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A CHARACTERIZATION OF *KK*-THEORY

NIGEL HIGSON

We characterize the *KK*-groups of G. G. Kasparov, along with the Kasparov product $KK(A,B) \times KK(B,C) \to KK(A,C)$, from the point of view of category theory (in a very elementary sense): the product is regarded as a law of composition in a category and we show that this category is the universal one with "homotopy invariance", "stability" and "split exactness". The third property is a weakened type of half-exactness: it amounts to the fact that the *KK*-groups transform split exact sequences of C^*-algebras to split exact sequences of abelian groups. The method is borrowed from Joachim Cuntz's approach to *KK*-theory, in which cycles for $KK(A,B)$ are regarded as generalized homomorphisms from A to B: the results follow from an analysis of the Kasparov product in this light.

Introduction. This paper is a study of the groups $KK(A,B)$, where A and B are separable C^*-algebras, introduced by G. G. Kasparov in [**15**]. These groups have received widespread attention since their introduction, due mainly to the possibilities they afford for the application of C^*-algebra techniques to problems in geometry and topology, but also because of their utility within the field of C^*-algebras. The groups arise from, and generalize, the topological K-theory of spaces—thus if X is a compact metric space then the groups $KK(\mathbf{C}, C(X))$ and $KK(C(X), \mathbf{C})$ are respectively the topological K-theory and K-homology of X—and as such their introduction has led to, for example, simplifications and a conceptualization of the proof of the Atiyah Singer Index Theorem (see [**4**], [**13**], [**9**]). More importantly, by using non-commutative C^*-algebras as arguments for the *KK*-groups the index theorem can be generalized in a number of interesting directions (for example, to foliations [**9**]). As another example of an application of these groups in topology, the group $KK(C^*(G), \mathbf{C})$ serves as an appproximation to the group $K(BG)$ (where G is say the fundamental group of a manifold) and a study of it as such has led to progress in the generalized Novikov conjecture (see e.g. [**18**] for a discussion of this). As a tool in the study of C^*-algebras, they are of importance as a relatively computable invariant, as well as in the study of extensions of C^*-algebras developed in [**6**] and [**15**].

The definition of the *KK*-groups originates in the close relationship between K-theory and the index theory of elliptic operators. An elliptic differential (or pseudodifferential) operator on a smooth closed manifold

M determines a homomorphism

$$\gamma_D\colon K(X \times M) \to K(X) \quad (X \text{ a compact space}),$$

as follows: given a vector bundle E over $X \times M$, by "twisting" D by E we obtain a family of elliptic operators over M, parameterized by X: the index of the family is $\gamma_D([E]) \in K(X)$ (see [2], [3]). It follows from duality theory in algebraic topology that D defines an element of the K-homology group $K_0(M)$ (in such a way that γ_D is obtained via the K-theory slant product). Furthermore, one can show that every element of $K_0(M)$ can be obtained in this fashion, and leading from this, Atiyah suggested the possibility of a *definition* of K-homology in terms of elliptic operators. He proposed the following notion of a "cycle" for $K_0(X)$ (from which $K_0(X)$ would be obtained by means of a suitable equivalence relation), consisting of a pair of Hilbert spaces, $H_\pm$, a representation $\phi_\pm$ of $C(X)$ on each of these spaces, and a Fredholm operator F from H_+ to H_- which essentially intertwines the two representations $\phi_\pm$ of $C(X)$, in the sense that the operators

$$F\phi_+(f) - \phi_-(f)F \quad \text{and} \quad \phi_-(f)F - \phi_+(f)F, \quad (\text{where } f \in C(X))$$

are compact. In the case of the elliptic operator D, $H_\pm$ are Sobolev spaces of functions, or sections of vector bundles over M; the algebra $C(M)$ acts on $H_\pm$ by pointwise multiplication of functions; F is D of course (the Fredholm character of F following from the ellipticity of D); and the intertwining condition expresses the following "pseudolocal" property of a pseudodifferential operator: if f_1 and f_2 in $C(M)$ have disjoint support then $f_1 D f_2$ is a compact operator (compare [13], Proposition 3.4]; in fact $f_1 D f_2$ is a smoothing operator). The data $(H_\pm, \phi_\pm, F)$ comprising a cycle is sufficient to define an index homomorphism γ as in the elliptic operator case. If E is a bundle over $X \times M$ which is of the form $F \times M$—trivial in the M direction—then $\gamma([E])$ is the index of the family

$$I_x \otimes F \in B(F_x \otimes H_+, F_x \otimes H_-) \qquad (x \in X),$$

where F_x is the fibre of F over $x \in X$. Roughly speaking, in the case of a more general E, the bundle is at least trivial in the M direction locally in M, and the family whose index is $\gamma([E])$ can be constructed locally as above, and then the pieces glued together by means of a partition of unity for M—a procedure which works because of the pseudolocal nature of F, and which makes plausible this definition of cycle.

