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SCHRÖDINGER OPERATORS WITH A NONSPHERICAL RADIATION CONDITION

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The Schrödinger operators with potentials $p(x)$ which do not necessarily converge to a constant at infinity will be discussed. The potential $p(x) = x_1/|x|$, $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^N$, is an example. The radiation condition associated with such Schrödinger operators is shown to have the form $\nabla u - i\sqrt{\lambda}(\nabla R)u =$ small at infinity, where $R = R(x, \lambda)$ is a solution of the eikonal equation $|\nabla R|^2 = 1 - p(x)/\lambda$. This radiation condition is "nonspherical" in the sense that ∇R is not proportional to the vector $\tilde{x} = x/|x|$ in general. The limiting absorption principle will be obtained using a priori estimates for the radiation condition.

Introduction. Let us consider the inhomogeneous Schrödinger equation

$$(0.1) \qquad (T - \lambda)u = -\sum_{j=1}^{N} D_j^2 u + V(x)u - \lambda u = f \quad \text{in } \mathbf{R}^N,$$

where $D_j = \partial/\partial x_j + ib_j(x)$ with the "magnetic potentials" $b_j(x)$, λ is a positive number, the "potential" $V(x)$ is a real-valued function on $\mathbf{R}^N$ and $f(x)$ is a given function. In this paper we are going to consider a class of potentials $V(x)$ which contains potentials $V(x)$ such that $V(x) = O(1)$ and $\partial V/\partial x_j = O(|x|^{-1})$ at $x = \infty$. One example of such a function is $V(x) = x_1/|x|$ where x_1 is the first coordinate of $x = (x_1, x_2, \dots, x_N) \in \mathbf{R}^N$. We shall study the limiting absorption principle and the unique existence of the solution $u = u(\lambda, f)$ of the equation (0.1) introducing a "nonspherical" radiation condition

$$(0.2) \quad \left(D_j - i\sqrt{\lambda}\beta_j\right)u(x) \quad \text{is small at } x = \infty \qquad (j = 1, 2, \dots, N).$$

Condition (0.2) is nonspherical in the sense that $\beta = (\beta_1, \beta_2, \dots, \beta_N)$ is the outward normal of a surface which is not a sphere in general, whereas it seems that the outward normal $\tilde{x} = x/|x|$ of a sphere always appeared in the radiation conditions which were used up to now for various types of Schrödinger operators.

Let us first assume that $V(x)$ becomes small at $x = \infty$. Then the unique existence of the solution $u = u(\lambda, f)$ of the equation $(0, 1)$ with the appropriate boundary conditions at infinity $x = \infty$ has been studied in many papers, some of which will be mentioned later. The conditions at infinity have been various kinds of generalizations of the Sommerfeld radiation conditions

$$(0.3)\qquad \begin{cases} \dfrac{\partial u}{\partial |x|} - i\sqrt{\lambda}\, u(x) = o\left(|x|^{-(N-1)/2}\right), \\ u(x) = O\left(|x|^{-(N-1)/2}\right) \end{cases}$$

as $|x| \to \infty$.

In 1962 Eidus **[6]** showed that the unique existence of the solution $u(\lambda, f)$ of the equation (0.1) in $\mathbf{R}^3$ with

$$(0.4)\qquad \lim_{\rho\to\infty} \int_{S_\rho} \left| \frac{\partial u}{\partial |x|} - i\sqrt{\lambda}\, u \right|^2 dS = 0,$$

where $S_\rho = \{x \in \mathbf{R}^3/|x| = \rho\}$. Here $b_j(x)$ is assumed to be 0 in a neighborhood of $x = \infty$, $V(x)$ is assumed to satisfy

$$(0.5)\qquad V(x) = O\left(|x|^{-2-\alpha}\right) \qquad (|x| \to \infty)$$

with $\alpha > \frac{1}{6}$ and f satisfies $f(x) = O(|x|^{-3-\beta})$ with $\beta > 0$. The solution $u(\lambda, f)$ was constructed by the limiting absorption method, that is, $u(\lambda, f)$ is obtained as the limit

$$(0.6)\qquad u(\lambda, f) = \lim_{\varepsilon \downarrow 0} u(\lambda + i\varepsilon, f),$$

where $u(\lambda + i\varepsilon, f)$ is the solution of (0,1) with λ replaced by $\lambda + i\varepsilon$ ($\varepsilon > 0$).

Ikebe-Saitō **[8]** also used the limiting absorption method to show the unique existence of $u(\lambda, f)$ of the equation (0.1) in $\mathbf{R}^N$ with the radiation condition

$$(0.7)\qquad \int_{\mathbf{R}^N} (1 + |x|)^{2(\delta-1)} |(D - i\sqrt{\lambda}\,\tilde{x})u|^2\, dx < \infty,$$

$$(0.8)\qquad \int_{\mathbf{R}^N} (1 + |x|)^{-2\delta} |u(x)|^2\, dx < \infty,$$

where δ is a fixed constant with $\delta > \frac{1}{2}$,

$$|(D - i\sqrt{\lambda}\,\tilde{x})u|^2 = \sum_{j=1}^{N} \left|D_j u(x) - i\sqrt{\lambda}\,\tilde{x} u(x)\right|^2 \quad \text{and} \quad \tilde{x} = x/|x|.$$

Here $V(x)$ is assumed to be decomposed into the sum of the long-range potential $V_1(x)$ and the short-range potential $V_2(x)$, i.e., we have

$$(0.9)\qquad \begin{cases} V(x) = V_1(x) + V_2(x), \\ V_1(x) = O\left(|x|^{-\varepsilon}\right), \qquad \partial V/\partial |x| = O\left(|x|^{-1-\varepsilon}\right) \\ V_2(x) = O\left(|x|^{-1-\varepsilon}\right) \end{cases}$$

with $\varepsilon > 0$ at infinity. Let us note that $\tilde{x}$ is the outward unit normal of the sphere $S^{N-1} = \{x \in \mathbf{R}^N/|x| = 1\}$. When $b_j(x)$ are assumed to be identically 0, the operator $-\Delta + V(x)$ can be transformed into the ordinary differential operator

$$(0.10)\qquad L = -d^2/dr^2 + B(r) + C(r) \qquad (r \in (0, \infty))$$

with the operator valued coefficients $B(r)$, $C(r)$. For fixed $r > 0$ $B(r)$ and $C(r)$ are operators in $L_2(S^{N-1})$ of the forms

$$(0.11)\qquad \begin{cases} B(r) = \dfrac{1}{r^2}(\mathscr{L}_N + (N-1)(N-3)), \\ C(r) = V(r\omega) \qquad (\omega \in S^{N-1}) \end{cases}$$

with the Laplace-Beltrami operator $\mathscr{L}_N$ on S^{N-1}. In this case the limiting absorption method can be applied to the operator L (Saitō [**21**], [**22**]). The radiation condition for L has the form

$$(0.12)\qquad \begin{cases} \displaystyle\int_0^\infty (1+r)^{2(\delta-1)} \|v'(r) - i\sqrt{\lambda}\, v(r)\|^2_{L_2(S^{N-1})}\, dr < \infty, \\ \displaystyle\int_0^\infty (1+r)^{-2\delta} \|v(r)\|^2_{L^2(S^{N-1})}\, dr < \infty, \end{cases}$$

where $v(r) = r^{(n-1)/2} u(r\omega)$, $\omega \in S^{N-1}$, is regarded as an $L_2(S^{N-1})$-valued function on $(0, \infty)$.

There exists another type of radiation condition. Let $V(x)$ be the sum of a long-range potential $V_1(x)$ and the short-range potential $V_2(x)$ and let $b_j(x) = 0$. Saitō [**21**], [**22**] and Isozaki [**9**] proved that the estimate

$$(0.13)\qquad \int_{\mathbf{R}^N} (1+|x|)^{2(1-\delta)} |(\nabla - i(\nabla K))u(x)|^2\, dx \leq C \int_{\mathbf{R}^N} (1+|x|)^{2(2-\delta)} |f(x)|^2\, dx$$

is valid for the solution (0.1), where $C = C(\lambda)$ is a positive constant depending only on λ (and the operator T) and

$$(0.14)\qquad \nabla v = \operatorname{grad} v = \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_N}\right).$$

Here $K = K(x, \lambda)$ is an exact or approximate solution of the eikonal equation

$$(0.15) \qquad |\nabla K|^2 + V_1(x) = \lambda \qquad \left(|\nabla K|^2 = \sum_{j=1}^{N}\left(\frac{\partial K}{\partial x_j}\right)^2\right)$$

and the essential part of $(\nabla K)(x, \lambda)$ has the form

$$(0.16) \qquad \nabla K(x, \lambda) = \Phi(x, \lambda)\tilde{x} \qquad (\tilde{x} = x/|x|).$$

The transformation of the operator T to the ordinary differential operator (0.10) works in the proof of (0.13), too. On the other hand, Mochizuki-Uchiyama **[15]** constructed a similar type of radiation condition

$$(0.17) \qquad \int_{\mathbf{R}^N} (1 + |x|)^{2(\delta-1)}|(\nabla - ik(x, \lambda)\tilde{x})u|^2\, dx < \infty,$$

to get the limiting absorption principle for the Schrödinger operator T with the "oscillating" long-range potential $V(x)$. Here $V(x)$ satisfies

$$(0.18) \qquad \begin{cases} V(x) = O(1), \\ \partial V/\partial|x| = O\left(|x|^{-1}\right), \\ \partial^2 V/\partial|x|^2 + aV(x) = O\left(|x|^{-1-\varepsilon}\right) \end{cases}$$

as $|x| \to \infty$ with constants $a \geq 0$ and $\varepsilon > 0$. In addition $V(x)$ is assumed to behave uniformly as $|x| \to \infty$ (cf. Mochizuku-Uchiyama **[16]**, §8, (V2–4)).

In all these works the outward normal $\tilde{x}$ of the unit sphere appears in the radiation condition and the limiting absorption principle holds for the operator (0.10) as well as the Schrödinger operator T. Therefore all these radiation conditions may be classified as "spherical" radiation conditions.

