PACIFIC JOURNAL OF MATHEMATICS
Vol. 126, No. 1, 1987

ON SOME REFLEXIVE OPERATOR ALGEBRAS CONSTRUCTED FROM TWO SETS OF CLOSED OPERATORS AND FROM A SET OF REFLEXIVE OPERATOR ALGEBRAS

E. V. KISSIN

In an earlier article by Kissin a new class of reflexive algebras possessing non-inner derivations implemented by bounded operators was introduced. Its method supplies us with many examples of reflexive algebras which have non-inner derivations implemented by bounded operators and for which effective analysis appears to be possible.

0. Introduction. It is generally well-known that all the derivations of W^*-algebras are inner. Christensen [**1**] and Wagner [**5**] have proved that the same is true of nest and quasitriangular algebras. Furthermore, although Gilfeather, Hopenwasser and Larson [**2**] have shown that some CSL-algebras may have non-inner derivations, none of these derivations are implemented by bounded operators. The present paper extends the approach adopted in the earlier article [**3**] and considers a new method of constructing reflexive operator algebras $\mathscr{A}$ from two given sets of closed operators $\{F_i\}_{i=1}^{n-1}$, $\{G_i\}_{i=1}^{n-1}$ and from a given set of reflexive operator algebras $\{\mathscr{T}_i\}_{i=1}^{n}$ (n can be a finite number or infinity).

The structure of these algebras and their properties are very interesting. For example, one can show that, if certain conditions are applied to the operators $\{F_i\}$ and $\{G_i\}$, then the algebras $\mathscr{A}$ are semi-simple and totally symmetric without, however, becoming C^*-algebras [**4**]. These algebras also possess the following property: if A is reversible and belongs to $\mathscr{A}$, then A^{-1} also belongs to $\mathscr{A}$. But in this paper we shall confine our discussion to two subjects:

(i) Under what conditions on $\{F_i\}$ and $\{G_i\}$ are the algebras $\mathscr{A}$ reflexive?

(ii) What is the structure of Lat $\mathscr{A}$?

Usually, when studing CSL-algebras, one considers the pairs $(\mathscr{A}, \operatorname{Lat}\mathscr{A})$ in the same way as one considers the pairs $(\mathscr{A}, \mathscr{A}')$ when studing W^*-algebras. However, it has been suggested [**3**] that in the general case of operator algebras $\mathscr{A}$ it would be more useful to consider the triplets $(\mathscr{A}, \operatorname{Lat}\mathscr{A}, \operatorname{Ad}\mathscr{A})$ where $\operatorname{Ad}\mathscr{A}$ consists of all bounded operators which generate derivations on $\mathscr{A}$. As well as the obvious connection between $\mathscr{A}$ and $\operatorname{Ad}\mathscr{A}$, there is also a close link between $\operatorname{Lat}\mathscr{A}$ and $\operatorname{Ad}\mathscr{A}$:

(i) All operators A in $\operatorname{Ad}\mathscr{A}$ generate one-parameter groups of homeomorphisms of $\operatorname{Lat}\mathscr{A}$ $(M \to \exp(tA)M)$.

(ii) For every subspace M in $\operatorname{Lat}\mathscr{A}$, the set $\operatorname{Ad}\mathscr{A}_M = \{B \in \operatorname{Ad}\mathscr{A}: BM \subseteq M\}$ is a Lie subalgebra of $\operatorname{Ad}\mathscr{A}$ and

$$\mathscr{A} = \bigcap_{M \in \operatorname{Lat}\mathscr{A}} \operatorname{Ad}\mathscr{A}_M$$

if $\mathscr{A}$ is reflexive.

A knowledge of the structure of $\operatorname{Ad}\mathscr{A}$ enables us to obtain a clearer description of the nature of $\operatorname{Lat}\mathscr{A}$. This can be done by establishing the structure of the orbits in $\operatorname{Lat}\mathscr{A}$ with respect to $\operatorname{Ad}\mathscr{A}$.

In many cases, however, these triplets degenerate into pairs. For example, if $\mathscr{A}$ is a W^*-algebra, then $\operatorname{Lat}\mathscr{A}$ is the set of all projections in $\mathscr{A}'$, and $\operatorname{Ad}\mathscr{A} = \mathscr{A} + \mathscr{A}'$; as a result the triplet turns into the pair $(\mathscr{A}, \mathscr{A}')$. If $\mathscr{A}$ is a CSL-algebra, then $\operatorname{Ad}\mathscr{A} = \mathscr{A}$ and the triplet becomes the pair $(\mathscr{A}, \operatorname{Lat}\mathscr{A})$. But, in the case of an arbitrary operator algebra, $\operatorname{Ad}\mathscr{A}$ is not usually equal to $\mathscr{A} + \mathscr{A}'$ and $\operatorname{Ad}\mathscr{A}$ does not contain $\operatorname{Lat}\mathscr{A}$; in this case, therefore, the triplet does not degenerate into a pair.

One of the simplest classes of this type of algebras is $\mathscr{R}_1$ [3]. This class consists of all the reflexive algebras $\mathscr{A}$ which satisfy the following conditions:

(a) The quotient Lie algebra $\operatorname{Ad}\mathscr{A}/\mathscr{A}$ is non-trivial;

(b) For every M in $\operatorname{Lat}\mathscr{A}$ the codimension of $\operatorname{Ad}\mathscr{A}_M$ in $\operatorname{Ad}\mathscr{A}$ is less than or equal to 1.

According to these conditions, no CSL- or W^*-algebras (except for the factors $B(H) \otimes I_2$) belong to $\mathscr{R}_1$. For algebras from $\mathscr{R}_1$, effective analysis appears to be possible. The structure of the quotient Lie algebra $\operatorname{Ad}\mathscr{A}/\mathscr{A}$, for $\mathscr{A} \in \mathscr{R}_1$, is quite simple and enables us to obtain a description of $\operatorname{Lat}\mathscr{A}$ in terms of the orbits in $\operatorname{Lat}\mathscr{A}$ with respect to $\operatorname{Ad}\mathscr{A}$ [3].

The new method introduced in the article provides us with a wide variety of algebras from $\mathscr{R}_1$, although not all the algebras obtained by this method belong to $\mathscr{R}_1$ (see Example 2). There is reason to think that this method may in fact provide us with all the algebras from $\mathscr{R}_1$ which satisfy some extra conditions on $\operatorname{Lat}\mathscr{A}$.

Theorem 2.4 investigates the structure of $\operatorname{Lat}\mathscr{A}$ and Theorem 2.5 considers some sufficient conditions for the algebras $\mathscr{A}$ to be reflexive. Section 3 deals with a particular case when all $\mathscr{T}_i = B(H_i)$ and a detailed description of Lat $\mathscr{A}$ is obtained in Theorem 3.5. Two examples of algebras $\mathscr{A}$ when $n = 2$ are also considered. In Example 1, $\dim(\mathrm{Ad}\,\mathscr{A}/\mathscr{A}) = 2$ and all operators from Ad $\mathscr{A}$ which do not belong to $\mathscr{A}$ generate non-inner derivations on $\mathscr{A}$. In Example 2, $\mathrm{Ad}\,\mathscr{A} = \mathscr{A}$, although the structure of Lat $\mathscr{A}$ is the same as in Example 1.

I would like to thank the referee of this article for his helpful suggestions, and am grateful to Dr. J. A. Erdos for his useful advice.