A cycle for the group $KK(A, B)$ consists of a pair of representations of A on "Hilbert B-modules" $H_{B\pm}$, together with an operator F from H_{B+} to H_{B-} which is Fredholm, in the appropriate sense, and which intertwines the two representations of A, modulo compact operators, appropriately defined (a precise definition is given in §2). This is the natural extension of notion of cycle discussed above, and of course, in the case $B = \mathbf{C}$ and $A = C(X)$ we get the K-homology cycles of Atiyah. Consider also the case $A = \mathbf{C}$, and $B = C(X)$: a cycle amounts to a "Fredholm" operator between $C(X)$-modules, and by "localizing" at each point $x \in X$, this is the same thing as a family of Fredholm operators between Hilbert spaces. We can obtain a more complicated example by combining these two cases: a family of elliptic operators on a smooth closed manifold M parameterized by a compact space X determines a cycles for $KK(C(M), C(X))$ (see [**9**]).

Central to Kasparov's work, and this paper, is the construction (in [**15**], §4) of a fundamental product mapping

$$KK(A_1, C_1 \otimes B) \otimes KK(A_2 \otimes B, C_2) \to KK(A_1 \otimes A_2, C_1 \otimes C_2).$$

This product contains, and generalizes, a number of constructions from K-theory and index theory. For example, we sketched above how an elliptic operator on a manifold M gives rise to an index homomorphism

$$K(X \times M) \to K(X),$$

After identifying K-theory with $KK(\mathbf{C}, -)$, and associating with the elliptic operator a cycle for $KK(C(M), \mathbf{C})$, this map is given by the Kasparov product. More generally, the same is true for the index homomorphism

$$K(X \times M) \to K(X \times Y)$$

determined by a family of elliptic operators over M parameterized by Y. Indeed, the product can be regarded as a sort of systematic calculus for these index maps (generalized beyond the case of spaces to the context of arbitrary C^*-algebras), and this is its importance.

The idea behind the definition of the product lies again, not surprisingly, in the index theory of elliptic operators. We do not wish to go to great lengths discussing this but let us quickly illustrate the idea with the following special case:

$$KK(C(M_1), \mathbf{C}) \otimes KK(C(M_2), \mathbf{C}) \to KK(C(M_1 \times M_2), \mathbf{C}).$$

Recalling the connection between elliptic operators and cycles, the problem roughly amounts to finding a suitable "product" operator on $M_1 \times M_2$, given elliptic operators D_1 and D_2 on M_1 and M_2. The solution is to construct from D_1 and D_2 an operator on $M_1 \times M_2$ whose principal symbol is the product, in the sense of K-theory, of the symbols of D_1 and

D_2. (For more information see [**13**], especially Remark 4.3, and [**9**].) With the help of some rather technical C^*-algebra results, this construction may be abstracted, first to the case of abstract K-homology cycles, and then to general KK-cycles (see [**15**], or the appendix to this paper).

We have presented KK-theory so far from the point of view of elliptic operators and index theory, in line with its origins and most applications. However, our approach in this paper has much more in common with the account of KK-theory given by Joachim Cuntz, in which elements of $KK(A, B)$ are regarded not as generalized elliptic operators but as (homotopy classes of) generalized homomorphisms from A to B: a $*$-homomorphism from A to B determines in a very simple way an element of $KK(A, B)$ (see §2.8), and building from this, Cuntz has given a description of KK-theory in terms of so-called "quasihomomorphisms" (see [**10**] and [**11**]). Our starting point is the observation that the product, specialized slightly to the pairing

$$KK(A, B) \otimes KK(B, C) \to KK(A, C),$$

defines a law of composition in a category **K** whose objects are separable C^*-algebras, and for which the set of morphisms from A to B is the abelian group $KK(A, B)$. It is natural to ask for some sort of description of this category, and our goal is to characterize **K** by isolating three simple properties it possesses, and showing that it is the "universal" category with them. The first two are *homotopy invariance* and *matrix stability* both of which are natural and basic properties in the K-theory of C^*-algebras. The third is *split exactness*. This means that if

$$0 \to J \to D \to D/J \to 0$$

is a short exact sequence of separable C^*-algebras and $*$-homomorphisms, and if $D \to D/J$ has a right inverse $D/J \to D$, then the corresponding sequences of KK-groups are split exact. The main technical device that we use is borrowed from Cuntz's work on quasihomomorphisms. This is the construction from a $KK(A, B)$-cycle Φ of a split short exact sequence

$$0 \to \mathscr{K} \otimes B \to A_\Phi \to A \to 0,$$

($\mathscr{K}$ denotes the compact operators), together with two specific section $*$-homomorphisms $A \to A_\Phi$, and from this (using the three properties of the KK-groups listed above), a homomorphism from $KK(D, A)$ to $KK(D, B)$. Our results follow from an analysis of this map, which turns out to be the same as taking the Kasparov product of the cycle Φ with elements of $KK(D, A)$.

The plan of the paper is as follows. After dealing with some preliminaries on multiplier algebras in §1, we give in §2 a definition of $KK(A, B)$ suitable for our purposes, and discuss the functoriality properties of the KK-groups (although we will be using many features of Cuntz's treatment of KK, our definition will be essentially that of Kasparov). We state the existence of the Kasparov product, together with its functoriality and a normalization condition, and use this to obtain the properties of the KK-groups listed above. In §3 we study natural transformations from KK to arbitrary functors with these properties, and from this, in §4, we obtain the above mentioned characterization, and related results. For example, the associativity of the product is a rather difficult point in Kasparov's work, but here we get it as a simple consequence of the results of section three. Finally in an appendix we briefly review Kasparov's construction of the product, and prove those properties of it that we have used.