The potential that we are going to consider is "wilder" than a longe-range potential or an oscillating long-range potential in the sense that our potential $V(x)$ essentially satisfies only the first two conditions of (0.18). The Schrödinger operator with such a potential has been studied from various viewpoints. There are many papers discussing the essential selfadjointness of Schrödinger operators (see e.g., Kato **[13]**, Eastham-Evans-McLeod **[4]**, Read **[19]**). As for the nonexistence of the eigenvalues, the works by Mochizuki **[15]** and Eastham-Kalf **[5]** should be noted. We are now going to study the absolute continuous spectrum. Ben-Artzi **[3]** and Jäger-Rejto **[11]** proved the limiting absorption principle for a Schrödinger operator with an exploding potential $V(x)$ which is assumed to go to $+\infty$ at $x = \infty$, though our potential does not satisfy their conditions. On the other hand, the commutator method developed by Mourre [**18**] and Jensen-Mourre-Perry [**12**] can be applied to our potential to show the existence of the limit

$$(0.19) \qquad (T - (\lambda \pm i0))^{-1} = \lim_{\varepsilon \downarrow 0} (T - (\lambda \pm i\varepsilon))^{-1}$$

for large enough λ. In this sense the limiting absorption principle has been already established. What we are going to do in this paper is to introduce a radiation condition of "nonspherical" type

$$(0.20)_\pm \qquad \int_{\mathbf{R}^N} (1 + |x|)^{2(\delta - 1)} |(D \mp i\sqrt{\lambda}\,\beta) u(x)|^2 \, dx < \infty$$

to show that $u_\pm(x) = (T - (\lambda \pm i0))^{-1} f$ satisfy $(0.20)_\pm$, and that the equation (0.1) with $(0.20)_+$ (or $(0.20)_-$) and (0.8) has a unique solution $u_\pm = u_\pm(\lambda, f)$. Here $\beta = \beta(x, \lambda) = (\beta_1, \beta_2, \dots, \beta_N)$ is expressed as

$$(0.21) \quad \beta = \nabla R \qquad \left(\text{or } \beta_j(x, \lambda) = \partial R(x, \lambda)/\partial x_j, \quad j = 1, 2, \dots, N\right),$$

where $R(x, \lambda)$ is a solution of the eikonal equation

$$(0.22) \qquad |\nabla R|^2 = 1 - V(x)/\lambda.$$

Though β is the outward normal of the surface $R(x, \lambda) = r$, this surface is not necessarily a sphere. In fact, when $V(x) = x_1/|x|$, the surface $R(x, \lambda) = r$ is an ellipsoid. We can also see that the usage of the operator (0.10) instead of the Schrödinger operator T is inadequate. It seems that the radial variable $r = |x|$ should be replaced by $R(x, \lambda)$ in our situation. At the same time another proof for the limiting absorption principle for our potential along the line of Eidus [**6**], Jäger [**10**], Agmon [**1**], Ikebe-Saitō [**8**] will be obtained.

In the studies of the Schrödinger operator with a long-range or an oscillating long-range potential, after establishing the existence and uniqueness of the solution of the inhomogeneous Schrödinger equation, we could derive an asymptotic formula for the solution which turned out to be a starting point for spectral and scattering theory for the Schrödinger operator (see, e.g., Saitō [**21**], [**22**], Mochizuki-Uchiyama [**17**]). It is also expected that we could develop spectral and scattering theory for our potential. This will be discussed elsewhere.

We shall give the rigorous definition on the potentials $V(x)$ and $b_j(x)$ and state our main theorem in §1. In the following two sections (§2 and §3) we shall show two a priori estimates for a solution of the Schrödinger equation (0.1). These estimates will be used in §4 to show the limiting absorption principle for large enough λ, whence follows the uniform existence of the solution $u = u(\lambda, f)$ of the equation (0.1) with the radiation condition (0.20) and (0.8). In §5 we shall give two concluding remarks, one of which is related to a stronger estimate for the radiation condition (0.20) similar to (0.17).

1. Main result. Let us consider the differential operator

$$T = -\sum_{j=1}^{N} D_j^2 + p(x) + Q(x) \tag{1.1}$$

in $\mathbf{R}^N$, where

$$D_j = \partial_j + ib_j(x) \qquad \left(\partial_j = \partial/\partial x_j,\ i = \sqrt{-1},\quad j = 1, 2, \ldots, N\right) \tag{1.2}$$

and N is a positive integer with $N \geq 2$. The given functions $p(x)$, $Q(x)$ and $b_j(x)$ are presumed to satisfy the following two assumptions:

Assumption 1.1.

(p) $p(x)$ is a bounded, real-valued function on $\mathbf{R}^N$ such that $p \in C^2(\mathbf{R}^N - \{0\})$ with estimates

$$|\partial^\alpha p(x)| \leq c|x|^{-|\alpha|} \qquad \left(|\alpha| \leq 2,\ x \in \mathbf{R}^N - \{0\}\right),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is an arbitrary multi-index with non-negative integers α_j $(1 \leq j \leq N)$. $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$ and c is a positive constant. We set $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2} \cdots \partial_N^{\alpha_N}$.

(Q) $Q(x)$ is a real-valued, measurable function on $\mathbf{R}^N$ and there exists $\nu > 0$ such that

$$M_Q(x) = \int_{|x-y|\leq 1} \frac{|Q(x)|^2}{|x-y|^{N-4+\nu}}\, dy \tag{1.3}$$

is locally bounded on $\mathbf{R}^N$. Further, there exist positive constants ε, r_0 such that

$$|Q(x)| \leq C(1 + |s|)^{-1-\varepsilon} \qquad (|x| \geq r_0), \tag{1.4}$$

c being as in (p).

(B) $b_j(x)$, $j = 1, 2, \ldots, N$, are real-valued C^1 functions on $\mathbf{R}^N$ satisfying

$$|B_{jl}(x)| \leq c(1 + |x|)^{-1-\varepsilon} \qquad (|x| \geq r_0) \tag{1.5}$$

where

$$B_{jl} = \partial_j b_l - \partial_l b_j \tag{1.6}$$

and c, ε, r_0 are as above.

(UC) The unique continuation property holds for the differential operator $T = -\Delta + p + Q$.

As for the potential $p(x)$ we need another assumption.

Assumption 1.2.

(R) There exists λ_0 ($\geq \sup_{x\in\mathbf{R}^N} p(x)$) such that for any $\lambda > \lambda_0$ the differential equation

$$|\nabla R|^2 = 1 - \frac{p(x)}{\lambda} \qquad \left(|\nabla R|^2 = \sum_{j=1}^{N} (\partial_j R)^2\right) \tag{1.7}$$

has a solution $R = R(x, \lambda)$ for $|x| \geq r_0$ which satisfies the following (i)–(iii):

(i) For each $\lambda > \lambda_0$ $R(x, \lambda)$ is a real-valued, C^3 function for $|x| \geq r_0$, r_0 being as above.

(ii) Setting

$$g(x, \lambda) = |x|^{-1} R(x, \lambda) \tag{1.8}$$

we have

$$c_0 \leq g(x, \lambda) \leq c_1 \tag{1.9}$$

for all $|x| \geq r_0$ and $\lambda > \lambda_0$ with positive constants c_0 and c_1.

(iii) We have

$$|x|^j (\partial^j g)(x, \lambda) = \begin{cases} 1 & (j = 0), \\ 0 & (j = 1, 2, 3) \end{cases} \tag{1.10}$$

uniformly for $x \in \{x \in \mathbf{R}^N / |x| \geq r_0\}$. Here $\partial^j g$ denotes an arbitrary jth derivative of g.

REMARK 1.3.

(i) Let $p(x) = x_1/|x|$, where $x = (x_1, x_2, \ldots, x_N)$. Then $p(x)$ satisfies (p) in Assumption 1.1. Set $\lambda_0 = 1$ and define $R(x, \lambda)$ for $\lambda > \lambda_0$ by

$$R(x, \lambda) = a(\lambda)|x| - b(\lambda) x_1 \tag{1.11}$$

with

$$\begin{cases} a(\lambda) = \frac{1}{2}\{(1 + 1/\lambda)^{1/2} + (1 - 1/\lambda)^{1/2}\}, \\ b(\lambda) = \frac{1}{2}\{(1 + 1/\lambda)^{1/2} - (1 - 1/\lambda)^{1/2}\}. \end{cases} \tag{1.12}$$

Then it is easy to see that $R(x, \lambda)$ satisfies (R) in Assumption 1.2 with $p(x) = x_1/|x|$, $\lambda_0 > 1$ and an arbitrary $\lambda_0 > 0$.

(ii) Suppose that $p(x)$ satisfy (p) in Assumption 1.1. Then it follows from Lions ([**13**], Chapters 2 and 5) that there exists a solution $R(x, \lambda)$ of (1.7) which satisfies (1.9) and (1.10) with $j = 0$. It seems to be open whether $R(x, \lambda)$ satisfies (1.10) with $j = 1, 2, 3$ in general. (See Added in Proof.)

(iii) In the trivial case of $p(x) = 0$, we can take $R(x, \lambda) = |x|$.

In order to state the main results we need some definitions and notations including the extension of $R(x, \lambda)$ to the complex plane.

DEFINITION 1.4. Let $z = \lambda + i\mu \in \mathbf{C}$ with $\lambda > \lambda_0$, where λ_0 is as in Assumption 1.2. Let $R(x, \lambda)$ be as in Assumption 1.2. Then $R(x, z)$ is defined by

$$(1.13)\qquad R(x, z) = R\left(x, |z|^2/\lambda\right) \qquad (r \geq r_0).$$

We set

$$(1.14)\qquad \begin{cases} \beta = \beta(x, z) = \nabla R(x, z), \\ \beta_j = \beta_j(x, z) = \partial_j R(x, z) \qquad (j = 1, 2, \ldots, N). \end{cases}$$

Notation 1.5.

$\mathbf{R}$: real numbers,
$\mathbf{C}$: complex numbers,
$D_j = \partial_j + ib_j(x)$ $(j = 1, 2, \ldots, N,\ \partial_j = \partial/\partial x_j)$,
$Du = (D_1 u, D_2 u, \ldots, D_N u)$,
$\nabla u = (\partial_1 u, \partial_2 u, \ldots, \partial_N u)$,
$\operatorname{Re} z$: the real part of z,
$\operatorname{Im} z$: the imaginary part of z.