1. Preliminaries and notation. Let n be an integer or infinity, let H_i, for $1 \leq i \leq n$ ($1 \leq i < \infty$, if $n = \infty$), be Hilbert spaces and let $\mathscr{T}_i$ be reflexive operator algebras on H_i. (A subalgebra $\mathscr{T}$ of $B(H)$ is reflexive if $\mathscr{T} = \mathrm{Alg\,Lat}\,\mathscr{T}$, where Lat $\mathscr{T}$ is the set of all closed subspaces invariant under operators from $\mathscr{T}$, and Alg Lat $\mathscr{T}$ is the algebra of all operators in $B(H)$ which leave every member of Lat $\mathscr{T}$ invariant.) Let F_i and G_i, for $1 \leq i < n$, be closed operators from H_{i+1} into H_i. By $D(F_i)$ and $D(G_i)$ we shall denote their domains in H_{i+1}. Let F_i^* and G_i^* be the adjoint operators from H_i into H_{i+1} and let $D(F_i^*)$ and $D(G_i^*)$ be their domains in H_i. Set $D_1 = H_1$, $D_n^* = H_n$ (if $n < \infty$)

$$D_{i+1} = D(F_i) \cap D(G_i) \quad \text{and} \quad D_i^* = D\left(F_i^*\right) \cap D\left(G_i^*\right)$$

for $1 \leq i < n$. Then $D_i \subseteq H_i$ and $D_i^* \subseteq H_i$.

Let us impose some restrictions on the operators $\{F_i\}$ and $\{G_i\}$.

(R_1) D_i and D_i^* are dense in H_i for all i.

(R_2) $G_i \neq 0$ for all i.

By $\mathscr{U}$ we shall denote the set of all sequences $T = \{T_i\}_{i=1}^n$ such that

(A_1) $T_i \in \mathscr{T}_i$, $T_{i+1}D(G_i) \subseteq D(G_i)$ and $T_{i+1}D(F_i) \subseteq D(F_i)$;

(A_2) $T_iG_i\,|\,_{D(G_i)} = G_iT_{i+1}\,|\,_{D(G_i)}$;

(A_3) the operators $(F_iT_{i+1} - T_iF_i)\,|\,_{D(F_i)}$ extend to bounded operators T_{F_i} from H_{i+1} into H_i;

(A_4) $\sup\|T_i\| < \infty$ and $\sup\|T_{F_i}\| < \infty$.

From (R_1) it follows that for every i there only exists one bounded operator T_{F_i} which extends $(F_iT_{i+1} - T_iF_i)\,|\,_{D(F_i)}$. For every i let $\mathscr{U}_i$ be a subalgebra of $\mathscr{T}_i$ such that an operator B belongs to $\mathscr{U}_i$ if and only if there exists a sequence $\{T_k\} \in \mathscr{U}$ for which $B = T_i$.

Let $\mathscr{H}$ be the direct sum of all H_i. For every sequence $T = \{T_i\}$ from $\mathscr{U}$ let $A^T = (A_{ij})$ be the operator on $\mathscr{H}$ such that

$$(1) \qquad A_{ii} = T_i, \quad A_{ii+1} = T_{F_i} \quad \text{and all other} \quad A_{ij} = 0.$$

By (A_4), A^T is bounded. Put

$$\mathscr{U}(\mathscr{H}) = \{A^T\colon T \in \mathscr{U}\};$$

$$I(\mathscr{H}) = \left\{A = \left(A_{ij}\right) \in B(\mathscr{H})\colon A_{ij} = 0 \text{ if } i \geq j - 1\right\}.$$

By $\mathscr{A}$ we shall denote the set of operators on $\mathscr{H}$ generated by all sums of operators from $\mathscr{U}(\mathscr{H})$ and from $I(\mathscr{H})$.

For example, if $n = 2$, then F and G are closed operators from H_2 into H_1, $\mathscr{H} = H_1 \oplus H_2$, $\mathscr{T}_i$, for $i = 1, 2$, are reflexive subalgebras of $B(H_i)$, $I(\mathscr{H}) = \{0\}$ and

$$\mathscr{A} = \mathscr{U}(\mathscr{H}) = \left\{ A = \begin{pmatrix} T_1 & T_F \\ 0 & T_2 \end{pmatrix} \in B(\mathscr{H}) \colon (1)\ T_i \in \mathscr{T}_i, T_2 D(G) \subseteq D(G) \right.$$

$$\text{and } T_2 D(F) \subseteq D(F);\ (2)\ T_1 G|_{D(G)} = G T_2|_{D(G)};$$

$$\left. (3)\ T_F|_{D(F)} = (F T_2 - T_1 F)|_{D(F)}. \right\}$$

Let $\mathscr{A}$ be a subalgebra of $B(H)$. Then

$$\operatorname{Ad} \mathscr{A} = \{ B \in B(H) \colon [B, A] = BA - AB \in \mathscr{A} \text{ for all } A \in \mathscr{A} \}.$$

Operators from $\operatorname{Ad} \mathscr{A}$ generate bounded derivations on $\mathscr{A}$. It can be easily checked that $\operatorname{Ad} \mathscr{A}$ is a Lie algebra and that $\mathscr{A}$ and its commutant $\mathscr{A}'$ are Lie ideals in $\operatorname{Ad} \mathscr{A}$.

The rank one operator $z \mapsto (z, x)y$ will be denoted by $x \otimes y$.

2. Reflexivity of $\mathscr{A}$. In this section, in Theorem 2.4 we shall obtain some information about $\operatorname{Lat} \mathscr{A}$ and in Theorem 2.5 we shall state some sufficient conditions for an algebra $\mathscr{A}$ to be reflexive.

LEMMA 2.1. *$\mathscr{A}$ is an algebra and $I(\mathscr{H})$ is a weakly closed ideal in $\mathscr{A}$.*

Proof. It is obvious that $I(\mathscr{H})$ is a weakly closed ideal in $\mathscr{A}$. Let $T = \{T_i\}$ and $T' = \{T_i'\}$ belong to $\mathscr{U}$. It is easy to see that their linear combinations also belong to $\mathscr{U}$. Therefore linear combinations of operators A^T and $A^{T'}$ belong to $\mathscr{U}(\mathscr{H})$. Let $B = \{B_i\}$ where $B_i = T_i T_i'$. Then B satisfies conditions (A_1) and (A_2). Since the operators

$$(F_i B_{i+1} - B_i F_i)|_{D(F_i)}$$
$$= (F_i T_{i+1} - T_i F_i) T'_{i+1}|_{D(F_i)} + T_i (F_i T'_{i+1} - T_i' F_i)|_{D(F_i)}$$

extend to the bounded operators $T_{F_i} T'_{i+1} + T_i T'_{F_i}$, we get that B satisfies (A_3) and that

$$B_{F_i} = T_{F_i} T'_{i+1} + T_i T'_{F_i}. \tag{2}$$

From (2) it follows immediately that B satisfies (A_4) and hence $B \in \mathscr{U}$. From simple computations and from (1) and (2) it follows that

$$A^T A^{T'} \equiv A^B \mod I(\mathscr{H}).$$

Therefore $\mathscr{A}$ is an algebra and the lemma is proved.

LEMMA 2.2. (i) *The operators* $F_i + tG_i$ *and* $F_i^* + \bar{t}G_i^*$ *are closable for every complex* t.

(ii) *For every* $\{T_i\} \in \mathscr{U}$

(A_1^*) $T_i^* D(F_i^*) \subseteq D(F_i^*)$ *and* $T_i^* D(G_i^*) \subseteq D(G_i^*)$;

(A_2^*) $G_i^* T_i^* |_{D(G_i^*)} = T_{i+1}^* G_i^* |_{D(G_i^*)}$;

(A_3^*) $(T_{i+1}^* F_i^* - F_i^* T_i^*)|_{D(F_i^*)} = T_{F_i}^* |_{D(F_i^*)}$.

Proof. For every complex t the domain of the operator $F_i^* + \bar{t}G_i^*$ is D_i^*. Since D_i^* is dense in H_i, there exists the adjoint operator $(F_i^* + \bar{t}G_i^*)^*$. We also have that

$$\left(F_i^* + \bar{t}G_i^*\right)^* |_{D_{i+1}} = \left(F_i + tG_i\right) |_{D_{i+1}}.$$

Since $(F_i^* + \bar{t}G_i^*)^*$ is closed, the operator $F_i + tG_i$ is closable. Similarly we can prove that the operator $F_i^* + \bar{t}G_i^*$ is closable. Thus (i) is proved.