Part of the material presented here formed part of my M.Sc. thesis at Dalhousie University, 1983. In addition to expressing my indebtedness to Joachim Cuntz, I would like to thank Bob Paré for several discussions on category theory, and my supervisor, Peter Fillmore, for his guidance and encouragement.

Multiplier Algebras

1.1. Let D be a C^*-algebra and let $\mathscr{M}(D)$ denote its multiplier algebra (see [**1**], [7] or [**17**]). Recall that $\mathscr{M}(D)$ contains D as a (closed, two-sided) ideal, and is characterized by the property that if E is any C^*-algebra containing D as an ideal then the identity map on D extends uniquely to a $*$-homomorphism from E into $\mathscr{M}(D)$. The kernel of this $*$-homomorphism is the *annihilator ideal* of D in E:

$$\mathrm{Ann}(D) = \{x \in E : x \cdot D = D \cdot x = \{0\}\}.$$

Thus if D is an *essential ideal* in E, that is, $\mathrm{Ann}(D) = \{0\}$, then the canonical map $E \to \mathscr{M}(D)$ embeds E as a subalgebra of $\mathscr{M}(D)$. Let us note two instances of this. First, $D_1 \otimes D_2$ is an essential ideal in $\mathscr{M}(D_1) \otimes \mathscr{M}(D_2)$ (see [**1**]; we use the minimal tensor product), and so $\mathscr{M}(D_1) \otimes \mathscr{M}(D_2) \subset \mathscr{M}(D_1 \otimes D_2)$. Secondly, if D is an essential ideal in D' then in fact it is an essential ideal in $\mathscr{M}(D')$; hence $\mathscr{M}(D') \subset \mathscr{M}(D)$. We note that $\mathscr{M}(D')$ consists of those elements $x \in \mathscr{M}(D)$ such that $x \cdot D' \subset D'$ and $D' \cdot x \subset D'$.

1.2. EXAMPLES. (1) If $\mathcal{K}$ denotes the algebra of compact operators on a separable Hilbert space H then $\mathcal{M}(K) \cong \mathcal{B}(H)$. In general it will be useful to think of elements of $\mathcal{M}(D)$ and D as being respectively "bounded" and "compact" operators. (This analogy is made precise in **[14]**.)

(2) If M_n denotes $n \times n$ matrices then clearly $\mathcal{M}(M_n(D)) \cong M_n(\mathcal{M}(D))$. Also, if $p \in \mathcal{M}(D)$ is a projection then $\mathcal{M}(pDp) \cong p\mathcal{M}(D)p$.

1.3. *Functoriality of* $\mathcal{M}(D)$. If $f\colon D_1 \to D_2$ is a $*$-homomorphism, and if $f(D_1)$ contains an approximate unit for D_2 then f extends uniquely to a $*$-homomorphism from $\mathcal{M}(D_1)$ (see **[1]**, **[17]**). If there is a projection $p \in \mathcal{M}(D_2)$ such that $f(D_1) \subset pD_2p$ and $f(D_1)$ contains an approximate unit for pD_2p, then we obtain a canonical $*$-homomorphism from $\mathcal{M}(D_1)$ to $\mathcal{M}(D_2)$ by means of the composition

$$\mathcal{M}(D_1) \to \mathcal{M}(pD_2p) \cong p\mathcal{M}(D_2)p \subset \mathcal{M}(D_2).$$

This construction applies if $D_i = \mathcal{K} \otimes B_i$ $(i = 1, 2)$, where B_1 is unital, and $f = 1 \otimes g$ for some $g\colon B_1 \to B_2$. For then $1 \otimes g(1) \in \mathcal{M}(K \otimes B_2)$ plays the role of the projection p above. In this way $\mathcal{M}(\mathcal{K} \otimes B)$ is functorial for unital C^*-algebras B. To deal with non-unital algebras we need the following result. Recall that two $*$-homomorphisms f_0, f_1: $D \to D'$ are said to be homotopic if they are obtained from a $*$-homomorphism $f\colon D \to D' \otimes C[0,1]$ by evaluating at 0 and 1.

1.4. LEMMA (*cf.* **[5**, *Lemma* 2.4**]**). *Suppose that B has a countable approximate identity* (*for example, suppose that B is separable*). *There exists an isometry $v_1 \in \mathcal{M}(\mathcal{K} \otimes B)$ such that*: (i) *if B is an essential ideal in B' then $(\mathcal{K} \otimes B') \cdot v_1 \subset \mathcal{K} \otimes B$, and hence $v_1\mathcal{M}(\mathcal{K} \otimes B)v_1^* \subset \mathcal{M}(\mathcal{K} \otimes B')$; and* (ii) *the map* $\operatorname{Ad}(v_1)$: $\mathcal{M}(\mathcal{K} \otimes B') \to \mathcal{M}(\mathcal{K} \otimes B')$ *is homotopic through $*$-homomorphisms to* $\operatorname{Ad}(w_0 \otimes 1)$: $\mathcal{M}(\mathcal{K} \otimes B') \to \mathcal{M}(\mathcal{K} \otimes B')$, *where w_0 is some isometry in $\mathcal{M}(\mathcal{K})$.*