$L_{2,\gamma}(G)$ $(\gamma \in \mathbf{R})$ denotes the Hilbert space of all functions f on G such that $(1 + |x|)^\gamma f$ is square integrable over G. The norm and inner product of $L_{2,\gamma}(G)$ are denoted by $\|\ \|_{\gamma,G}$ and $(\ ,\ )_{\gamma,G}$, respectively. We set $L_{2,\gamma}(\mathbf{R}^N) = L_{2,\gamma}$, $\|\ \|_{\gamma,\mathbf{R}^N} = \|\ \|_\gamma$ and $(\ ,\ )_{\gamma,\mathbf{R}^N} = (\ ,\ )_\gamma$. When $\gamma = 0$, we shall omit the subscript 0 as in $L_2(G)$, $\|\ \|_G$ etc.

H_m is all L_2 functions with L_2 distribution derivatives up to the mth order, inclusive.

C^m is the class of m-times continuously differentiable functions.

C_0^∞ is the class of infinitely continuously differentiable functions with compact support in $\mathbf{R}^N$.

M_{loc} is the class of all locally M functions.

Let us consider the inhomogeneous Schrödinger equation

$$(1.15)\qquad \begin{cases} (T-z)u = -\sum_{j=1}^{N} D_j^2 u + (p(x) + Q(x) - z)u = f, \\ u \in L_{2,-\delta} \cap H_{2,\mathrm{loc}}, \end{cases}$$

where δ is a fixed constant such that

$$(1.16)\qquad \frac{1}{2} < \delta \leq \min\left(\frac{1+\varepsilon}{2}, 1\right)$$

with ε given in Assumption 1.1, (Q), and $f(x)$ is a given function. The following is our main result.

THEOREM 1.6. *Assume Assumptions* 1.1 *and* 1.2.

(i) *Then there exists a unique solution* $u = u(z, f)$ *of* (1.15) *with* $f \in L_2$ *for any* $z = \lambda + i\mu$ *with* $\mu \neq 0$.

(ii) *There exists* $\Lambda_0 > 0$ *such that* $\Lambda_0 > \lambda_0$ *and there exist limits*

$$(1.17)\qquad \lim_{\mu>0, \mu\to 0} u(\lambda \pm i\mu, f) = u_{\pm}(\lambda, f) \quad \text{in } L_{2,-\delta} \cap H_{1,\mathrm{loc}}$$

for any $\lambda > \Lambda_0$ *and any* $f \in L_{2,\delta}$. *The functions* $u_{\pm}(\lambda, f)$ *are unique solutions of the equation* (1.5) *with the generalized radiation conditions*

$$(1.18)\qquad \left\|(D \mp i\sqrt{\lambda}\beta)u\right\|_{\delta-1, E_{r_0}} < \infty,$$

respectively, where λ_0 *is as in Assumption* 1.2,

$$(1.19)\qquad \left\|(D \mp i\sqrt{\lambda}\beta)u\right\|^2_{\delta-1, E_{r_0}} = \int_{E_{r_0}} (1+|x|)^{2(\delta-1)} \sum_{j=1}^{N} \left|D_j u \mp i\sqrt{\lambda}\beta_j u\right|^2 dx,$$

and $E_{r_0} = \{x \in \mathbf{R}^N \mid |x| \geq r_0\}$.

(iii) *Let* $M = \{z = \lambda + i\mu / \lambda_1 \leq \lambda \leq \lambda_2, 0 < |\mu| < \mu_0\}$ *with* $\Lambda_0 < \lambda_1 < \lambda_2$ *and* $\mu_0 > 0$. *Then there exists a positive constant* $C = C(M)$ *depending only on* M (*and the potentials* p, Q, b_j) *such that the estimates*

$$(1.20)\qquad \|u(z, f)\|_{-\delta} \leq C\|f\|_{\delta},$$

$$(1.21)\qquad \left\|(D - i\sqrt{z}\beta)u\right\|_{\delta-1, E_{r_0}} \leq C\|f\|_{\delta},$$

$$(1.22)\qquad \|u(z, f)\|_{-\delta, E_r} \leq Cr^{-(\delta-1/2)}\|f\|_{\delta} \qquad (r \geq r_0),$$

for any pair $(z, f) \in M \times L_{2,\delta}$. *Therefore the estimates* (1.20)–(1.22) *are also valid for* $u_{\pm}(\lambda, f)$ *with* $\lambda_1 \leq \lambda \leq \lambda_2$ *and* $f \in L_{2,\delta}$.

(iv) *Set*

$$\overline{M}_{\pm} = \{z = \lambda \pm i\mu/\lambda_1 \le \lambda \le \lambda_2, 0 \le \mu < \mu_0\} \tag{1.23}$$

with $\lambda_0 < \lambda_1 < \lambda_2$ *and* $\mu_0 > 0$. *Let* $u(z, f)$ $(z = \lambda + i\mu,\ \mu \neq 0)$ *be as above. Set* $u(\lambda, f) = u_+(\lambda, f)$ *when* $\lambda \in [\lambda_1, \lambda_2]$ *is contained in* $\overline{M}_+$ *and set* $u(\lambda, f) = u_-(\lambda, f)$ *when* $\lambda \in [\lambda_1, \lambda_2]$ *is contained in* $\overline{M}_-$. *Then* $u(z, f)$ *is an* $L_{2,-\delta}$*-valued continuous function for* $(z, f) \in \overline{M}_+ \times L_{2,\delta}$ *and* $(z, f) \in \overline{M}_- \times L_{2,\delta}$, *respectively.*

In the following two sections we shall give a priori estimates for the solution of the equation (1.15). They will be used in §4 to show the proof of Theorem 1.6. Some concluding remarks will be given also in §5.

2. A priori estimate for $(D - i\sqrt{z}\,\beta)u$. Let us start with the definition of several functions and differential expressions which will be used when we get an a priori estimate for $(D - i\sqrt{z}\,\beta)u$.

DEFINITION 2.1.

(i) Let $z = \lambda + i\psi \in \mathbf{C}$ with $\lambda > \lambda_0$ and let $\beta(x, z) = \nabla R(x, z)$ be as in Definition 1.4. Then the functions $h(x, z)$ and $\eta(x, z)$ are defined by

$$h(x, z) = \frac{1}{2\beta^2}(\mathrm{div}\beta) \qquad (|x| \ge r_0), \tag{2.1}$$

$$\eta(x, z) = \frac{\sqrt{z}\,\mu p(x)}{2\beta^2|z|^2} \qquad (|x| \ge r_0), \tag{2.2}$$

where r_0 is as in Assumption 1.1 and the square root $\sqrt{z}$ of z is taken in such a way as $\mathrm{Im}\sqrt{z} \ge 0$.

(ii) We set

$$\mathscr{D}_j u = D_j u + \left(h + \eta - i\sqrt{z}\right)\beta_j u \qquad (j = 1, 2, \dots, N), \tag{2.3}$$

$$\mathscr{D} u = (\mathscr{D}_1 u, \mathscr{D}_2 u, \dots, \mathscr{D}_N u), \tag{2.4}$$

$$\mathscr{D}_\beta u = \sum_{j=1}^{N} (\mathscr{D}_j u)\beta_j. \tag{2.5}$$

REMARK 2.2. If $p(x) = 0$ and $R(x, \lambda) = |x|$ (cf. (iii) of Remark 1.3), then we have

$$\mathscr{D}_j u = D_j u + \frac{N-1}{2|x|}\tilde{x}_j u - i\sqrt{z}\,\tilde{x}_j u \qquad \left(\tilde{x}_j = \frac{x_j}{|x|}\right), \tag{2.6}$$

which is the same as $\mathscr{D}_j u$ given in Ikebe-Saitō [**5**].

By direct computation we have the following lemma.

LEMMA 2.3. *Let $u \in H_{2,\mathrm{loc}}$. Then for $|x| > r_0$ and $z = \lambda + i\mu$ with $\lambda > \lambda_0$ we have*

$$p(x) - z + z\beta^2 + 2i\sqrt{z}\,\eta\beta^2 = 0, \tag{2.7}$$

$$\begin{aligned}(T - z)u = &- \sum_{j=1}^{N} D_j\mathscr{D}_j u + (h + \eta - ik)(\mathscr{D}_\beta u)\\ &+ \tilde{Q}(x)u - q(x)u\end{aligned} \tag{2.8}$$

with

$$\begin{cases} \tilde{Q}(x) = Q(x) + \{\operatorname{div}(h\beta) - h^2\beta^2\}, \\ q(x) = (\eta^2 + 2h\eta)\beta^2, \end{cases} \tag{2.9}$$

and

$$\begin{aligned} D_j\mathscr{D}_l u - D_l\mathscr{D}_j u = &(h + \eta - ik)(\beta_l\mathscr{D}_j u - \beta_j\mathscr{D}_l u)\\ &+ M_{jl}u + iB_{jl}u \qquad (j, l = 1, 2, \ldots, N)\end{aligned} \tag{2.10}$$

with

$$\begin{cases} M_{jl} = \partial_j\{(h + \eta)\beta_l\} - \partial_l\{(h + \eta)\beta_j\}, \\ B_{jl} = (\partial_j b_l) - (\partial_l b_j). \end{cases} \tag{2.11}$$

Let $R_1 = c_1 r_0$ with c_1, r_0 given in Assumption 1.2. Take $\rho \in C^\infty(\mathbf{R})$ such that $0 \le \rho \le 1$, $d\rho/dr \ge 0$ and

$$\rho(r) = \begin{cases} 1, & r \ge R_1 + 1, \\ 0, & r \le R_1. \end{cases} \tag{2.12}$$

We define $\phi(r)$ by

$$\phi(r) = \rho(r)(1 + r)^{2\delta - 1} \tag{2.13}$$

with a fixed constant δ satisfying (1.16).