From (A_2) it follows that for every $\{T_k\} \in \mathscr{U}$, for every $y \in D(G_i)$ and for every $x \in D(G_i^*)$

$$(3) \qquad \left(G_i y, T_i^* x\right) = \left(T_i G_i y, x\right) = \left(G_i T_{i+1} y, x\right) = \left(y, T_{i+1}^* G_i^* x\right).$$

Hence for every $x \in D(G_i^*)$

$$(4) \qquad T_i^* x \in D\left(G_i^*\right) \quad \text{and} \quad G_i^* T_i^* |_{D(G_i^*)} = T_{i+1}^* G_i^* |_{D(G_i^*)}.$$

Thus (A_2^*) is proved.

From (A_3) it follows that for every $y \in D(F_i)$ and every $x \in D(F_i^*)$

$$(5) \qquad \left(F_i y, T_i^* x\right) = \left(T_i F_i y, x\right)$$
$$= \left(\left(F_i T_{i+1} - T_{F_i}\right) y, x\right) = \left(y, \left(T_{i+1}^* F_i^* - T_{F_i}^*\right) x\right).$$

Therefore for every $x \in D(F_i^*)$

$$(6) \qquad T_i^* x \in D\left(F_i^*\right) \quad \text{and} \quad T_{F_i}^* |_{D(F_i^*)} = \left(T_{i+1}^* F_i^* - F_i^* T_i^*\right) |_{D(F_i^*)}$$

Thus (A_3^*) is proved. From (4) and (6) it follows that (A_1^*) holds which concludes the proof of the lemma.

DEFINITION. By S_t^i we shall denote the closure of the operator $F_i + tG_i$ which is defined on D_{i+1} and by R_t^i we shall denote the closure of the operator $F_i^* + \bar{t}G_i^*$ which is defined on D_i^*. By $D(S_t^i)$ and by $D(R_t^i)$ we shall denote their domains.

It is easy to see that $(R_t^i)^* |_{D_i+1} = F_i + tG_i$. Since $(R_t^i)^*$ is closed, we get that

$$(7) \qquad S_t^i \subseteq \left(R_t^i\right)^*.$$

Since S_0^i is the closure of $F_i |_{D_{i+1}}$ and $(R_0^i)^* = (F_i^* |_{D_i^*})^*$, it follows that

$$(8) \qquad S_0^i \subseteq F_i \subseteq \left(R_0^i\right)^*.$$

By $\mathscr{H}_0$ we shall denote the null subspace in $\mathscr{H}$. For every $0 < i < n$ let $\mathscr{H}_i$ be the direct sum of $H_1, \dots, H_i$. We shall consider $\mathscr{H}_i$ as a subspace in $\mathscr{H}$. It is easy to see that $\mathscr{H}_i \in \operatorname{Lat} \mathscr{A}$.

For every $K \in \operatorname{Lat} \mathscr{T}_i$ let $\mathscr{K}$ be the direct sum of $\mathscr{H}_{i-1}$ and K. Then $\mathscr{K}$ can be considered as a subspace in $\mathscr{H}$, so that $\mathscr{K} \subseteq \mathscr{H}_i$ and $\mathscr{K} \in \operatorname{Lat} \mathscr{A}$.

Let S be a closed operator from H_{i+1} into H_i. Put

$$M_S^i = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : x \in D(S) \text{ and } y = Sx \right\}.$$

Then M_S^i is a closed subspace in $H_i \oplus H_{i+1}$ which can be considered as a closed subspace in $\mathscr{H}$. Therefore M_S^i is a closed subspace in $\mathscr{H}$. By $\mathscr{M}_S^i$ we shall denote the direct sum of $\mathscr{H}_{i-1}$ and M_S^i, and we shall consider $\mathscr{M}_S^i$ as a closed subspace in $\mathscr{H}$.

LEMMA 2.3. (i) *Let S be a closed operator from H_{i+1} into H_i and let D be a linear manifold in $D(S)$ such that*

1) *S is the closure of the oprator $S |_D$;*
2) *$TD \subseteq D$ for every $T \in \mathscr{U}_{i+1}$;*
3) *$T_{F_i} |_D = (ST_{i+1} - T_i S) |_D$ for every $\{T_k\} \in \mathscr{U}$.*

Then $\mathscr{M}_S^i \in \operatorname{Lat} \mathscr{A}$.

(ii) *Let S be a closed operator from H_i into H_{i+1} and let D be a linear manifold in $D(S)$ such that*

1) *D is dense in H_i;*
2) *S is the closure of the operator $S |_D$;*
3) *$T^*D \subseteq D$ for every $T \in \mathscr{U}_i$.*
4) *$(T_{i+1}^* S - ST_i^*) |_D = T_{F_i}^* |_D$ for every $\{T_k\} \in \mathscr{U}$.*

Then $\mathscr{M}_{S^*}^i \in \operatorname{Lat} \mathscr{A}$.

Proof. If an operator A belongs to $I(\mathscr{H})$, then it is easy to see that $A\xi \in \mathscr{H}_{i-1}$ for every $\xi \in \mathscr{M}^i_S$.

Let $T = \{T_k\} \in \mathscr{U}$ and $A^T \in \mathscr{U}(\mathscr{H})$. Then $A^T\xi \in \mathscr{H}_{i-1}$ for every $\xi \in \mathscr{H}_{i-1}$. Suppose that $\xi = \binom{y}{x} \in M^i_S$. Then

$$A^T\xi \equiv \xi' \mod \mathscr{H}_{i-1}$$

where

$$\xi' = \begin{pmatrix} y' \\ x' \end{pmatrix} \in H_i \oplus H_{i+1}, \quad x' = T_{i+1}x \quad \text{and} \quad y' = T_i y + T_{F_i}x.$$

Let $x \in D$. Then, by 2), $x' \in D$. Since $y = Sx$, we get, by 3), that

$$y' = T_i Sx + (ST_{i+1} - T_i S)x = ST_{i+1}x.$$

Hence $\xi' \in M^i_S$. Thus, if $\xi = \binom{y}{x} \in M^i_{S'}$ and if $x \in D$, then $A^T\xi \in \mathscr{M}^i_S$. But, by 1), the elements $\xi = \binom{y}{x}$, where $x \in D$, are dense in M^i_S. Therefore $A^T\xi \in \mathscr{M}^i_S$ for every $\xi \in M^i_S$ which completes the proof of (i).

Now let S be a closed operator from H_i into H_{i+1}. We only need condition 3) for condition 4) to be defined correctly. By 1), S^* is a closed operator from H_{i+1} into H_i. Let $x \in D$ and $y \in D(S^*)$. Then for every $\{T_k\} \in \mathscr{U}$, by 4),

$$\begin{aligned}(T_{i+1}y, Sx) &= \left(y, T^*_{i+1}Sx\right) \\ &= \left(y, \left[ST_i^* + T^*_{F_i}\right]x\right) = \left(\left[T_i S^* + T_{F_i}\right]y, x\right).\end{aligned}$$

By 2),

$$T_{i+1}y \in D(S^*) \quad \text{and} \quad S^*T_{i+1}\,|_{D(S^*)} = \left(T_i S^* + T_{F_i}\right)|_{D(S^*)}.$$

Applying (i) to S^* we obtain that $\mathscr{M}^i_{S^*} \in \operatorname{Lat}\mathscr{A}$. The proof is complete.