Proof. Let $s_1, s_2, \ldots$ be a sequence of elements in $\mathcal{K} \otimes B$ such that $\sum_{n=1}^{\infty} s_n^* s_n = 1$ (convergence in the strict topology [7]) and let $w_0, w_1, \ldots$ be a sequence of isometries in $\mathcal{M}(\mathcal{K})$ with disjoint range projections (i.e. $w_i^* w_j = 0$ if $i \neq j$). Let $u_k = \sum_{n=1}^{k}(w_n \otimes 1)s_n$ $(k = 1, 2, \ldots)$ and note that $u_k^* u_k = \sum_{n=1}^{k} s_n^* s_n$ so that $\|u_k\| \leq 1$ and $u_k^* u_k \to 1$ in the strict topology. If $x \in \mathcal{K} \otimes B$ then

$$\|u_{k+j}x - u_k x\|^2 = \left\|x^*\left(u_{k+j}^* - u_k^*\right)\left(u_{k+j} - u_k\right)x\right\| = \left\|x^* \sum_{n=k+1}^{k+j} s_n^* s_n x\right\|$$

and so the sequence $\{u_k x\}_{k=1}^{\infty}$ converges in norm. If $y \in \mathscr{K} \otimes B'$ then the sequence $\{yu_k\}_{k=1}^{\infty}$ converges in norm (and so the limit is in $K \otimes B$ since the yu_k are). To see this, note that if $y(w_n \otimes 1) = 0$ for all but finitely many n then the result is obvious, and then note that such y are dense in $\mathscr{K} \otimes B'$, from which the result follows since $\{u_k\}_{k=1}^{\infty}$ is a bounded sequence. The strict topology is complete (see [7]), and therefore $\{u_k\}_{k=1}^{\infty}$ converges to some $v_1 \in \mathscr{M}(\mathscr{K} \otimes B)$; since multiplication is strictly continuous on bounded sets, $v_1^* v_1 = 1$. Also, $(\mathscr{K} \otimes B') \cdot v_1 \subset \mathscr{K} \otimes B$ and $v_0^* v_1 = 0$, where $v_0 = w_0 \otimes 1$. The path of isometries $v_t = (1-t)^{1/2} v_0 + t^{1/2} v_1$ gives the desired homotopy. □

1.5. Remark. There seems to be a slight problem of Kasparov's treatment of the analogous point in [**15**] (see §1.19 of that paper): using the notation of [**15**], it is not clear what the "restriction homomorphism $\mathscr{L}(H_D \oplus H_D) \to \mathscr{L}(H_B \oplus H_D)$" is. In terms of algebras this would be a map

$$\mathscr{M}(M_2(\mathscr{K} \otimes D)) \to \mathscr{M}\begin{pmatrix} \mathscr{K} \otimes B & \mathscr{K} \otimes B \\ \mathscr{K} \otimes B & \mathscr{K} \otimes D \end{pmatrix}$$

(here B is an ideal in D, and

$$\begin{pmatrix} \mathscr{K} \otimes B & \mathscr{K} \otimes B \\ \mathscr{K} \otimes B & \mathscr{K} \otimes D \end{pmatrix}$$

is the obvious subalgebra of $M_2(\mathscr{K} \otimes D)$— it is *not* an ideal).

The Kasparov Groups

2.1. Definition. (i) Let A and B be separable C^*-algebras. A *$KK(A,B)$-cycle* is a triple (ϕ_+, ϕ_-, U), where $\phi_\pm\colon A \to \mathscr{M}(\mathscr{K} \otimes B)$ are $*$-homomorphisms, and U is an element of $\mathscr{M}(\mathscr{K} \otimes B)$ such that $U\phi_+(a) - \phi_-(a)U$, $\phi_+(a)(U^*U - 1)$ and $\phi_-(a)(UU^* - 1)$ are elements of $\mathscr{K} \otimes B$ for each $a \in A$.

(ii) Two $KK(A,B)$-cycles $(\phi_+^i, \phi_-^i, U^i)$ $(i = 0,1)$ are *homotopic* if there is a $KK(A, B \otimes C[0,1])$-cycle (ϕ_+, ϕ_-, U) (a *homotopy*) such that $(\varepsilon_i^0 \phi_+, \varepsilon_i^0 \phi_-, \varepsilon_i(U)) = (\phi_+^i, \phi_-^i, U^i)$, where $\varepsilon_i\colon \mathscr{M}(\mathscr{K} \otimes B \otimes C[0,1]) \to \mathscr{M}(\mathscr{K} \otimes B)$ is evaluation at i.

(iii) A $KK(A,B)$-cycle (ψ_+, ψ_-, V) is *degenerate* if the elements $V\psi_+(a) - \psi_-(a)V$, $\psi_+(a)(V^*V - 1)$, and $\psi_-(a)(VV^* - 1)$ are all zero.

This definition of cycle was motivated in the introduction. Note that we require U to be not only "Fredholm", but in fact "essentially unitary" in the sense that $\phi_+(a)(U^*U - 1)$ and $\phi_-(a)(UU^* - 1)$ are elements of

$\mathcal{K} \otimes B$ for all $a \in A$. This is not really necessary but is included here so as to conform to Kasparov's definition in [15]. (For a comparison with Kasparov's defintion, which differs from ours in some minor ways, see the appendix.)