PROPOSITION 2.4. *Let $u \in H_{2,\mathrm{loc}}$ and set $f = (T - z)u$ with $z = \lambda + i\mu \in \mathbf{C}$. Let ϕ be given in* (2.13) *and set*

$$\begin{cases} \sqrt{z} = \sigma + i\tau \qquad (\rho \ge 0), \\ B = B(\lambda, R_1, R_*) = \{x/R_1 < R(x, z) < R_*\}, \\ \phi(R) = \phi(R(x, z)), \end{cases} \tag{2.14}$$

where $R(x, z)$ is as in Definition 1.4 *and R_* $(> R_1 = c_1 r_0)$ is a constant. Then there exists a constant Λ_1 $(\geq \lambda_0)$ such that for $z = \lambda + i\mu$ with $\lambda > \Lambda_1$ we have*

$$\begin{aligned}(2.15)\qquad &\int_B \left\{ \phi(R)\left(1 + \frac{\sigma^2 p(x)}{\beta^2(\sigma^2+\tau^2)^2}\right)\tau - \frac{1}{2}\phi'(R)\right\}\beta^2 |\mathscr{D}u|^2\,dx \\ &\quad + \int_B \phi'(R)|\mathscr{D}_\beta u|^2\,dx \\ &\quad + \mathrm{Re}\int_B \phi(R)\sum_{j,l=1}^{N}(\partial_j\beta_l)(\mathscr{D}_j u)(\overline{\mathscr{D}_l u})\,dx \\ &\quad + \mathrm{Re}\int_B \phi(R)\sum_{j,l=1}^{N}(\overline{M}_{jl} - iB_{jl})\beta_l(\mathscr{D}_j u)\bar{u}\,dx \\ &\quad + \mathrm{Re}\int_B \phi(R)(\tilde{Q}+q)u(\overline{\mathscr{D}_\beta u})\,dx \\ &= \mathrm{Re}\int_B \phi(R) f(\overline{\mathscr{D}_\beta u})\,dx \\ &\quad + \int_{R(x,z)=R_*} \frac{\phi(R)}{|\beta|}\left(|\mathscr{D}_\beta u|^2 - \frac{\beta^2}{2}|\mathscr{D}u|^2\right) dS,\end{aligned}$$

where

$$(2.16)\qquad |\mathscr{D}u(x)|^2 = \sum_{j=1}^{N} |\mathscr{D}_j u(x)|^2,$$

and the second term in the right-hand side is the surface integral on the surface $\{x \in \mathbf{R}^N \mid R(x, z) = R_*\}$.

Proof. First we have to consider the surface $\Sigma_T = \Sigma_{T,Z} = \{x \in \mathbf{R}^N / R(x, z) = T\}$ $(T \geq R_1)$ and the surface integral on it. Expressing $R(x, z)$ as $R(x, z) = |x| g(x, z)$, we have

$$\begin{aligned}(2.17)\qquad \frac{\partial R}{\partial |x|} &= \sum_{j=1}^{N}(\partial_j R)\cdot \tilde{x}_j = \sum_{j=1}^{N}\left(\tilde{x}_j^2 g + \tilde{x}_j |x|(\partial_j g)\right) \\ &= g(x, z) + |x|\partial g/\partial|x|.\end{aligned}$$

Using (1.9) and (1.10) with $j = 1$, we can see that there exists $\Lambda_1 > \lambda_0$ such that for $z = \lambda + i\mu$ with $\lambda > \Lambda_1$

$$(2.18)\qquad \partial R(x, z)/\partial|x| \geq \tfrac{1}{2}c_0 > 0 \qquad (|x| \geq r_0).$$

Therefore $R(r\omega, z)$ is an increasing function of r for fixed $\omega \in S^{N-1}$ and $z = \lambda + i\mu$ with $\lambda > \Lambda_1$. Since $R(r_0\omega, z) \le R_1 \le T$, there exists a unique $r = \psi(T, \omega) = \psi_z(T, \omega)$ for each $(\omega, T) \in S^{N-1} \times [R_1, \infty)$ such that $R(\psi_z(T, \omega)\omega, z) = T$. Thus we have

$$(2.19)\quad \Sigma_T = \{x \in \mathbf{R}^N / R(x, z) = T\} = \{\mu_z(T, \omega)\omega/\omega \in S^{N-1}\}$$

and the surface Σ_T is shown to be smooth by the use of the implicit function theorem. Thus Σ_T is diffeomorphic to the unit sphere. Let us denote by a the angle between the outward normal $\nabla R/|\nabla R| = \beta/|\beta|$ of Σ_T and $\tilde{x} = x/|x|$, the outward normal of the unit sphere S^{N-1} in $\mathbf{R}^N$. Let dS and $d\omega$ be the surface elements on Σ_T and S^{N-1}, respectively. Then, since

$$(2.20)\qquad \frac{\psi(T, \omega)^{N-1} d\omega}{dS} = \cos a = \frac{\beta}{|\beta|} \cdot \tilde{x} = \frac{\partial R/\partial |x|}{|\beta|},$$

we have the formula for the integral surface on Σ_T:

$$(2.21)\quad \int_{\Sigma_T} f\, dS = \int_{S^{N-1}} \left[\frac{|\beta(x, z)|}{\partial R/\partial |x|}\right]_{x = \psi(T, \omega)\omega} f(\psi(T, u)\omega)\psi(T, \omega)^{N-1}\, d\omega.$$

Let us integrate the surface integral on (T_1, T_2) with respect to T. Then, chaging the variable T to $r = \psi(T, \omega)$, we get

$$(2.22)\quad \int_{T_1}^{T_2} h(T) \int_{\Sigma_T} f\, dS\, dT = \int_{S^{N-1}} \int_{R(T_1\omega, z)}^{R(T_2\omega, z)} h(R(r\omega, z)) f(r\omega) |\beta(r\omega, z)| r^{N-1}\, dr\, d\omega$$

where we should note that

$$(2.23)\qquad \begin{cases} R(r\omega, z) = T, \\ dT = \dfrac{\partial R}{\partial |x|}(r\omega, z)\, dr. \end{cases}$$

If $h(R(x, z))f(x)$ is a nonnegative function, then by making use of the inequalities $c_0 T_j \le R(T_j\omega, z) \le c_1 T_j$ $(j = 1, 2)$, the integral in (2.22) may be estimated as

$$(2.24)\quad \int_{C_1T_1\le|x|\le C_0T_2} h(R(x,z))f(x)|\beta|\,dx$$
$$\le \int_{T_1}^{T_2} h(T)\int_{\Sigma_T} f\,dS\,dT \le \int_{C_0T_1\le|x|\le C_1T_2} h(R(x,z))f(x)|\beta|\,dx.$$

Thus it follows that if $g(x)$ is integrable on $\{x\,|\,|x|>r\}$ with $r>0$, then

$$(2.25)\qquad \lim_{T\to\infty}\left\{T\int_{\Sigma_T}|g|\,dS\right\}=0.$$

Let us show (2.15). Multiply $f=(T-z)u$ by $\phi(R)\overline{\mathscr{D}_\beta u}$ and take the real part after integrating it on B. Then, using (2.8), we have

$$(2.26)\quad \operatorname{Re}\int_B \phi(R)f\left(\overline{\mathscr{D}_\beta u}\right)dx$$
$$= -\operatorname{Re}\sum_{j=1}^N\int_B \phi(R)\left(D_j\mathscr{D}_j u\right)\left(\overline{\mathscr{D}_\beta u}\right)dx$$
$$+\operatorname{Re}\int_B \phi(R)(h+\eta-ik)|\mathscr{D}_\beta u|^2\,dx$$
$$+\operatorname{Re}\int_B \phi(R)(\tilde{Q}+q)u\left(\overline{\mathscr{D}_\beta u}\right)dx \equiv I_1+I_2+I_3.$$

Since

$$(2.27)\qquad \operatorname{Re}\eta=\frac{\sigma^2\tau p(x)}{\beta^2(\sigma^2+\tau^2)^2},$$

it follows that

$$(2.28)\quad I_2=\int\phi(R)\left\{h+\left(\frac{\sigma^2 p(x)}{\beta^2(\sigma^2+\tau^2)^2}+1\right)\tau\right\}|\mathscr{D}_\beta u|^2\,dx.$$

By partial integration we have

$$(2.29)\quad I_1=-\operatorname{Re}\int_{R(x,\lambda)=R_*}\phi(R)\frac{1}{|\nabla R|}|\mathscr{D}_\beta u|^2\,dS+\int_B\phi'(R)|\mathscr{D}_\beta u|^2\,dx$$
$$+\operatorname{Re}\int_B\phi(R)\sum_{j=1}^N(\mathscr{D}_\beta u)\overline{D_j(\mathscr{D}_\beta u)}\,dx=I_{11}+I_{12}+I_{13},$$

where we should note that the outward normal of the surface $R(x,\lambda)=R_*$ is $\beta/|\beta|=\beta/|\nabla R|$ and that $\partial_j\phi(R)=\phi'(R)\beta_j$ with $\phi'=d\phi/dr$. By the use of (2.10) the term I_{13} is expressed as

$$
\begin{aligned}
(2.30)\quad I_{13} &= \mathrm{Re}\int_B \phi(R) \sum_{j,l=1}^{N} (\mathscr{D}_j u)(\partial_j \beta_l)(\overline{\mathscr{D}_l u})\, dx \\
&+ \int_B \phi(R)\left\{h + \left(\frac{\sigma^2 p(x)}{\beta^2(\sigma^2+\tau^2)^2} + 1\right)\tau\right\}\left(\beta^2|\mathscr{D}u|^2 - |\mathscr{D}_\beta u|^2\right) dx \\
&+ \mathrm{Re}\int_B \phi(R) \sum_{j,l=1}^{N} (\mathscr{D}_j u)(\overline{M}_{jl} - iB_{jl})\bar{u}\, dx \\
&+ \mathrm{Re}\int_B \phi(R) \sum_{j,l=1} \beta_l(\mathscr{D}_j u)\overline{D_l(\mathscr{D}_j u)}\, dx = \sum_{m=1}^{4} I_{13m}.
\end{aligned}
$$

Using partial integration again, we get

$$
\begin{aligned}
(2.31)\quad I_{134} &= \frac{1}{2}\int_{R(x,\lambda)=R_*} \phi(R)|\nabla R|\,|\mathscr{D}u|^2\, dS \\
&- \frac{1}{2}\int_B \phi'(R)\beta^2|\mathscr{D}u|^2\, dx - \frac{1}{2}\int_B \phi(R)(\mathrm{div}\beta)|\mathscr{D}u|^2\, dx.
\end{aligned}
$$

The relation (2.15) is obtained from (2.26), (2.28), (2.29), (2.30) and (2.31). □

The third term in the left-hand side of (2.15) may be simplified by using the next lemma.

