1).$$

Now let us remember that the constant which defines the class of uniqueness in § 3 is $M_0 = (N+8)/3\sqrt{N-1}$, and observe that $4\sqrt{(2/3(N-1))} < M_0$ for any $N \geq 2$. Then from Theorem 1, we deduce that z is equal to Z, hence u is equal to U, near the origin, and on the whole interval $]0, +\infty[$.

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