THEOREM 2.4. *Subspaces $\mathscr{M}^i_{S^i_t}$, $\mathscr{M}^i_{(R^i_t)^*}$ and $\mathscr{M}^i_{F_i}$ belong to* Lat $\mathscr{A}$ *for $1 \le i < n$ and for all complex t.*

Proof. Put $D = D_{i+1}$. Then $D \subseteq D(S^i_t)$ and it follows from the definition of S^i_t that S^i_t is the closure of $S^i_t\,|_D$. It follows from (A_1) that $TD_{i+1} \subseteq D_{i+1}$ for every $T \in \mathscr{U}_{i+1}$. Finally, by (A_2), and by (A_3), we get

$$\begin{aligned}\left(S^i_t T_{i+1} - T_i S^i_t\right)|_{D_{i+1}} &= \left(F_i T_{i+1} - T_i F_i + t(G_i T_{i+1} - T_i G_i)\right)|_{D_{i+1}} \\ &= \left(F_i T_{i+1} - T_i F_i\right)|_{D_{i+1}} = T_{F_i}\,|_{D_{i+1}}.\end{aligned}$$

Therefore, by Lemma 2.3, $\mathscr{M}^i_{S^i_t} \in \operatorname{Lat}\mathscr{A}$.

Now put $D = D_i^*$. By the definition of R_t^i, we have that $D \subseteq D(R_t^i)$ and that the closure of $R_t^i|_D$ is R_t^i. By (R_1), D is dense in H_i. It follows from Lemma 2.2 (A_1^*) that $T^*D \subseteq D$ for every $T \in \mathscr{U}_i$. Thus, conditions 1), 2) and 3) of Lemma 2.3 (ii) hold. By Lemma 2.2 (A_2) and (A_3),

$$\left(T_{i+1}^* R_t^i - R_t^i T_i^*\right)|_{D_i^*}$$
$$= \left(T_{i+1}^* F_i^* - F_i^* T_i^* + \bar{t}\left(T_{i+1}^* G_i^* - G_i^* T_i^*\right)\right)|_{D_i^*} = T_{F_i}^*|_{D_i^*}.$$

Therefore condition 4) of Lemma 2.3(ii) holds and $\mathscr{M}^i_{(R_t^i)^*} \in \operatorname{Lat}\mathscr{A}$.

At last, if $S = F_i$ and $D = D(F_i)$, then it can be easily seen that conditions 2) and 3) of Lemma 2.3(i) follows from (A_1) and (A_3). Therefore $\mathscr{M}^i_{F_i} \in \operatorname{Lat}\mathscr{A}$ and this completes the proof of the theorem.

Now we shall prove the main result of the section.

THEOREM 2.5. *If for every i, $1 \le i < n$, either*
(a) $\bigcap_{t\in\mathbf{C}} D(S_t^i) = D_{i+1}$ *and the closure of* $G_i|_{D_{i+1}}$ *is* G_i,
or
(b) $\bigcap_{t\in\mathbf{C}} D(R_t^i) = D_i^*$ *and the closure of* $G_i^*|_{D_i^*}$ *is* G_i^*,
then $\mathscr{A}$ is reflexive.

Proof. Let $B = (B_{ij}) \in \operatorname{Alg}\operatorname{Lat}\mathscr{A}$. Since $\mathscr{H}_i \in \operatorname{Lat}\mathscr{A}$, we obtain that $B_{ij} = 0$ if $i > j$. For every $K \in \operatorname{Lat}\mathscr{T}_i$ the subspace $\mathscr{K} = \mathscr{H}_{i-1} \oplus K$ is contained in $\mathscr{H}_i$ and belongs to $\operatorname{Lat}\mathscr{A}$. Since all algebras $\mathscr{T}_i$ are reflexive, we obtain that

$$B_{ii} \in \mathscr{T}_i. \tag{9}$$

Now let

$$z = \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} F_i x \\ x \end{pmatrix} \in M_{F_i}^i$$

where $x \in D(F_i)$. Considering $M_{F_i}^i$ as a subspace in $\mathscr{H}$ we obtain that $Bz \equiv z' \bmod \mathscr{H}_{i-1}$ where

$$z' = \begin{pmatrix} y' \\ x' \end{pmatrix} \in H_i \oplus H_{i+1}, \qquad x' = B_{i+1\,i+1}x$$

and $y' = B_{ii}y + B_{i\,i+1}x$.

Since $M_{F_i}^i \subseteq \mathscr{M}^i_{F_i}$ and since, by Theorem 2.4, $\mathscr{M}^i_{F_i} \in \operatorname{Lat}\mathscr{A}$, we have that $z' \in M_{F_i}^i$. Therefore

$$x' = B_{i+1\,i+1}x \in D(F_i), \tag{10}$$
$$y' = B_{ii}F_i x + B_{i\,i+1}x = F_i x' = F_i B_{i+1\,i+1}x.$$

Thus

$$B_{ii+1}|_{D(F_i)} = (F_i B_{i+1i+1} - B_{ii}F_i)|_{D(F_i)}. \tag{11}$$

Now let (a) hold for some i and let

$$z = \begin{pmatrix} S_t^i x \\ x \end{pmatrix} \in M_{S_t^i}^i \quad \text{where } x \in D(S_t^i).$$

Then repeating the argument above we obtain that

$$B_{i+1i+1}x \in D(S_t^i),$$

$$B_{ii}S_t^i x + B_{ii+1}x = S_t^i B_{i+1i+1}x.$$

If $x \in D_{i+1}$, then $x \in D(S_t^i)$ and, by (a),

$$B_{i+1i+1}x \in \bigcap_{t \in \mathbf{C}} D(S_t^i) = D_{i+1}.$$

Therefore

$$B_{ii}(F_i + tG_i)x + B_{ii+1}x = (F_i + tG_i)B_{i+1i+1}x.$$

From this and from (11) we immediately obtain that

$$B_{ii}G_i|_{D_{i+1}} = G_i B_{i+1i+1}|_{D_{i+1}}. \tag{12}$$

Let $x \in D(G_i)$. Since, by (a), the closure of $G_i|_{D_{i+1}}$ is G_i, there exists a sequence $\{x_n\}$ such that $x_n \in D_{i+1}$, $\{x_n\}$ converges to x and $\{G_i x_n\}$ converges to $G_i x$. Then, by (12),

$$B_{ii}G_i x = \lim B_{ii}G_i x_n = \lim G_i B_{i+1i+1}x_n.$$

Since the sequence $\{B_{i+1i+1}x_n\}$ converges to $B_{i+1i+1}x$ and since G_i is closed, we obtain that

$$B_{i+1i+1}x \in D(G_i) \quad \text{and} \quad B_{ii}G_i x = G_i B_{i+1i+1}x. \tag{13}$$

Now let (b) hold for some i and let

$$z = \begin{pmatrix} (R_t^i)^* & x \\ x & \end{pmatrix} \quad \text{where } x \in D\left((R_t^i)^*\right).$$

Repeating the same argument as for F_i we obtain that

$$B_{i+1i+1}x \in D\left((R_t^i)^*\right),$$

$$B_{ii}(R_t^i)^* x + B_{ii+1}x = (R_t^i)^* B_{i+1i+1}x.$$

Therefore for every $y \in D_i^*$

$$\begin{aligned}
\left(B_{ii}^* y, (R_t^i)^* x\right) &= \left(y, B_{ii}(R_t^i)^* x\right) \\
&= \left(y, \left[-B_{ii+1} + (R_t^i)^* B_{i+1i+1}\right] x\right) = \left(\left[-B_{ii+1}^* + B_{i+1i+1}^* R_t^i\right] y, x\right) \\
&= \left(\left[-B_{ii+1}^* + B_{i+1i+1}^*(F_i^* + \bar{t}G_i^*)\right] y, x\right).
\end{aligned}$$

Repeating the same argument as in Lemma 2.2 we obtain from (11) that

$$B_{ii}^* D(F_i^*) \subseteq D(F_i^*)$$

and that

$$B_{ii+1}^* |_{D(F_i^*)} = \left(B_{i+1i+1}^* F_i^* - F_i^* B_{ii}^*\right)|_{D(F_i^*)}.$$

Taking this into account and since $D_i^* \subseteq D(F_i^*)$, we obtain

$$\left(B_{ii}^* y, (R_t^i)^* x\right) = \left(\left[F_i^* B_{ii}^* + \bar{t} B_{i+1i+1}^* G_i^*\right] y, x\right).$$