LEMMA 2.5. *The functions* $\partial_j \beta_l$ $(1 \le j,\ l \le N)$ *are expressed as*

$$
(2.32)\qquad \partial_j \beta_l = \frac{\beta^2 \delta_{jl}}{R} - \frac{\beta_j \beta_l}{R} + \frac{1}{R}F_{jl}(x, z) \qquad \left(|x| \ge r_0\right),
$$

where $\beta_l = \beta_l(x, z)$, $\beta_j = \beta_j(x, z)$, $R = R(x, z)$ *with* $z = \lambda + i\mu$, $\lambda > \lambda_0$, *and* $F_{jl}(x, z)$ *is a bounded function of* x *for* $|x| \ge r_0$ *such that*

$$
(2.33)\qquad \lim_{\lambda\to\infty} \sup_{|x|\ge r_0} |F_{jl}(x, z)| = 0 \qquad (j, l = 1, 2, \ldots, N).
$$

Here r_0 *and* λ_0 *are given in Assumptions* 1.1 *and* 1.2, *and*

$$
(2.34)\qquad \delta_{jl} = \begin{cases} 1 & (j = l), \\ 0 & (j \ne l). \end{cases}
$$

Proof. Setting $R(x, z) = |x|g(x, z)$, we have

$$
(2.35)\qquad \beta_l = \partial_l R = \tilde{x}_l g + |x|(\partial_l g) \qquad \left(\tilde{x}_l = x_l/|x|\right)
$$

and

$$\partial_j \beta_l = \left(\frac{\delta_{jl}}{R} - \frac{\tilde{x}_l \tilde{x}_j}{R} \right) g^2 + \frac{1}{R} G_{jl}(x, z) \tag{2.36}$$

with

$$G_{jl}(x, z) = g\left\{ \tilde{x}_l |x| (\partial_j g) + \tilde{x}_j |x| (\partial_l g) + |x|^2 (\partial_j \partial_l g) \right\}. \tag{2.37}$$

On the other hand it follows from (2.35) that

$$\begin{cases} \tilde{x}_l g = \beta_l - |x| (\partial_l g), \\ \beta^2 = g^2 + 2|x| \tilde{x} \cdot \nabla g + |x|^2 |\nabla g|^2, \end{cases} \tag{2.38}$$

which, together with (2.36), gives (2.32) with

$$\begin{aligned} F_{jl} = G_{jl} - \Big\{ &\delta_{jl}\left(2|x| \tilde{x} \cdot (\nabla g) + |x|^2 |\nabla g|^2 \right) - |x| (\partial_l g) \beta_j \\ &- |x| (\partial_j g) \beta_l + |x|^2 (\partial_j g)(\partial_l g) \Big\}. \end{aligned} \tag{2.39}$$

The relation (2.33) follows from (1.10) in Assumption 1.2. □

By the use of (2.32) we get a corollary of Proposition 2.4.

COROLLARY 2.6. *Let u, $f = (T - z)u$, $\phi(R)$ and B be as in Proposition 2.4. Then we have*

$$\begin{aligned} &\int_B \left\{ \phi(R) \left(1 + \frac{\sigma^2 p(x)}{\beta^2 (\sigma^2 + \tau^2)^2} \right) \tau + \frac{1}{2} \phi'(R) \right\} \beta^2 |\mathscr{D} u|^2 \, dx \\ &\quad + \int_B \left(\frac{\phi(R)}{R} - \phi'(R) \right) \left\{ \beta^2 |\mathscr{D} u|^2 - |\mathscr{D}_\beta u|^2 \right\} dx \\ &\quad + \mathrm{Re} \int_B \frac{\phi(R)}{R} \sum_{j,l=1}^N F_{jl} (\mathscr{D}_j u) \left(\overline{\mathscr{D}_l u} \right) dx \\ &\quad + \mathrm{Re} \int_B \phi(R) \sum_{j,l=1}^N \left(\overline{M}_{jl} - i B_{jl} \right) \beta_l (\mathscr{D}_j u) \bar{u} \, dx \\ &\quad + \mathrm{Re} \int_B \phi(R) (\tilde{Q} + q) u \left(\overline{\mathscr{D}_\beta u} \right) dx \\ &= \mathrm{Re} \int_B \phi(R) f \left(\overline{\mathscr{D}_\beta u} \right) dx \\ &\quad + \int_{\Sigma_{R_*}} \frac{\phi(R)}{|\beta|} \left\{ |\mathscr{D}_\beta u|^2 - \frac{1}{2} \beta^2 |\mathscr{D} u|^2 \right\} dS, \end{aligned} \tag{2.40}$$

where $\Sigma_{R_*} = \{ x \in \mathbf{R}^N / R(x, z) = R_* \}$.

In order to estimate $\mathscr{D}u$ from Corollary 2.6, we need several lemmas. Let $V(x)$ be a real-valued, measurable function such that

$$M_V(x) = \int_{|x-y|<1} \frac{|V(x)|^2}{|x-y|^{N-4+\nu}}\, dy \tag{2.41}$$

is locally bounded on $\mathbf{R}^N$ with $\nu > 0$ and

$$|V(x)| \le C_1 \qquad (|x| \ge r_1) \tag{2.42}$$

with constants C_1, $r_1 > 0$. We set

$$T_1 = -\Delta + V(x). \tag{2.43}$$

LEMMA 2.7. *Let $V(x)$ and T_1 be as above and let $u \in L_{2,\gamma} \cap H_{2,\mathrm{loc}}$ with some $\gamma \in \mathbf{R}$.*

(i) *Let u be a solution of the equation $(T_1 - z)u = f$ with $z \in \mathbf{C}$ and $f \in L_{2,\gamma}$. Then we have $Du \in L_{2,\gamma}$, i.e., $D_j u \in L_{2,\gamma}$ for each $j = 1, 2, \ldots, N$.*

(ii) *Let u be a solution of the equation $(T_1 - z) = f$ with $z \in \mathbf{C}$, $\operatorname{Im} z \ne 0$ and $f \in L_{2,\gamma+1/2}$. Then $u \in L_{2,\gamma+1/2}$.*

The proof is essentially the same as the proof of Lemma 2.4 in Ikebe-Saitō [7], so we omit it.

Let

$$K = \{z = \lambda + iu/\lambda_1 \le \lambda \le \lambda_2, 0 < |\mu| < \mu_0\} \tag{2.44}$$

with $\lambda_0 < \lambda_1 < \lambda_2$ and $\mu_0 > 0$, λ_0 being in Assumption 1.2.

LEMMA 2.8. *Let $T = -\Delta + p(x) + Q(x)$ and let K be as above. Let $\alpha \in \mathbf{R}$ and let $u \in L_{2,\alpha} \cap H_{2,\mathrm{loc}}$ be a solution of the equation $(T - z)u = f$ with $z \in K$ and $f \in L_{2,\alpha}$. Then for any $\theta \le \alpha$ and $r \ge r_0$ there exist constants $C_j = C_j(K, r, \theta)$ $(j = 1, 2)$, depending only on K, r, θ, such that the estimate*

$$|\mu|\, \|u\|_\theta \le |2\theta|(1+r)^{\theta-\alpha} \|\mathscr{D}u\|_{\alpha-1,E_{r_0}} + C_1 \|u\|_{\theta-1} + c_2 \|f\|_\theta, \tag{2.45}$$

r_0 being as in Assumption 1.1 *and $E_{r_0} = \{x/|x| \ge r_0\}$.*

Proof. Let $\rho_r(x) = \rho(s - r)$, where $\rho \in C^\infty(\mathbf{R})$, $0 \le \rho \le 1$ and

$$\rho(s) = \begin{cases} 1 & (s \ge 1), \\ 0 & (s \le 0). \end{cases} \tag{2.46}$$

Set $\phi(x) = \phi(|x|) = (1 + |x|)^{2\theta}\rho_R(|x|)$. Multiplying both sides of $(T - z)u = f$ by $\phi(x)u$, integrating on $\mathbf{R}^N$ and using partial integration, we get

$$(2.47)\qquad \int_{\mathbf{R}^N} \phi|Du|^2\,dx + \int_{\mathbf{R}^N} \frac{\partial\phi}{\partial r}(D_r u)\bar{u}\,dx + \int_{\mathbf{R}^N} \phi(p + Q - z)|u|^2\,dx = \int_{\mathbf{R}^N} \phi f\bar{u}\,dx,$$

where $D_r u = \sum_{j=1}^N D_j u \cdot \tilde{x}_j$ and it should be noted that $Du \in L_{2,\alpha}$ by (i) of Lemma 2.7 so that the surface integral at infinity will vanish. Since $\mathscr{D}_j u = D_j u + (h + \eta - i\sqrt{z})\beta_j u$ by (2.3), we get by taking the imaginary part of both sides of (2.47)

$$(2.48)\qquad \mu\int_{\mathbf{R}^N} \phi|u|^2\,dx = \operatorname{Im}\int_{\mathbf{R}^N} \frac{\partial\phi}{\partial|x|}\left\{\mathscr{D}u \cdot \tilde{x}\bar{u} - (h + \eta)\beta\tilde{x}|u|^2 - i\sqrt{z}\,\beta\tilde{x}|u|^2\right\} dx - \operatorname{Im}\int_{\mathbf{R}^N} \phi f\bar{u}\,dx,$$

whence it follows that

$$(2.49)\qquad |\mu|\int_{\mathbf{R}^N} \phi|u|^2\,dx \le \int_{\mathbf{R}^N} \left|\frac{\partial\phi}{\partial|x|}\right| |\mathscr{D}u|\,|u|\,dx + C_3\int_{\mathbf{R}^N} \left|\frac{\partial\phi}{\partial|x|}\right| |u|^2\,dx + \int_{\mathbf{R}^N} \phi|f|\,|u|\,dx \equiv J_1 + J_2 + J_3 \qquad (z \in K)$$

with a constant $C_3 = C_3(K)$ depending only on K. The first term J_1 is estimated as

$$(2.50)\qquad J_1 = \int_{\mathbf{R}^N} \left|2\theta\phi(1 + |x|)^{-1} + \rho_r'(|x|)(1 + |x|)^{2\theta}\right| |\mathscr{D}u|\,|u|\,dx \le |2\theta|\int_{|x|\ge r} (1 + |x|)^{2\theta-1}|\mathscr{D}u|\,|u|\,dx + \left(\sup_S |\rho'(s)|\right)\int_{r\le|x|\le r+1} (1 + r)^{2\theta}|\mathscr{D}u|\,|u|\,dx = J_{11} + J_{12}.$$