2.2. DEFINITION. (i) The sum $(\phi_+, \phi_-, U) \oplus (\psi_+, \psi_-, V)$ of two $KK(A, B)$-cycles is the $KK(A, B)$-cycle

$$\left(\begin{pmatrix} \phi_+ & 0 \\ 0 & \psi_+ \end{pmatrix}, \begin{pmatrix} \phi_- & 0 \\ 0 & \psi_- \end{pmatrix}, \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right)$$

(where the algebra $M_2(\mathcal{M}(\mathcal{K} \otimes B))$ is identified with $\mathcal{M}(\mathcal{K} \otimes B)$ by means of some isomorphism $M_2(\mathcal{K}) \cong \mathcal{K}$—unique up to homotopy).

(ii) Two cycles $(\phi^i_+, \phi^i_-, U^i)$ $(i = 0, 1)$ are said to be *equivalent* if there exist degenerate cycles $(\psi^i_+, \psi^i_-, V^i)$ $(i = 0, 1)$ such that

$$(\phi^0_+, \phi^0_-, U^0) \oplus (\psi^0_+, \psi^0_-, V^0) \quad \text{and} \quad (\phi^1_+, \phi^1_-, U^1) \oplus (\psi^1_+, \psi^1_-, V^1)$$

are homotopic. The set of equivalence classes is denoted by $KK(A, B)$; we will write $\{(\phi_+, \phi_-, U)\}$ for the class of (ϕ_+, ϕ_-, U) in $KK(A, B)$.

It is clear that the above is indeed an equivalence relation, in fact the least one which incorporates homotopy and the equivalence of a cycle with the sum of itself and a degenerate cycle. The following lemma shows that the relation includes unitary equivalence.

2.3. LEMMA (*cf.* [15], §4, *Theorem* 1). *The set $KK(A, B)$ is an abelian group, via addition of cycles. If $v_+, v_- \in \mathcal{M}(\mathcal{K} \otimes B)$ are isometries then (ϕ_+, ϕ_-, U) is equivalent to the cycle* $(\mathrm{Ad}(v_+)\phi_+, \mathrm{Ad}(v_-)\phi_-, v_- U v_+^*)$.

Proof. For the first statement we will just verify that inverses exist, simultaneously with proving the second statement. The sum

$$\left(\begin{pmatrix} \phi_- & 0 \\ 0 & \mathrm{Ad}(v_+)\phi_+ \end{pmatrix}, \begin{pmatrix} \phi_+ & 0 \\ 0 & \mathrm{Ad}(v_-)\phi_- \end{pmatrix}, \begin{pmatrix} U^* & 0 \\ 0 & v_- U v_+^* \end{pmatrix} \right)$$

is homotopic to a degenerate cycle via the operator homotopy

$$W_t = \begin{pmatrix} \cos(t) U^* & -\sin(t) v_+^* \\ \sin(t) v_- & \cos(t) v_- U v_+^* \end{pmatrix}, \qquad t \in \left[0, \frac{\pi}{2}\right]$$

Consequently $\{(\phi_-, \phi_+, U^*)\}$ is an inverse for both $\{(\phi_-, \phi_+, U)\}$ (set $v_+ = v_- = 1$) and that class conjugated with any v_+ and v_-. □

2.4. *Functoriality*. From a $*$-homomorphism $f\colon A' \to A$ we obtain a mapping $f^*\colon KK(A, B) \to KK(A', B)$ by the formula $f^*\{(\phi_-, \phi_+, U)\} = \{(\phi_+ \circ f, \phi_- \circ f, U)\}$. This makes $KK(-, B)$ into a contravariant functor from (separable) C^*-algebras to abelian groups.

If g: $B \to B^*$ is a $*$-homomorphism between separable C^*-algebras and if B is *unital* then we obtain a homomorphism g_*: $KK(A, B) \to KK(A, B')$ by the formula $g_*\{(\phi_-, \phi_+, U)\} = \{(g_\sharp \circ \phi_+, g_\sharp \circ \phi_-, g_\sharp(U))\}$, where $g_\sharp$: $\mathscr{M}(\mathscr{K} \otimes B) \to \mathscr{M}(\mathscr{K} \otimes B')$ is the canonical extension of g (see 1.3). This makes $KK(A, B)$ covariantly functorial for unital B. Non-unital algebras are dealt with by means of the following device.

2.5. DEFINITION. Denote by $\tilde{B}$ the C^*-algebra obtained by adding a unit to B (if B is already unital we set $\tilde{B} = B$). A *special* $KK(A, B)$-cycle is a cycle (ϕ_+, ϕ_-, U) for which $\phi_\pm(A)$ and U are contained in the subalgebra $\mathscr{M}(\mathscr{K} \otimes \tilde{B})$ of $\mathscr{M}(\mathscr{K} \otimes B)$.