From this formula it follows that

$$B_{ii}^* y \in D\left(R_t^i\right) \quad \text{and} \quad R_t^i B_{ii}^* y = \left(F_i^* B_{ii}^* + \bar{t} B_{i+1i+1}^* G_i^*\right) y.$$

Therefore, by (b), for every $y \in D_i^*$

$$B_{ii}^* y \in \bigcap_{t \in \mathbf{C}} D\left(R_t^i\right) = D_i^*$$

and

$$\left(F_i^* + \bar{t} G_i^*\right) B_{ii}^* y = \left(F_i^* B_{ii}^* + \bar{t} B_{i+1i+1}^* G_i^*\right) y.$$

Thus

$$G_i^* B_{ii}^* |_{D_i^*} = B_{i+1i+1}^* G_i^* |_{D_i^*}.$$

Let $y \in D_i^*$ and $z \in D(G_i)$. Then

$$\left(G_i^* y, B_{i+1i+1} z\right) = \left(B_{i+1i+1}^* G_i^* y, z\right) = \left(G_i^* B_{ii}^* y, z\right) = \left(y, B_{ii} G_i z\right).$$

Since, by (b), the closure of $G_i^* |_{D_i^*}$ is G_i^*, we obtain from this formula that

(13′) $\quad B_{i+1i+1} D(G_i) \subseteq D(G_i) \quad$ and $\quad B_{ii} G_i |_{D(G_i)} = G_i B_{i+1i+1} |_{D(G_i)}.$

Put $T_i = B_{ii}$. It follows from (9), (10), (11), (13) and (13′) that conditions (A_1), (A_2) and (A_3) hold for the sequence $T = \{T_i\}$ and that $B_{ii+1} = T_{F_i}$. Since B is bounded, T also satisfies condition (A_4). Therefore the sequence $T = \{T_i\}$ belongs to $\mathcal{U}$ and $B - A^T \in I(\mathcal{H})$. Thus $B \in \mathcal{A}$ which concludes the proof of the theorem.

COROLLARY 2.6. *If for every i at least one of the operators F_i or G_i is bounded, then $\mathcal{A}$ is reflexive.*

Proof. We obtain easily that $D_{i+1} = D(S_t^i)$ for every i and for $t \neq 0$. Therefore, by Theorem 2.5(a), $\mathcal{A}$ is reflexive.

3. Structure of Lat $\mathscr{A}$. In Lemma 2.3 and Theorem 2.4 we obtained some information about the structure of Lat $\mathscr{A}$. But further investigation of its structure in the general case of arbitrary reflexive algebras $\{\mathscr{T}_i\}$ is very difficult. Therefore in this section we shall consider the simplest case when all $\mathscr{T}_i = B(H_i)$. In Lemma 3.1 we shall show that if all $\mathscr{U}_i$ are weakly dense in $B(H_i)$, then the sufficient conditions of Lemma 2.3 for a subspace $\mathscr{M}$ to belong to Lat $\mathscr{A}$ are also necessary. Imposing some further restriction (R_3) on the operators $\{F_i\}$ and $\{G_i\}$ we shall obtain the main result of the section (Theorem 3.5) which describes the structure of Lat $\mathscr{A}$.

LEMMA 3.1. *Let all $\mathscr{T}_i = B(H_i)$ and let all $\mathscr{U}_i$ be weakly dense in $B(H_i)$. If $\mathscr{M} \in$ Lat $\mathscr{A}$, then $\mathscr{M}$ is either $\mathscr{H}$ or one of the subspaces $\mathscr{H}_i$ for $0 \leq i < n$, or there exist an integer $1 \leq i < n$ and a closed operator S from H_{i+1} into H_i such that*

(1) *$D(S)$ is dense in H_{i+1};*

(2) *$TD(S) \subseteq D(S)$ for every $T \in \mathscr{U}_{i+1}$;*

(3) *$T_{F_i}|_{D(S)} = (ST_{i+1} - T_iS)|_{D(S)}$ for every sequence $\{T_K\} \in \mathscr{U}$;*

and that $\mathscr{M} = \mathscr{M}_S^i$.

Proof. Let $z \in \mathscr{M}$. If $z \in \mathscr{H}_{i+1}$ but $z \notin \mathscr{H}_i$, then $\mathscr{H}_{i-1} \subset \mathscr{M}$, since $I(\mathscr{H}) \subset \mathscr{A}$. Therefore if $n = \infty$ and if for every i there exists $z_i \in \mathscr{M}$ such that $z_i \in \mathscr{H}_{i+1}$ but $z_i \notin \mathscr{H}_i$, then $\mathscr{M} = \mathscr{H}$.

Suppose that $\mathscr{M} \neq \mathscr{H}$. Then there exists an integer i such that $\mathscr{M} \subseteq \mathscr{H}_{i+1}$ but $\mathscr{M} \subsetneqq \mathscr{H}_i$. (If $n < \infty$, then it is obvious. If $n = \infty$, then it follows from the argument above.) Hence $\mathscr{H}_{i-1} \subseteq \mathscr{M}$ and we get that $\mathscr{M} = \mathscr{H}_{i-1} \oplus M$, where M is a closed subspace in $H_i \oplus H_{i+1}$ which is considered as a subspace in $\mathscr{H}$.

Suppose that $\mathscr{M} \neq \mathscr{H}_{i+1}$. Let us show that $M \cap H_i = \{0\}$. Let $z \neq 0$ belong to $M \cap H_i$. Then for every $T = \{T_k\} \in \mathscr{U}$ we have that

$$A^T z \equiv T_i z \mod \mathscr{H}_{i-1} \in \mathscr{M}.$$

Since $\mathscr{H}_{i-1} \subseteq \mathscr{M}$, we obtain that $T_i z \in \mathscr{M}$. Hence $Tz \in \mathscr{M}$ for every $T \in \mathscr{U}_i$. Since $\mathscr{U}_i$ is weakly dense in $B(H_i)$, the set $\{Tz\colon T \in \mathscr{U}_i\}$ is dense in H_i. Therefore, since $\mathscr{M}$ is closed, we obtain that $H_i \subseteq \mathscr{M}$. Hence $\mathscr{H}_i = \mathscr{H}_{i-1} \oplus H_i$ is contained in $\mathscr{M}$. Since $\mathscr{M} \neq \mathscr{H}_i$, there exists $x \in \mathscr{M}$ such that $x \in H_{i+1}$. Using that $\mathscr{U}_{i+1}$ is weakly dense in $B(H_{i+1})$ and repeating the above argument we obtain that $H_{i+1} \subseteq \mathscr{M}$. Hence $\mathscr{M} = \mathscr{H}_{i+1}$ which contradicts the assumption that $\mathscr{M} \neq \mathscr{H}_{i+1}$. Thus $M \cap H_i = \{0\}$.

Since M is closed, there exists a closed operator S from H_{i+1} into H_i such that

$$M = M_S^i = \left\{ z = \begin{pmatrix} y \\ x \end{pmatrix} : x \in D(S) \subseteq H_{i+1} \text{ and } y = Sx \in H_i \right\}.$$

Therefore $\mathcal{M} = \mathcal{M}_S^i$.

Now for every $T = \{T_k\} \in \mathcal{U}$ and for every $z = \binom{y}{x} \in M_S^i$ we have that $A^T z \equiv z' \bmod \mathcal{H}_{i-1}$, where

$$z' = \begin{pmatrix} y' \\ x' \end{pmatrix}, \quad x' = T_{i+1}x \quad \text{and} \quad y' = T_i y + T_{F_i} x.$$

Since $\mathcal{M} \in \operatorname{Lat} \mathcal{A}$ and since $\mathcal{H}_{i-1} \subset \mathcal{M}$, we have that $z' \in M_S^i$. Hence

(14) $$T_{i+1}x \in D(S) \quad \text{and} \quad T_i Sx + T_{F_i} x = ST_{i+1}x$$

for every $x \in D(S)$. Thus conditions (2) and (3) of the lemma hold. From weak density of $\mathcal{U}_{i+1}$ in $B(H_{i+1})$ and from (14) it follows that $D(S)$ is dense in H_{i+1}. Hence condition (1) holds and the lemma is proved.