Here we have

$$(2.51)\qquad \begin{aligned} J_{11} &\le |2\theta| \left[\int_{|x|\ge r} (1+|x|)^{2\theta-2} |\mathcal{D}u|^2\, dx \right]^2 \|u\|_\theta \\ &\le |2\theta|(1+r)^{\theta-\alpha} \|\mathcal{D}u\|_{\alpha-1,E_{r_0}} \|u\|_\theta, \end{aligned}$$

where we should note that $r \ge r_0$. By the use of the interior estimate (see, e.g., [4], Lemma 2.1) and the Schwarz inequality, it is easy to see that there exists a constant $C_4 = C_4(K, r, \theta)$ such that

$$(2.52)\qquad J_{12} \le C_4\{\|u\|_{\theta-1} + \|f\|_\theta\}\|u\|_\theta.$$

In a similar manner J_2 and J_3 are estimated as

$$(2.53)\qquad J_2 + J_3 \le C_5\{\|u\|_{\theta-1} + \|f\|_\theta\}\|u\|_\theta$$

with $c_5 = c_5(K, r, \theta)$. Thus, we get from (2.49)–(2.53) that

$$(2.54)\qquad \begin{aligned} &|\mu| \int_{\mathbf{R}^N} \phi |u|^2\, dx \\ &\quad \le \Big\{ |2\theta|(1+r)^{\theta-\alpha} \|\mathcal{D}u\|_{\alpha-1,E_{r_0}} \\ &\qquad\qquad + C_4(\|u\|_{\theta-1} + \|f\|_\theta) + C_5(\|u\|_{\theta-1} + \|f\|_\theta) \Big\}. \end{aligned}$$

It follows from (2.54) that

$$(2.55)\qquad \begin{aligned} |\mu|\, \|u\|_\theta^2 &\le \Big\{ |2\theta|(1+r)^{\theta-\alpha} \|\mathcal{D}u\|_{\alpha-1,E_{r_0}} \\ &\qquad + (C_4 + C_5)\|u\|_{\theta-1} + (C_4 + C_5)\|f\|_\theta \Big\} \\ &\qquad + |\mu| \int_{|x|\le r+1} (1-\rho_r)(1+r)^{2\theta} |u|^2\, dx, \\ &= \Big\{ |2\theta|(1-r)^{\theta-\alpha} \|\mathcal{D}u\|_{\alpha-1,E_{r_0}} \\ &\qquad + (C_4 + C_5 + C_6)\|u\|_{\theta-1} + (C_4 + C_5)\|f\|_\theta \Big\} \|u\|_\theta \end{aligned}$$

with the constant $C_6 = C_6(K, r, \theta)$. The inequality (2.45) directly follows from (2.55) by dividing both sides by $\|u\|_\theta$. □

Lemma 2.8 will be used in the following forms.

COROLLARY 2.9. *Let K be as above and let $u \in L_{2,-\delta} \cap H_{2,\mathrm{loc}}$ be a solution of the equation $(T - z)u = f$ with $z = \lambda + i\mu \in K$ and $f \in L_{2,\delta}$. Let $r \ge r_0$. Then there exist constants $C_7 = C_7(K, r)$ and $C_8 = C_8(K)$*

such that

$$(2.56)\quad |\mu|\,\|u\|_{\delta-1} \le 2(1-\delta)r^{-1}\|\mathscr{D}u\|_{\delta-1,E_{r_0}} + C_7\{\|u\|_{-\delta} + \|f\|_{\delta}\}$$

and

$$(2.57)\quad |\mu|\,\|u\|_{\delta} \le 2\delta\|\mathscr{D}u\|_{\delta-1,E_{r_0}} + C_8\{\|u\|_{\delta-1} + \|f\|_{\delta}\}$$

are valid.

Proof. In order to show (2.56) we have only to set $\theta = \delta - 1$, $\alpha = \delta$ in (2.45) and notice that $\|u\|_{\delta-2} \le \|u\|_{-\delta}$ and $\|f\|_{\delta-1} \le \|f\|_{\delta}$. As for (2.57) we set $r = r_0$, $\theta = \alpha = \delta$ in (2.45). □

Now we are in a position to obtain a priori estimate for $(D - i\sqrt{z}\,\beta)u$.

PROPOSITION 2.10. *Suppose that Assumptions* 1.1 *and* 1.2 *hold. Let* δ *be a fixed constant which satisfies* (1.16). *Then there exists* λ_2 $(\ge \lambda_0)$ *such that for M of the form*

$$(2.58)\quad M = \{z = \lambda + i\mu/\lambda_1 \le \lambda \le \lambda_2, 0 < |\mu| < \mu_0\}$$

with $\Lambda_2 < \lambda_1 < \lambda_2$ *and* $\mu_0 > 0$ *there exists a positive conjstant* $C = C(M)$ *such that*

$$(2.59)\quad \|(D - i\sqrt{z}\,\beta)u\|_{\delta-1} \le C\{\|u\|_{-\delta} + \|f\|_{\delta}\}$$

is valid for a solution $u \in L_{2,-\delta} \cap H_{2,\mathrm{loc}}$ *of the equation* $(T - z)u = f$ *with* $z = \lambda + i\mu \in M$ *and* $f \in L_{2,\delta}$.

Proof. It follows from Lemma 2.7 with $V(x) = p(x) + Q(x)$ that u, $Du \in L_{2,\delta}$. Let us estimate each term of (2.40). In the following J_{Lj} (J_{Rj}) means the jth term of the left (right)-hand side of (2.40).

(1) Take a constant Λ_1' $(\ge \Lambda_1)$ so that

$$(2.60)\quad 1 + \frac{\sigma^2 p(x)}{\beta^2(\sigma^2+\tau^2)^2} \ge 0 \qquad (x \in \mathbf{R}^N)$$

for all $\lambda \ge \lambda_1'$; here Λ_1 is given in Proposition 2.4. Since

$$(2.61)\quad \begin{aligned}\phi'(R) &= \rho'(R)(1+R)^{2\delta-1} + (2\delta-1)\rho(R)(1+R)^{2\delta-2}\\ &\ge (2\delta-1)\rho(R)(1+R)^{2\delta-1},\end{aligned}$$

we have

$$(2.62)\quad J_{L1} \ge \left(\delta - \frac{1}{2}\right)\int_B \rho(R)(1+R)^{2\delta-2}\beta^2|\mathscr{D}u|^2\,dx.$$

(2) We have $\beta^2|\mathscr{D}u|^2 \geq |\mathscr{D}_\beta u|^2$ and

$$(2.63)\quad \begin{aligned} \frac{\phi(R)}{R} - \phi'(R) &\geq \frac{\phi(R)}{1+R} - \phi'(R) \\ &= 2(1-\delta)\rho(R)(1+R)^{2\delta-2} - \rho'(R)(1+R)^{2\delta-1} \\ &\geq -\rho'(R)(1+R)^{2\delta-1}. \end{aligned}$$

Therefore we obtain

$$(2.64)\quad \begin{aligned} J_{L2} &\geq -\int_B \rho'(R)(1+R)^{2\delta-1}\left(\beta^2|\mathscr{D}u|^2 - |\mathscr{D}_\beta u|^2\right) dx \\ &\geq -\int_B \rho'(R)(1+R)^{2\delta-1}\beta^2|\mathscr{D}u|^2\, dx. \end{aligned}$$

(3) It follows from Lemma 2.5 that

$$(2.65)\quad \begin{aligned} &\left|\sum_{j,l=1}^N F_{jl}(\mathscr{D}_j u)(\overline{\mathscr{D}_l u})\right| \\ &\leq N\left(\sup_{|x|\geq r_0,\, 1\leq j,l\leq N} |F_{jl}(x)|\right)|\mathscr{D}u|^2 = o(1)|\mathscr{D}u|^2 \end{aligned}$$

as $\lambda \to \infty$. Therefore we have

$$(2.66)\quad |J_{L3}| \leq o(1)\int_B \rho(R)(1+R)^{2\delta-2}|\mathscr{D}u|^2\, dx \qquad (\lambda \to \infty).$$

(4) It follows from Assumptions 1.1 and 1.2 (and the definitions of M_{jl}, B_{jl}, $\tilde{Q}$ and q (Lemma 2.3)) that we have the estimate

$$(2.67)\quad \begin{aligned} |J_{L4} + J_{L5}| \leq c_0\bigg\{&\int_{E_{r_0}} (1+|x|)^{2\delta-2-\varepsilon}|\beta|\,|\mathscr{D}u|\,|u|\, dx \\ &+|\mu|\int_{E_{r_0}} (1+|x|)^{2\delta-2}|\beta|\,|\mathscr{D}u|\,|u|\, dx \\ &+|\mu|^2\int_{E_{r_0}} (1+|x|)^{2\delta-1}|\beta|\,|\mathscr{D}u|\,|u|\, dx\bigg\} \\ \equiv c_0\{&I_1 + I_2 + I_3\} \end{aligned}$$

where $c_0 = c_0(M)$ is a constant depending only on M and we should note that $E_{r_0} = \{x \in \mathbf{R}^N/|x| \geq r_0\} \supset B = \{x \in \mathbf{R}^N/R_1 \leq R(x,z) \leq R_*\}$, and $R(x,z)$ can be replaced by $|x|$ by the assumption (1.9). Since $2\delta - 2 - \varepsilon \leq \delta - 1 + (-\delta)$ by (1.16), the first term I_1 is evaluated as

$$(2.68)\quad I_1 \leq \kappa\|\,|\beta|\mathscr{D}u\|^2_{\delta-1,E_{r_0}} + \frac{1}{4\kappa}\|u\|^2_{-\delta}$$

for $\kappa > 0$. Using (2.53) in Corollary 2.9, we have

$$(2.69)\qquad \begin{aligned} I_2 &\le \| \, |\beta| \mathscr{D}u \|_{\delta-1,E_{r0}} \big(|\mu| \, \|u\|_{\delta-1} \big) \\ &\le \| \, |\beta| \mathscr{D}u \|_{\delta-1,E_{r0}} \Big\{ 2(1-\delta)^{-1} r^{-1} \| \mathscr{D}u \|_{\delta-1,E_{r0}} \\ &\qquad\qquad + C_7 \big(\|u\|_{-\delta} + \|f\|_{\delta} \big) \Big\} \\ &\le c_2 \big\{ 2(1-\delta)^{-1} r^{-1} + \kappa \big\} \| \, |\beta| \mathscr{D}u \|^2_{\delta-1,E_{r0}} \\ &\qquad + C_7' \big\{ \|u\|_{-\delta} + \|f\|^2_{\delta} \big\} \end{aligned}$$

for $r \ge r_0$, $\kappa > 0$ with the constants $c_2 = 1/\inf_{x,\lambda} |\beta(x, \lambda)|$ and $C_7' = C_7'(M, r, \kappa)$. Using (2.53) and (2.54) in Corollary 2.9 and proceeding as in the estimate for I_2, we obtain

$$(2.70)\qquad \begin{aligned} I_3 \le {} & \big\{ (2c_2\delta + C_8)|\mu| + 2C_8(1-\delta) r^{-1} + C_8 \kappa \big\} \| \, |\beta| \mathscr{D}u \|^2_{\delta-1,E_{r0}} \\ & + C_8' \big\{ \|u\|^2_{-\delta} + \|f\|^2_{\delta} \big\} \end{aligned}$$

for $r \ge r_0$, $\kappa > 0$ with c_2 given in (2.69), $C_8 = C_8(M)$ as in (2.57) and $C_8' = C_8'(M, r, \kappa)$.