2.6. LEMMA. *Any $KK(A, B)$-cycle is equivalent to a special one, and if two special cycles are equivalent then there exist degenerate special cycles and a homotopy in $\mathscr{M}(\mathscr{K} \otimes \tilde{B} \otimes C[0,1])$ which give the equivalence.*

Proof. Conjugation with the isometry v_1 of Lemma 1.4 sends any cycle to a special cycle to which, by Lemma 2.3 (with $v_+ = v_- = v_1$) it is equivalent. If $(\phi^i_+, \phi^i_-, U^i)$ $(i = 0, 1)$ are equivalent special cycles then by conjugating the homotopy in the equivalence with

$$v_1 \otimes 1 \in \mathscr{M}(\mathscr{K} \otimes B \otimes C[0,1])$$

we obtain a "special" equivalence as desired, but between the cycles $(\mathrm{Ad}(v_1)\phi^i_+, \mathrm{Ad}(v_1)\phi^i_-, v_1 U^i v_1^*)$. However, by Lemma 1.4, these cycles are homotopic within $\mathscr{M}(\mathscr{K} \otimes \tilde{B} \otimes C[0,1])$ to the $(\phi^i_+, \phi^i_-, U^i)$ conjugated with some $w_0 \otimes 1$, and then by connecting $w_0 \in \mathscr{M}(\mathscr{K})$ to 1 by a strictly continuous path of isometries we obtain the desired equivalence. □

We can now define g_*: $KK(A, B) \to KK(A, B')$, where B is non-unital by $g_*\{(\phi_-, \phi_+, U)\} = \{(\tilde{g}_\sharp \circ \phi_+, \tilde{g}_\sharp \circ \phi_-, \tilde{g}_\sharp(U))\}$, where (ϕ_-, ϕ_+, U) is a special cycle, and $\tilde{g}_\sharp$: $\mathscr{M}(\mathscr{K} \otimes \tilde{B}) \to \mathscr{M}(\mathscr{K} \otimes \tilde{B}')$ is the map induced by $\tilde{g}$: $\tilde{B} \to \tilde{B}'$. By the lemma this is well defined, and $KK(A, -)$ becomes functorial for non-unital B. It remains to be seen that the unital and non-unital cases are compatible, so consider a composition of $*$-homomorphisms $B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} B_3$. If B_1 is unital and B_2 is nonunital then since the composition of $\tilde{g}_2$ with g_1 is equal to $g_2 \circ g_1$ it follows that $g_{2*} \circ g_{1*} = (g_2 \circ g_1)_*$. In the reverse case, where B_1 is nonunital and B_2 is unital, $g_{2*} \circ g_{1*}$ and $(g_2 \circ g_1)_*$ are obtained from the two homomorphisms

$$g_2 g_1 \colon \tilde{B}_1 \to \tilde{B}_3 \quad \text{and} \quad (g_2 g_1)_p \colon \tilde{B}_1 \to p B_3 p,$$

where $p \in B_3$ is the projection $g_2(1)$. So functoriality follows from the following proposition.

2.7. PROPOSITION. *If $p \in \mathcal{M}(B')$ is a projection and g: $B \to B'$ maps B into $pB'p$ then the maps $KK(A, B) \to KK(A, B')$ induced by the $*$-homomorphisms*

$$\tilde{g}_{\sharp}\colon \mathcal{M}(\mathcal{K} \otimes \tilde{B}) \to \mathcal{M}(\mathcal{K} \otimes \tilde{B})$$

and

$$\tilde{g}_{p\sharp}\colon \mathcal{M}(\mathcal{K} \otimes \tilde{B}) \to \mathcal{M}(\mathcal{K} \otimes pB'p) \subset \mathcal{M}(\mathcal{K} \otimes B')$$

are equal.

Proof. The projection $q = 1 \otimes p \in \mathcal{M}(\mathcal{K} \otimes B')$ reduces $\tilde{g}_{\sharp}$ in the sense that it commutes with $\tilde{g}_{\sharp}(b)$ for every $b \in B$. Thus we may write: $\tilde{g}_{\sharp} = \tilde{g}_{\sharp q} + \tilde{g}_{\sharp(1-q)}$. Note that $\tilde{g}_{\sharp q} = \tilde{g}_{p\sharp}$, so conjugating with the unitary

$$\begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \in M_2(\mathcal{M}(\mathcal{K} \otimes B'))$$

shows that for any special cycle (ϕ_+, ϕ_-, U), the sum of $\tilde{g}_{\sharp}(\phi_+, \phi_-, U)$ and a zero cycle is equivalent to the sum of $\tilde{g}_{p\sharp}(\phi_+, \phi_-, U)$ and $\tilde{g}_{\sharp(1-q)}(\phi_+, \phi_-, U)$. But this last cycle is degenerate since $\tilde{g}_{\sharp(1-q)}$ maps $\mathcal{K} \otimes B$ to zero, and so $\tilde{g}_{\sharp}(\phi_+, \phi_-, U)$ is equivalent to $\tilde{g}_{p\sharp}(\phi_+, \phi_-, U)$. □

2.8. *Cycles obtained from $*$-homomorphisms*. We now come to an examination of the simplest KK-cycles, those obtained from $*$-homomorphisms. First of all, we introduce some notation which will be used frequently from now on. Define a $*$-homomorphism e: $B \to \mathcal{K} \otimes B$ by $e(b) = e \otimes b$, where e is a rank one projection in $\mathcal{K}$. (The particular choice of the projection e has no real relevance since all such e's are homotopic, in fact unitarily equivalent, and the same can be said about the associated homomorphisms.) Now, if $\phi_{\pm}$: $A \to B$ are $*$-homomorphisms then define $\langle \phi_+, \phi_- \rangle$ to be the $KK(A, B)$-cycle $(e \circ \phi_+, e \circ \phi_-, 1)$. Let us note that the choice $F = 1$ in this cycle is unimportant because a choice of any other F would give an equivalent cycle (since the $*$-homomorphisms $e \circ \phi_{\pm}$ map into $\mathcal{K} \otimes B$ a straight line between F and 1 would be an operator homotopy). If 1: $A \to A$ denotes the identity map then define $1_A \in KK(A, A)$ to be the class of the cycle $\langle 1, 0 \rangle$. We will need the following simple relation:

$$\{\langle \phi_+, \phi_- \rangle\} = \phi_{+*}(1_A) - \phi_{-*}(1_A).$$

To see this, note first that $\{\langle\phi,0\rangle\} = \phi_*(1_A) = \phi^*(1_B)$; also, $\{\langle 0,\phi\rangle\} = -\{\langle\phi,0\rangle\}$. The result follows since $\langle\phi_+,0\rangle \oplus \langle 0,\phi_-\rangle$ is homotopic to $\langle\phi_+,\phi_-\rangle \oplus \langle 0,0\rangle$ by rotating

$$\begin{pmatrix} 0 & 0 \\ 0 & \phi_- \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \phi_- & 0 \\ 0 & 0 \end{pmatrix}.$$

In a sense, our goal is to obtain a similar expression for an arbitrary cycle (compare Theorem 3.5).

Next we discuss the Kasparov product. The following theorem summarizes those of its properties that we need. These amount to the existence of a functorial pairing, which satisfies a normalization condition, so as to rule out the zero product: the particular structure of the product is of no interest to us yet. We will use the product principally to obtain the split exactness of the *KK*-groups; later on we will reverse the procedure and recover the product from split exactness. For a proof of Theorem 2.9, see **[15]**, **[19]**, or the appendix.

2.9. THEOREM. *There exists a bilinear pairing from* $KK(A,B) \times KK(B,C)$ *to* $KK(A,C)$, *denoted* $(x,y) \mapsto x \otimes_B y$, *with the following properties*:

(1) *If* $f\colon A' \to A$ *then* $f^*(x \otimes_B y) = f^*(x) \otimes_B y$;

(2) *If* $g\colon B \to B'$ *then* $g_*(x) \otimes_{B'} z = x \otimes_B g^*(z)$, *where* $z \in KK(B',C)$;

(3) *If* $h\colon C \to C'$ *then* $h_*(x \otimes_B y) = x \otimes_B h_*(y)$; *and*

(4) $1_A \otimes_A x = x \otimes_B 1_B = x$. □

We now turn to the three properties of the *KK*-groups mentioned in the introduction. Since homotopy is built into the definition of $KK(A,B)$ we obviously have:

2.10. PROPOSITION (*Homotopy*). *The functor KK is homotopy invariant in both variables.* □

2.11. PROPOSITION (*Stability, see* **[15]** §5, *Theorem* 1). *The homomorphisms* $e_*\colon KK(A,B) \to KK(A,\mathscr{K}\otimes B)$ *and* $e^*\colon KK(\mathscr{K}\otimes B,C) \to KK(B,C)$ *are isomorphisms.*

Proof. Let $j\colon \mathscr{K}\otimes B \to \mathscr{M}(\mathscr{K}\otimes B)$ be the inclusion and let $\alpha = \{(j,0,1)\} \in KK(\mathscr{K}\otimes B, B)$. Then $e_*(\alpha) = 1_{\mathscr{K}\otimes B}$ and $e^*(\alpha) = 1_B$, so by Theorem 2.9, Kasparov product with α on the right and left is inverse to e_* and e^* respectively. □

2.12. PROPOSITION (*Split exactness*). *If*

$$0 \to J \xrightarrow{j} D \underset{p}{\overset{s}{\leftrightarrows}} D/J \to 0$$

is a split exact sequence of separable C^*-algebras and $*$-homomorphisms then the following are split exact sequences of abelian groups:

$$0 \to KK(A,J) \xrightarrow{j_*} KK(A,D) \underset{p_*}{\overset{s_*}{\leftrightarrows}} KK(A,D/J) \to 0$$

and

$$0 \to KK(D/J,B) \underset{p^*}{\overset{s^*}{\leftrightarrows}} KK(D,B) \xrightarrow{j^*} KK(J,B) \to 0$$

We require a simple computation:

2.13. LEMMA. *Let J be an ideal in D and let (ϕ_+, ϕ_-, U) be a $KK(C,D)$-cycle for which $U\phi_+(a) - \phi_-(a)U$, $\phi_+(a)(U^*U - 1)$, and $\phi_-(a)(UU^* - 1)$ are elements of $\mathscr{K} \otimes J$. If r: $\mathscr{M}(\mathscr{K} \otimes D) \to \mathscr{M}(\mathscr{K} \otimes J)$ denotes the canonical map (see §1.1) then $j_*\{(r \circ \phi_+, r \circ \phi_-, r(U))\} = \{(\phi_+, \phi_-, U)\}$.*

Proof. The lemma is easily verified if D is of the form $J \oplus I$, or (using Lemma 1.4) if J is an essential ideal in D. The general case follows from considering the sequence $J \lhd J \oplus \mathrm{Ann}(J) \lhd D$. □

Proof of 2.12. Note that D maps into $\mathscr{M}(\mathscr{K} \otimes J)$ via

$$D \xrightarrow{e} \mathscr{K} \otimes D \xrightarrow{r} \mathscr{M}(\mathscr{K} \otimes J).$$