From this lemma and from Lemma 2.3 we obtain the following corollary.

COROLLARY 3.2. *Let all $\mathcal{T}_i = B(H_i)$ and let all $\mathcal{U}_i$ be weakly dense in $B(H_i)$. Then* Lat $\mathcal{A}$ *consists of $\mathcal{H}$, of all subspaces $\mathcal{H}_i$ for $0 \le i < n$, and of all subspaces $\mathcal{M}_S^i$ for $1 \le i < n$, where S are closed operators from H_{i+1} into H_i which satisfy the conditions of Lemma* 3.1.

Now let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be sequences such that

(B_1) $y_i \in D_i \subseteq H_i$, ($B_1^*$) $x_i \in D_i^* \subseteq H_i$,

(B_2) $y_i = G_i y_{i+1}$, (B_2^*) $x_{i+1} = G_i^* x_i$,

(B_3) $\sup\|y_i\| < \infty$, $\sup\|F_i y_{i+1}\| < \infty$;

(B_3^*) $\sup\|x_i\| < \infty$, $\sup\|F_i^* x_i\| < \infty$.

By X we shall denote the set of sequences $\{x_i\}$ which satisfy conditions (B_1^*)–(B_3^*), and by Y we shall denote the set of sequences $\{y_i\}$ which satisfy conditions (B_1)–(B_3). It is obvious that X and Y are linear manifolds.

LEMMA 3.3. *Let all $\mathcal{T}_i = B(H_i)$. If $\{x_i\} \in X$ and $\{y_i\} \in Y$, then the sequence of operators $\{x_i \otimes y_i\}$ belongs to $\mathcal{U}$.*

Proof. Put $T_i = x_i \otimes y_i$. For every $x \in H_i$, by (B_1), we have that

$$T_i x = (x, x_i) y_i \in D_i.$$

Hence condition (A_1) holds. By (B_2) and by (B_2^*), for every $x \in D(G_i)$

$$T_iG_ix = (G_ix, x_i)y_i = (x, G_i^*x_i)G_iy_{i+1}$$
$$= (x, x_{i+1})G_iy_{i+1} = G_iT_{i+1}x.$$

Hence condition (A_2) holds. Next, for every $x \in D(F_i)$ we have that

$$(F_iT_{i+1} - T_iF_i)x = (x, x_{i+1})F_iy_{i+1} - (F_ix, x_i)y_i$$
$$= (x, x_{i+1})F_iy_{i+1} - (x, F_i^*x_i)y_i = T_{F_i}x,$$

where the operator

$$T_{F_i} = x_{i+1} \otimes F_iy_{i+1} - F_i^*x_i \otimes y_i \tag{15}$$

is bounded. Hence condition (A_3) holds. Finally, by (B_3), (B_3^*) and (15),

$$\sup\|T_i\| = \sup\|x_i \otimes y_i\| \leq \sup\|x_i\|\sup\|y_i\| < \infty$$

and

$$\sup\|T_{F_i}\| = \sup\|x_{i+1} \otimes F_iy_{i+1} - F_i^*x_i \otimes y_i\|$$
$$\leq \sup\|x_{i+1}\|\sup\|F_iy_{i+1}\| + \sup\|y_i\|\sup\|F_i^*x_i\| < \infty.$$

Thus condition (A_4) holds and therefore the sequence $\{x_i \otimes y_i\}$ belongs to $\mathcal{U}$. The lemma is proved.

DEFINITION. For every k let $Y_k(X_k)$ be the set of elements in $D_k(D_k^*)$ such that $y \in Y_k(x \in X_k)$ if there exists a sequence $\{y_i\} \in Y(\{x_i\} \in X)$ for which $y = y_k$ $(x = x_k)$.

Since X and Y are linear manifolds, X_k and Y_k are also linear manifolds.

LEMMA 3.4. (i) *If* $\{x_i\} \in X$ *and* $\{y_i\} \in Y$ *and if* $\{T_i\} \in \mathcal{U}$, *then* $\{T_i^*x_i\} \in X$ *and* $\{T_iy_i\} \in Y$.

(ii) *If all* $\mathcal{U}_i$ *are weakly dense in* $B(H_i)$ *and if* $X \neq \{0\}$ *and* $Y \neq \{0\}$, *then all* X_i *and* Y_i *are dense in* H_i.

Proof. Let us prove that $\{T_iy_i\} \in Y$. Since $y_i \in D_i$, we have, by (A_1), that $T_iy_i \in D_i$. Hence (B_1) holds. By (A_2) and by (B_2),

$$G_i(T_{i+1}y_{i+1}) = T_i(G_iy_{i+1}) = T_iy_i.$$

Thus (B_2) holds for $\{T_iy_i\}$. By (A_3), by (A_4) and by (B_3),

$$\sup\|T_iy_i\| \leq \sup\|T_i\|\sup\|y_i\| < \infty$$

and

$$\sup\|F_iT_{i+1}y_{i+1}\| = \sup\|(T_iF_i + T_{F_i})y_{i+1}\|$$
$$\leq \sup\|T_i\|\sup\|F_iy_{i+1}\| + \sup\|T_{F_i}\|\sup\|y_{i+1}\| < \infty.$$

Hence (B_3) holds for $\{T_iy_i\}$. Thus the sequence $\{T_iy_i\}$ satisfies conditions (B_1)–(B_3) and therefore $\{T_iy_i\} \in Y$. In the same way, using conditions (A_1^*)–(A_3^*) and (B_1^*)–(B_3^*), we obtain that $\{T_ix_i\} \in X$, and (i) is proved.

Now suppose that $Y \neq \{0\}$. Then there exists a sequence $\{y_i\} \in Y$ and the smallest k such that $y_k \neq 0$. It follows from (B_2) that $y_i \neq 0$ for $i \geq k$. By (i), $\{T_iy_i\} \in Y$ for every $\{T_i\} \in \mathscr{U}$. Since $\mathscr{U}_i$ are weakly dense in $B(H_i)$ and since $y_i \neq 0$ for $i \geq k$, the linear manifolds Y_i are dense in H_i for $i \geq k$. Suppose that $1 < k$. Then $y_{k-1} = G_{k-1}y_k = 0$. Hence, by (A_2),

$$G_{k-1}T_ky_k = T_{k-1}G_{k-1}y_k = 0,$$

and therefore $T_ky_k \in \operatorname{Ker} G_{k-1}$ for every $\{T_i\} \in \mathscr{U}$. Since $\mathscr{U}_k$ is weakly dense in $B(H_k)$, $\operatorname{Ker} G_{k-1}$ is dense in $B(H_k)$. Hence $G_{k-1} = 0$ which contradicts (R_2). Therefore $y_{k-1} \neq 0$ which contradicts the assumption that $1 < k$ is the smallest number such that $y_k \neq 0$. Hence $k = 1$ and all Y_i are dense in H_i. In the same we obtain that if $X \neq \{0\}$, then all X_i are dense in H_i, and the lemma is proved.

Let us impose further restrictions on the operators $\{F_i\}$ and $\{G_i\}$.

(R_3) Let all X_i and Y_i are dense in H_i.

Since the operators S_t^i are closed, the operators $S_t^i|_{Y_{i+1}}$ are closable.

DEFINITION. By Q_t^i we shall denote the closed operator $(R_t^i|_{X_i})^*$ and by P_t^i we shall denote the closure of $S_t^i|_{Y_{i+1}}$.