(5) Let $R_* \to \infty$ in (2.40) along a suitable sequence so that the surface integral J_{R2} goes to zero (cf. (2.25)). Then we have from (1)–(4)

$$(2.71)\qquad \begin{aligned} & \big\{ c_3(\delta - \tfrac{1}{2}) - c_4 \big(|\mu| + r^{-1} + \kappa \big) - \xi(\lambda) \big\} \| \, |\beta| \mathscr{D}u \|_{\delta-1,E_{r0+1}} \\ & \quad - C_9 \big(\|u\|^2_{-\delta} + \|f\|^2_{\delta} \big) \le \|f\|_{\delta} \| \, |\beta| \mathscr{D}u \|_{\delta-1,E_{r0}}, \end{aligned}$$

where c_3 is a constant which comes from the replacement of $R(x, \lambda)$ by $|x|$, $c_4 = c_4(M)$, $\xi(\lambda) \to 0$ as $\lambda \to \infty$, $C_9 = C_9(M, r, \kappa)$, and we have used the interior estimate to estimate the integrals containing $|\mathscr{D}u|^2$ on a bounded region. Take Λ_2 $(\ge \Lambda_1')$ so large that $\xi(\lambda) \le \frac{1}{3} c_3(\delta - \frac{1}{2})$ for $\lambda \ge \lambda_0$. Then take r^{-1}, κ, $\mu_1 > 0$ so small that $c_4(|\mu| + r^{-1} + \kappa) \le \frac{1}{3} c_3(\delta - \frac{1}{2})$ for μ such that $|\mu| \le \mu_1$. Note again that $|\beta|$ is bounded below from a positive constant. Thus we get

$$(2.72)\qquad c_5 \| \mathscr{D}u \|^2_{\delta-1,E_{r0+1}} \le C_{10} \big(\|u\|^2_{-\delta} + \|f\|^2_{\delta} \big)$$

for $z = \lambda + i\mu \in M$ with $0 < |\mu| \le \mu_1$, where $C_{10} = C_{10}(M, r, \kappa, \mu_1)$ and $c_5 = c_5(M)$.

(6) Since $|(D - i\sqrt{z}\,\beta)u| \le |\mathscr{D}u| + (|h| + |\eta|)|u|$, we have only to estimate $\|hu\|_{\delta-1}$ and $\||\eta|u\|_{\delta-1}$ by using Corollary 2.9 to get (2.59) from (2.72) for $z = \lambda + i\mu$ with $0 < |\mu| < \mu_1$.

(7) The case that $z = \lambda + i\mu \in M$ with $\mu_1 \leq |\mu| \leq \mu_0$ is easy. Take the imaginary part of the relation $((T - z)u, u)_0 = (f, u)_0$. Then we have

$$\|u\|_0 \leq \frac{1}{|\mu|}\|f\|_0 \leq \frac{1}{\mu_1}\|f\|_0. \tag{2.73}$$

Take the real part and use (2.73) and the interior estimate. Then we have

$$\|Du\|_0 \leq C_{10}\|f\|_0 \tag{2.74}$$

with a constant C_{10}, whence, together with (2.73), (2.59) follows. □

3. A priori estimate for $\|u\|_{-\delta,E_r}$. In this section we shall show that for a solution u of the equation (1.15) the norm $\|u\|_{-\delta,E_r}$ is decreasing with some positive exponent when $r \to \infty$.

PROPOSITION 3.1. *Assume Assumptions* 1.1 *and* 1.2. *Let δ be a fixed constant which satisfies* (1.16). *Let Λ_2 be as in Proposition* 2.10 *and let M be defined by* (2.58) *with $\Lambda_2 < \lambda_1 < \lambda_2$ and $\mu_0 > 0$. Then there exists a positive constant $C = C(M)$ such that*

$$\|u\|_{-\delta,E_r} \leq Cr^{-(\delta-1/2)}\{\|u\|_{-\delta} + \|f\|_\delta\} \qquad (r < c_1 r_0) \tag{3.1}$$

is valid for a solution $u \in L_{2,-\delta} \cap H_{2,\mathrm{loc}}$ of the equation $(T - z)u = f$ with $z = \lambda + i\mu \in M$ and $f \in L_{2,\delta}$. Here c_1 and r_0 are as in Assumption 1.2 *and $E_r = \{x \in \mathbf{R}^N/|x| \geq r\}$.*

Proof. Set $\sqrt{z} = \sigma + i\tau$ $(\tau > 0)$ and let B_T be the inside of the closed surface $\{x \mid R(x, z) = T\}$ $(T > r_0,\ \lambda \geq \Lambda_2)$. Integrating $(T - z)u \cdot \bar{u} = f\bar{u}$ on B_T and taking the imaginary part, we have

$$-\mathrm{Im}\int_{\Sigma_T} \frac{1}{|\beta|}(Du \cdot \beta)\bar{u}\, dS - 2\sigma\tau\int_{B_T} |u|^2\, dx = \mathrm{Im}\int_{B_T} f\bar{u}\, dx, \tag{3.2}$$

where $\Sigma_T = \{x \in \mathbf{R}^N/R(x, z) = T\}$ and we have used the fact that $\mu = 2\sigma\tau$. Thus it follows that

$$2\sigma\,\mathrm{Im}\int_{R(x,z)=T} \frac{1}{|\beta|}(Du \cdot \beta)\bar{u}\, ds \leq -2\sigma\,\mathrm{Im}\int_{B_T} f\bar{u}\, dx. \tag{3.3}$$

On the other hand, since

$$\begin{aligned} |D_j u - i\sqrt{z}\beta_j u|^2 &= |D_j u + \tau\beta_j u - i\sigma\beta_j u|^2 \\ &= |D_j u + \tau\beta_j u|^2 + \sigma^2\beta_j^2|u|^2 - 2\sigma\,\mathrm{Im}\{(D_j u)\beta_j\bar{u}\}, \end{aligned} \tag{3.4}$$

we get

$$\sigma^2|\beta|\,|u|^2 \leq \frac{1}{|\beta|}|(D - i\sqrt{z}\beta)u|^2 + 2\sigma\,\mathrm{Im}\left\{\frac{1}{|\beta|}(Du)\cdot\beta\bar{u}\right\}. \tag{3.5}$$

Integrate (3.5) on the surface Σ_T and use (3.3). Then we have

$$(3.6)\quad \sigma^2\int_{\Sigma_T}|\beta||u|^2\,dS \le \int_{\Sigma_T}\frac{1}{|\beta|}\left|(D-i\sqrt{z}\,\beta)u\right|^2 dS - 2\sigma\,\mathrm{Im}\int_{B_T} f\bar{u}\,dx$$
$$\le \int_{\Sigma_T}\frac{1}{|\beta|}\left|(D-i\sqrt{z}\,\beta)u\right|^2 dS + 2|\sigma|\|f\|_\delta\|u\|_{-\delta}.$$

Multiply both sides of (3.6) by $(1+T)^{-2\delta}$ and integrate on $(r/c_1,\infty)$ with respect to T. It follows from (2.24) and the estimate

$$(3.7)\quad d_1(1+|x|)^{-2\delta} \le (1+R(x,z))^{-2\delta} \le d_2(1+|x|)^{-2\delta}$$
$$\left(d_1=\{\max(1,c_1)\}^{-2\delta},\ d_2=\{\min(1,c_0)\}^{-2\delta}\right)$$

with c_0 and c_1 in (1.9) that

$$(3.8)\quad \int_{r/c_1}^{\infty}(1+T)^{-2\delta}\int_{\Sigma_T}|\beta||u|^2\,dS\,dT$$
$$\ge \sigma^2 d_1\int_{|x|>r}(1+|x|)^{-2\delta}|\beta|^2|u|^2\,dx$$
$$\ge \sigma^2 d_1 d_3\|u\|_{-\delta,E_r}\left(d_3=\inf_{|x|\ge c_1 r_0}|\beta(x,z)|^2\right),$$

$$(3.9)\quad \int_{r/c_1}^{\infty}(1+T)^{-2\delta}\int_{\Sigma_T}\frac{1}{|\beta|}\left|(D-i\sqrt{z}\,\beta)u\right|^2 ds\,dT$$
$$+2|\sigma|\,\|f\|_\delta\|u\|_{-\delta}\int_{r/c_1}^{\infty}(1+T)^{-2\delta}\,dT$$
$$\le d_2\int_{|x|\ge c_1 r/c_0}(1+|x|)^{-2\delta}\left|(D-i\sqrt{z}\,\beta)u\right|^2 dx$$
$$+\frac{1}{2\delta-1}\left(1+\frac{r}{c_1}\right)^{-(2\delta-1)}2|\sigma|\,\|f\|_\delta\|u\|_{-\delta}$$
$$\le d_2\left(1+\frac{c_1 r}{c_0}\right)^{-(4\delta-2)}\left\|(D-i\sqrt{z}\,\beta)u\right\|^2_{\delta-1,E_t}$$
$$+\frac{|\sigma|}{2\delta-1}\left(1+\frac{r}{c_1}\right)^{-(2\delta-1)}\left\{\|f\|_\delta^2+\|u\|_{-\delta}^2\right\}\qquad\left(t=\frac{c_1 r}{c_0}\right).$$

The estimate (3.1) is obtained from (3,8), (3.9) and (2.59) in Proposition 2.10. □

4. Proof of the main theorem. The a priori estimates obtained in the preceding sections will be used to show Theorem 1.6. The proof of Theorem 1.6 will be divided into several steps.