Then $P_t^i \subseteq S_t^i$ and, since $R_t^i|_{X_i} \subseteq R_t^i$, we have that $(R_t^i)^* \subseteq Q_t^i$. Taking (7) into account we obtain that

$$P_t^i \subseteq S_t^i \subseteq (R_t^i)^* \subseteq Q_t^i. \tag{16}$$

THEOREM 3.5. *Let* (R_3) *hold. Then* $\operatorname{Lat}\mathscr{A}$ *consists of* $\mathscr{H}$, *of all subspaces* $\mathscr{H}_i$ *for* $0 \leq i < n$, *and of all subspaces* $\mathscr{M}_S^i$ *for* $1 \leq i < n$, *where* S *can be* P_t^i, S_t^i, F_i, $(R_t^i)^*$, Q_t^i *or any closed operator from* H_{i+1} *into* H_i *such that*

(1) $P_t^i \subseteq S \subseteq Q_t^i$ *for some* t;

(2) $TD(S) \subseteq D(S)$ *for every* $T \in \mathscr{U}_{i+1}$.

Proof. It was already proved in Theorem 2.4 that subspaces $\mathscr{M}^i_{S^i_t}$, $\mathscr{M}^i_{(R^i_t)^*}$ and $\mathscr{M}^i_{F_i}$ belong to Lat $\mathscr{A}$. Repeating the same argument and using Lemma 2.3 we obtain that the subspaces $\mathscr{M}^i_{P^i_t}$ and $\mathscr{M}^i_{Q^i_t}$ also belong to Lat $\mathscr{A}$. Now let S be a closed operator which satisfies the conditions of the theorem. Since $Y_{i+1} \subseteq D(P^i_t) \subseteq D(S)$, condition (1) of Lemma 3.1 holds. Condition (2) of Lemma 3.1 follows from condition (2) of the theorem. Since $\mathscr{M}^i_{Q^i_t}$ belongs to Lat $\mathscr{A}$, Q^i_t satisfies condition (3) of Lemma 3.1. Therefore taking into account that $S = Q^i_t|_{D(S)}$, we obtain

$$\begin{aligned}(T_iS + T_{F_i})|_{D(S)} &= (T_iQ^i_t + T_{F_i})|_{D(S)}\\ &= Q^i_tT_{i+1}|_{D(S)} = ST_{i+1}|_{D(S)},\end{aligned}$$

so that condition (3) of Lemma 3.1 holds. Therefore $\mathscr{M}^i_S \in \operatorname{Lat}\mathscr{A}$.

Now let S be a closed operator from H_{i+1} into H_i which satisfies the conditions of Lemma 3.1 and let us prove that it satisfies the conditions of this theorem. It obviously satisfies condition (2) of the theorem.

Let $\{x_k\} \in X$ and $\{y_k\} \in Y$. Then, by Lemma 3.3, the operator $x_{i+1} \otimes y_{i+1}$ belongs to $\mathscr{U}_{i+1}$. It follows from condition (2) of Lemma 3.1 that for every $z \in D(S)$

$$(x_{i+1} \otimes y_{i+1})z = (z, x_{i+1})y_{i+1} \in D(S).$$

Since, by condition (1) of Lemma 3.1, $D(S)$ is dense in H_{i+1}, we get that $Y_{i+1} \subseteq D(S)$. It follows from condition (3) of Lemma 3.1 and from (15) that for every $z \in D(S)$

$$(x_i \otimes y_i)Sz + (x_{i+1} \otimes F_iy_{i+1})z - (F_i^*x_i \otimes y_i)z = S(x_{i+1} \otimes y_{i+1})z.$$

Hence

$$(Sz, x_i)y_i + (z, x_{i+1})F_iy_{i+1} - (z, F_i^*x_i)y_i = (z, x_{i+1})Sy_{i+1} \tag{17}$$

Let $z \in Y_{i+1}$. Then $(z, F_i^*x_i) = (F_iz, x_i)$. Put $V = S - F_i$. We obtain from (17) that

$$(Vz, x_i)y_i = (z, x_{i+1})Vy_{i+1} \tag{18}$$

By (B_2), $y_i = G_iy_{i+1}$. Since X_{i+1} is dense in H_{i+1}, we can choose x_{i+1} such that $(z, x_{i+1}) \neq 0$. Then it follows from (18) that for every $y \in Y_{i+1}$

$$Vy = tG_iy,$$

where $t = (Vz, x_i)/(z, x_{i+1})$. Therefore we obtain that

$$S|_{Y_{i+1}} = (F_i + tG_i)|_{Y_{i+1}} = S^i_t|_{Y_{i+1}}. \tag{19}$$

Thus $P^i_t \subseteq S$. Using (19) we obtain from (17) that for every $z \in D(S)$

$$(Sz, x_i)y_i - (z, F_i^*x_i)y_i = (z, x_{i+1})tG_iy_{i+1}.$$

By (B_2), $y_i = G_i y_{i+1}$ and, by (B_2^*), $x_{i+1} = G_i^* x_i$. Hence

$$(Sz, x_i) - (z, F_i^* x_i) = t(z, G_i^* x_i).$$

Therefore $(Sz, x_i) = (z, R_t^i x_i)$ which means that

$$S \subseteq \left(R_t^i|_{X_i}\right)^* = Q_t^i.$$

Thus $P_t^i \subseteq S \subseteq Q_t^i$ and S satisfies condition (1) of this theorem which completes the proof.

Now suppose that $n < \infty$, that all $H_i = H$, that all $G_i = I$ and that all $\mathcal{T}_i = B(H)$. Then

$$D_{i+1} = D(F_i), \qquad D_i^* = D(F_i^*),$$

all $Y_i = D = \bigcap_{i=1}^{n-1} D_{i+1}$ and all $X_i = D^* = \bigcap_{i=1}^{n-1} D_i^*$. If D and D^* are dense in H, then $\mathcal{U}$ consists of all sequences $\{T_i\}_{i=1}^n$ such that $T_1 = \cdots = T_n = T$, where T belongs to

$$\mathbf{A} = \{T \in B(H)\colon \text{(a) } TD_i \subseteq D_i;$$

$$\text{(b) the operators } (F_i T - T F_i)|_{D_{i+1}} \text{ extend to bounded operators } T_{F_i}\}.$$

From Corollary 2.6 it follows that $\mathcal{A}$ *is reflexive*. We also have that the operators P_t^i are the closures of the operators $(F_i + tI)_D = F_i|_D + tI$, that $S_t^i = F_i + tI$, that $R_t^i = F_i^* + \bar{t}I$ and that

$$Q_t^i = \left((F_i^* + \bar{t}I)|_{D^*}\right)^* = \left(F_i^*|_{D^*}\right)^* + tI.$$

Therefore $(R_t^i)^* = S_t^i$, $S_0^i = F_i$ and it follows from Theorem 3.5 that Lat $\mathcal{A}$ *consists of* $\mathcal{H}_i$ *for* $i = 0, \ldots, n$, *and of all subspaces* $\mathcal{M}_S^i$ *for* $i = 1, \ldots, n-1$, *where* S *can be* P_t^i, S_t^i, Q_t^i *or any closed operator such that*

(1) $P_t^i \subset S \subset Q_t^i$ *for some* t;

(2) $TD(S) \subseteq D(S)$ *for every* $T \in \mathbf{A}$.

If the operators $\{F_i\}$ are such that for every i the closure of $F_i|_D$ is F_i and the closure of $F_i^*|_{D^*}$ is F_i^*, then

$$P_t^i = F_i + tI = S_t^i$$

and

$$Q_t^i = \left(F_i^*|_{D^*}\right)^* + tI = \left(F_i^*\right)^* + tI = F_i + tI = S_t^i.$$

Therefore we obtain the following theorem which was proved in [3] (Theorem 4.4(ii)) (the theorem was erroneously stated without condition (b)).