(I) Let us consider the equation (1.15) with $z = \lambda + i\mu$, $\mu \neq 0$. Let us first show the uniqueness of the solution. Let $u \in L_{2,-\delta} \cap H_{2,\mathrm{loc}}$ be a solution of the equation $(T - z)u = 0$ with $z = \lambda + i\mu$, $\mu \neq 0$. Then it follows from Lemma 2.7 that we have u, $Du \in L_2$. Therefore, multiplying $(T - z)u = 0$ by $\bar{u}$ and integrating by parts and taking the imaginary part, we get $\mu\|u\|_0 = 0$, i.e., $u = 0$.

It is known that, under Assumption 1.1, the differential operator T restricted on $C_0(\mathbf{R}^N)$ is essentially self-adjoint (Ikebe-Kato [7]). Let us denote its unique self-adjoint extension by H. Let $f \in L_2$ and then $u(z, f) = (H - z)^{-1}f$ $(z = \lambda + i\mu,\ \mu \neq 0)$ belongs to $H_{2,\mathrm{loc}} \cap L_2$ (see Ikebe-Kato [6]). Thus we have proved the unique existence of the solution of the equation (1.15) for $f \in L_2$ and $z = \lambda + i\mu$, $\mu \neq 0$.

(II) Let us next assume that $u \in L_{2,-\delta} \cap H_{2,\mathrm{loc}}$ is a solution of the equation $(T - \lambda)u = 0$ with $\lambda > \lambda_0$ and that u satisfies

$$\|(D - i\sigma\beta)u\|_{\delta-1,E_{r_0}} < \infty$$

with $\sigma = \sqrt{\lambda}$ or $-\sqrt{\lambda}$. Here λ_0 is as in Assumption 1.2. Then it follows from (3.2) with $\tau = 0$ and $f = 0$ that we have

$$\mathrm{Im}\int_{\Sigma_T} \frac{1}{|\beta|}(du \cdot \beta)\bar{u}\,dS = 0 \qquad (T > r_0) \tag{4.1}$$

(4.1) and (3.4) with $\tau = 0$, $\sqrt{z} = \sqrt{\lambda}$ and $\sigma^2 = \lambda$ give

$$\int_{\Sigma_T} \left\{|Du|^2 + (\lambda - p(x))|u|^2\right\} dS = \int_{\Sigma_T} |(D - i\sigma\beta)u|^2\,dS, \tag{4.2}$$

where we have used the relation $\beta^2 = 1 - p(x)/\lambda$. Multiply both sides of (4.2) by $(1 + T)^{2(\delta-1)}$ and integrate it on (T_0, ∞) with T_0 large enough. Then, using (2.24), (3.7) and the condition $\|(D - i\sigma\beta)u\|_{\delta-1,E_{r_0}} < \infty$, we get

$$\int_{|x|>T} (1 + |x|)^{2(\delta-1)}\left\{|Du|^2 + (\lambda - p(x))|u|^2\right\} dx < \infty \tag{4.3}$$

with T large enough, whence directly follows that

$$\lim_{r\to\infty} r^{2\delta-1}\int_{S_r} \left\{|Du|^2 + (\lambda - p(x))|u|^2\right\} dx = 0, \tag{4.4}$$

where S_r is the sphere in $\mathbf{R}^N$ with radius r. Now we can apply Theorem 1.1 in Mochizuki [**15**] to conclude that the support of u is compact in $\mathbf{R}^N$ if

$$\lambda > \overline{\lim_{|x|\to\infty}} \left\{ p(x) + \frac{1}{2(2\delta-1)}|x|\frac{\partial p}{\partial |x|} \right\} \equiv \Lambda_3 \tag{4.5}$$

(and $\lambda > \Lambda_0$). Thus, by using the unique continuation property (UC) in Assumption 1.1, we have $u = 0$.

(III) Set $\Lambda_0 = \max(\Lambda_2, \Lambda_3)$. Now that we have established the uniqueness of the solution of the equation (1.15)–(1.18) and the estimates in Propositions 2.10 and 3.1, the rest of the proof of Theorem 1.6 can be done in the same way as in many works on the limiting absorption principle (see, e.g., Ikebe-Saitō [**8**], Saitō [**20**, **21**, **22**], etc.) First the estimate (1.20) will be shown. In fact if we assume that there exists a sequence $\{u_n\}_{n=1} \subset L_{2,-\delta} \cap H_{2,\text{loc}}$ such that $\|u_n\|_{-\delta} = 1$ and $\|u_n\|_{-\delta} > n\|f_n\|_\delta$ $(n = 1,2,\ldots)$ with $f_n = (T - z_n)u_n$, then we can obtain a contradiction. Thus (1.20) will be established. Propositions 2.10 and 3.1, together with (1.20) will give (1.21) and (1.22). Using (1.20)–(1.22), we can show that $u(z,f)$ $(z = \lambda + i\mu,\ \mu \neq 0)$ has its limit on the real axis $\lambda > \Lambda_0$. We can also easily prove the continuity of $u(z,f)$ on the upper or lower half plane.

5. Concluding remarks.

1°. Let us define the operator $(T-z)^{-1}$ by

$$(T-z)^{-1}f = u(z,f) \tag{5.1}$$

where $z = \lambda + i\mu$ with $\lambda > \Lambda_0$, $\mu \neq 0$ and $f \in L_{2,\delta}$. Then it follows from Theorem 1.6 that $(T-z)^{-1} \in \mathbf{B}(L_{2,\delta}, L_{2,-\delta})$, where $\mathbf{B}(X,Y)$ denotes all bounded linear transforms from X into Y. Also $(T-z)^{-1}$ is a compact operator from $L_{2,\delta}$ to $L_{2,-\delta}$. $(T-z)^{-1}$ is of course the restriction of $(H-z)^{-1}$ into $L_{2,\delta}$ where $(H-z)^{-1}$ is the resolvent of the self-adjoint operator H defined in §4. Further, if we define $(T - (\lambda \pm i0))^{-1}$ by

$$(T - (\lambda \pm i0))^{-1}f = \mu_{\pm}(\lambda, f) \tag{5.2}$$

for $\lambda > \Lambda_0$, then it follows from Theorem 1.6 that

$$\lim_{\mu\downarrow 0}(T - (\lambda \pm i\mu))^{-1} = (t - (\lambda \pm i0))^{-1} \tag{5.3}$$

in $\mathbf{B}(L_{2,\delta}, L_{2,-\delta})$ (cf. Mourre [**18**], Jensen-Mourre-Perry [**12**]). $(T - (\lambda \pm i0))^{-1}$ are also compact operators from $L_{2,\delta}$ into $L_{2,-\delta}$.

2°. Under even stronger conditions on the potential $Q(x)$ and $\beta_j(x)$ we can show a stronger estimate for $(D - i\sqrt{z}\,\beta)u)$ (cf. Saitō **[21]**, **[22]**, Isozaki **[9]**).

PROPOSITION 5.1. *Let us assume Assumption* 1.1 *with* (1.4) *and* (1.5) *replaced by*

$$(5.4) \qquad |Q(x)| \le c(1 + |x|)^{-2} \qquad (|x| \ge r_0)$$

and

$$(5.5) \qquad |B_{jl}(x)| \le c(1 + |x|)^{-2}.$$

respectively. Let us assume Assumption 1.2. *Let* $\overline{M}_{\pm}$ *and* $u(z, f)$ *be as in* (iv) *of Theorem* 1.6. *Then there exists a positive constant* C *which depends only on* $\overline{M}_{\pm}$ *such that*

$$(5.6) \qquad \|(D - i\sqrt{z}\,\beta)u\|_{1-\delta} \le C\|f\|_{2-\delta}$$

is valid for $u = u(z, f)$ *with* $(z, f) \in \overline{M}_+$ *or* $(z, f) \in \overline{M}_-$.

For the proof of this proposition we have only to show the a priori estimates

$$(5.7) \qquad \|(D - i\sqrt{z}\,\tau b)u\|_{1-\delta} \le \tilde{C}\{\|u\|_{-\delta} + \|f\|_{2-\delta}\}$$

with $\tilde{C} = \tilde{C}(M)$, because (5.7) is combined with (1.20) to give (5.6). (5.7) will be shown starting with (2.40) in Corollary 2.6, with $\phi(R) = \rho(R)(1 + R)^{3-2\delta}$ where we should notice that the first tow terms of the left-hand side of (2.40) are expressed as

(5.8) the first two terms of the left-hand side of (2.40)

$$= \int_B \left\{\phi(R)\left(\frac{\sigma^2 p(x)}{\beta^2(\sigma^2 + \tau^2)} + 1\right)\tau + \frac{\phi(R)}{R} - \frac{1}{2}\phi'(R)\right\}\beta^2|\mathscr{D}u|^2\,dx$$

$$+ \int_B \left(\phi'(R) - \frac{\phi(R)}{R}\right)|\mathscr{D}_\beta u|^2\,dx.$$

We shall have to use Lemma 2.8 as in the proof of Proposition 2.10. Thus the proof will be quite similar to that of Proposition 2.10.

Added in proof (*March* 1, 1986). Recently G. Barles **[2]** has shown that Assumption 1.2 follows from Assumption 1.1, (p), namely, that the eikonal equation

$$|\nabla R|^2 = 1 + p(x)/\lambda$$

has a solution $R(x, \lambda)$ for all sufficiently large λ, and $R(x, \lambda)$ satisfies all requirements given in Assumption 1.2 if $p(x)$ satisfies Assumption 1.1, (p). Along the line of Lions [**14**], Barles defined $R(x, \lambda)$ by

$$R(x, \lambda)^2 = \inf\left\{\int_0^1 \left(1 - \frac{p(\xi(s))}{\lambda}\right)\left|\frac{d\xi(s)}{ds}\right|^2 ds / \xi(0) = 0,\ \xi(1) = x\right\}$$

and proved that $R(x, \lambda)$ is a solution of the eikonal equation and also that $R(x, \lambda)$ has the smoothness required in Assumption 1.2. Thus only Assumption 1.1 is needed to guarantee that all the results given in this work hold.

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