THEOREM 3.6. *If* (a) *D and D^* are dense in H*; (b) *for every i the closure of $F_i \mid_D$ is F_i and the closure of $F_i^* \mid_{D^*}$ is F_i^*, then* Lat $\mathscr{A}$ *consists of $\mathscr{H}_i$ for $i = 0, \dots, n$, and of all subspaces $\mathscr{M}^i_{S_t^i}$ for $i = 1, \dots, n-1$ and for $t \in \mathbf{C}$.*

If the conditions of Theorem 3.6 do not hold, then the structure of Lat $\mathscr{A}$ is more complicated, and even in comparatively simple cases it is difficult to describe it fully.

EXAMPLE. Let $F_1 \subset F_2 \subset \cdots \subset F_{n-1}$. Then $D = D(F_1)$ and $D^* = D(F^*_{n-1})$. Hence all $P^i_t = F_1 + tI$ and all

$$Q^i_t = (F_i^* \mid_{D^*})^* + tI = (F^*_{n-1})^* + tI = F_{n-1} + tI.$$

Then for every $1 < k < n-1$ and for every $t \in \mathbf{C}$ we have that

$$F_1 + tI \subset F_k + tI \subset F_{n-1} + tI.$$

By property (a) of $\mathbf{A}$, $TD(F_k) \subseteq D(F_k)$ for every $T \in \mathbf{A}$. Therefore Lat $\mathscr{A}$ contains all subspaces $\mathscr{H}_i$ for $i = 0, \dots, n$, and all subspaces $\mathscr{M}^i_S$ for $i = 1, \dots, n-1$, where S can be any of the operators $F_k + tI$ for $1 \le k \le n-1$ and for $t \in \mathbf{C}$. The following question arises: do other operators R exist, apart from F_k, $k = 2, \dots, n-2$, such that

(1) $F_1 \subset R \subset F_{n-1}$;

(2) $TD(R) \subseteq D(R)$ for every $T \in \mathbf{A}$.

If such operators do not exist, then we have a full description of Lat $\mathscr{A}$. If they do exist, then each of them generates a set of subspaces $\mathscr{M}^i_{R+tI}$ for $i = 1, \dots, n-1$ and for $t \in \mathbf{C}$, which belong to Lat $\mathscr{A}$.

Finally, we shall briefly consider two examples of algebras $\mathscr{A}$ for $n = 2$ and provide full descriptions of Lat $\mathscr{A}$ and of Ad $\mathscr{A}$. The case when the operator G is the identity was investigated in [3]. In Theorem 4.3 it was shown that Ad $\mathscr{A} \neq \mathscr{A}$. In Example 2 a closed operator F was considered such that Ad $\mathscr{A} = \mathscr{A} + \{N\} + \{B\}$, where N and B do not belong to $\mathscr{A}$, so that dim(Ad $\mathscr{A}/\mathscr{A}$) = 2. It was also proved that $\mathscr{A}' = \{I\} + \{N\}$ so that B generates a non-inner derivation on $\mathscr{A}$. Now we shall consider an example of a reflexive algebra $\mathscr{A}$ constructed from two closed operators F and G such that Ad $\mathscr{A} = \mathscr{A} + \{N\} + \{B\}$. But for this algebra $\mathscr{A}' = \{I\}$, so that all operators from Ad $\mathscr{A}$ which do not belong to $\mathscr{A}$ generate non-inner derivations on $\mathscr{A}$.

EXAMPLE 1. Let $H_1 = H_2 = H = K \oplus K$, where K is an infinite-dimensional Hilbert space and let $\mathscr{H} = H \oplus H$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthogonal basis in K and let W be an unbounded operator on K such that

$$We_n = ne_n.$$

For a complex a set

$$F = \begin{pmatrix} aW^2 & W^2 \\ 0 & aW \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} W^2 & 0 \\ 0 & W \end{pmatrix}.$$

Then

$$D(F) = D(W^2) \oplus D(W^2), \qquad D(G) = D(W^2) \oplus D(W),$$

$$D_2 = D(F), \qquad D_1^* = D(G).$$

Therefore restrictions (R_1), (R_2) and (R_3) on operators F and G hold. Obviously G is the closure of $G|_{D_2}$ and F is the closure of $F|_{D_2}$. Also

$$P_t = S_t = F + tG = \begin{pmatrix} (a+t)W^2 & W^2 \\ 0 & (a+t)W \end{pmatrix} \quad \text{for } t \neq -a$$

and

$$S_{-a} = \begin{pmatrix} 0 & W^2 \\ 0 & 0 \end{pmatrix} = P_{-a}.$$

We also have that $D(S_t) = D_2$, if $t \neq -a$ and $D(S_{-a}) = K \oplus D(W^2)$. So $\bigcap_{t \in \mathbb{C}} D(S_t) = D_2$ and, by Theorem 2.5, $\mathscr{A}$ is reflexive.

We have that

$$R_t = F^* + \bar{t}G^* = \begin{pmatrix} (\bar{a}+\bar{t})W^2 & 0 \\ W^2 & (\bar{a}+\bar{t})W^2 \end{pmatrix} \quad \text{for } t \neq -a$$

and

$$R_{-a} = \begin{pmatrix} 0 & 0 \\ W^2 & 0 \end{pmatrix}.$$

It is easy to check that $S_t = R_t^* = Q_t$. Therefore, by Theorem 3.5, Lat $\mathscr{A}$ consists of $\mathscr{H}_0$, $\mathscr{H}_1$, $\mathscr{H}$ and of all M_{S_t}, for $t \in \mathbb{C}$.

Set

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ W^{-2} & 0 & 0 & -2I \\ 0 & W^{-1} & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $B, N \in B(\mathscr{H})$ and it is easy to check that $[N, B] = NB - BN = N$. It can be proven that Ad $\mathscr{A} = \mathscr{A} + \{N\} + \{B\}$ and that $\mathscr{A}' = \{I\}$, so that all linear combinations of the operators N and B generate non-inner derivations on $\mathscr{A}$. One can also show that $\mathscr{A} \in R_1$.

In the following example we shall consider a reflexive algebra $\mathscr{A}$ constructed from two closed operators F and G such that $\operatorname{Ad}\mathscr{A}=\mathscr{A}$, although the structure of $\operatorname{Lat}\mathscr{A}$ is the same as in Example 1.

EXAMPLE 2. Let $\mathscr{H}$ and W be the same as in Example 1. Set

$$F=\begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \quad \text{and} \quad G=\begin{pmatrix} W & 0 \\ 0 & W^{-1} \end{pmatrix}.$$

Then

$$D(F)=D(W)\oplus D(W), \qquad D(G)=D(W)\oplus K,$$
$$D_2=D(F) \quad \text{and} \quad D_1^*=D_2.$$

The operators F and G satisfy restrictions (R_1), (R_2) and (R_3). Repeating the same argument as in Example 1 we obtain that $\mathscr{A}$ is reflexive, that $\operatorname{Lat}\mathscr{A}$ consists of $\mathscr{H}_0$, $\mathscr{H}_1$, $\mathscr{H}$ and of all M_{S_t}, for $t\in\mathbf{C}$, and that G is the closure of $G|_{D_2}$ and F is the closure of $F|_{D_2}$. It can be proven that $\operatorname{Ad}\mathscr{A}=\mathscr{A}$, so that all derivations on $\mathscr{A}$ implemented by bounded operators are inner.

REFERENCES

[1] E. Christensen, *Derivations of nest algebras*, Math. Ann., **229** (1977), 155–166.

[2] F. Gilfeather, A. Hopenwasser and D. R. Larson, *Reflexive algebras with finite width lattices: tensor products, cohomology, compact perturbations*, J. Funct. Anal., **55** (1984), 176–199.

[3] E. V. Kissin, *On some reflexive algebras of operators and the operator Lie algebras of their derivations*, Proc. London Math. Soc., (3), **49** (1984), 1–35.

[4] ______, *Totally symmetric algebras and the similarity problem*, (1985), preprint.

[5] B. H. Wagner, *Derivations of quasitriangular algebras*, preprint.

Received March 21, 1984 and in revised form February 12, 1986.

THE POLYTECHNIC OF NORTH LONDON
HOLLOWAY, LONDON N7 8DB, ENGLAND