Denote by $\pi \in KK(D,J)$ the class of the cycle $(1, s \circ p, 1)$. It is clear that $j^*(\pi) = 1_J$, whilst by the lemma, $j_*(\pi) = \{\langle 1, s \circ p \rangle\}$, which equals $1_D - s_* p_*(1_D)$. So by Theorem 2.9, taking the Kasparov product with π gives a homomorphism $KK(A,D) \to KK(A,J)$ which is left inverse to j_* and whose kernel is the image of the homomorphism s_*: $KK(A,D/J) \to KK(A,D)$; therefore the covariant sequence is exact. The contravariant sequence is dealt with similarly. □

NATURAL TRANSFORMATIONS

In this section we will be studying covariant functors F from separable C^*-algebras to abelian groups, all of which will be assumed to be *homotopy invariant*, *stable*, and *split exact*. In other words:

(i) F is a homotopy functor;

(ii) for every separable C^*-algebra B the homomorphism e_*: $F(B) \to F(\mathscr{K} \otimes B)$ is an isomorphism; and

(iii) if $0 \to J \to D \leftrightarrows D/J \to 0$ is a split exact sequence of separable C^*-algebras then the sequence $0 \to F(J) \to F(D) \leftrightarrows F(D/J) \to 0$ is also split exact.

Define a *natural transformation* between two such functors, F_1 and F_2, to be a collection of functions α_A: $F_1(A) \to F_2(A)$ (not homomorphisms, *a priori*), one for each separable C^*-algebra, such that if f: $A \to B$ then the diagram

$$\begin{array}{ccc} F_1(A) & \xrightarrow{f_*} & F_1(B) \\ {\scriptstyle\alpha_A}\downarrow & & \downarrow{\scriptstyle\alpha_B} \\ F_2(A) & \xrightarrow[f_*]{} & F_2(B) \end{array}$$

commutes (i.e., $f_*\alpha_A = \alpha_B f_*$). This is the same as the conventional definition:

3.2. LEMMA. *The functions α_A: $F_1(A) \to F_2(A)$ are group homomorphisms.*

Proof. Denote by j_1, j_2, π_1 and π_2 the inclusions into, and projections onto, the first and second factors of $A \oplus A$, and define δ: $A \oplus A \to M_2(A)$ by

$$\delta(a_1 \oplus a_2) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

The $*$-homomorphisms $\delta \circ j_1, \delta \circ j_2$: $A \to M_2(A)$ are homotopic via rotation and therefore $\delta_* j_{1*} = \delta_* j_{2*}$; furthermore, by stability, $\delta_* j_{i*}$ is an isomorphism. From the split exactness of F_1 applied to

$$A \xrightarrow{j_1} A \oplus A \underset{\pi_2}{\overset{j_2}{\leftrightarrows}} A$$

we see that $j_{1*}\pi_{1*} + j_{2*}\pi_{2*} = 1_{F_1(A \oplus A)}$; by composing with δ_* on the left we see that $\pi_{1*} + \pi_{2*} = (\delta j_1)_*^{-1} \circ \delta_*$ and from this we see that $(\pi_{1*} + \pi_{2*})\alpha_{A \oplus A} = \alpha_A(\pi_{1*} + \pi_{2*})$. Given $x_1, x_2 \in F_1(A)$, let $x = j_{1*}(x_1) + j_{2*}(x_2)$. Then $\pi_{i*}(x) = x_i$, and so

$$\begin{aligned} \alpha_A(x_1 + x_2) &= \alpha_A((\pi_{1*} + \pi_{2*})(x)) = (\pi_{1*} + \pi_{2*})(\alpha_{A \oplus A}(x)) \\ &= \pi_{1*}\alpha_{A \oplus A}(x) + \pi_{2*}\alpha_{A \oplus A}(x) = \alpha_A(x_1) + \alpha_A(x_2). \qquad \square \end{aligned}$$

We will classify natural transformations $KK(A, -) \to F$, using the following constructions, which are essentially due to Cuntz [**11**].

3.3. Definition. If $\Phi = (\phi_+, \phi_-, 1)$ is a $KK(A, B)$-cycle (for which $U = 1$), then let

$$A_\Phi = \{a \oplus x \in A \oplus \mathscr{M}(\mathscr{K} \otimes B) \colon \phi_+(a) = x, \text{ modulo } \mathscr{K} \otimes B\}.$$

Define $\hat{\phi}_\pm \colon A \to A_\Phi$ by $\hat{\phi}_\pm(a) = a \oplus \phi_\pm(a)$; define $j \colon \mathscr{K} \otimes B \to A_\Phi$ by $j(x) = 0 \oplus x$; and define $p \colon A_\Phi \to A$ by $p(a \oplus x) = a$.

These maps combine to form a short exact sequence

$$0 \to \mathscr{K} \otimes B \xrightarrow{j} A_\Phi \xrightarrow{p} A \to 0$$

which is split by either of the maps $\hat{\phi}_\pm \colon A \to A_\Phi$. Using this construction, we get a homomorphism from $F(A)$ to $F(B)$ as follows.

3.4. Definition. Let $\Phi_* \colon F(A) \to F(B)$ be the following composition of homomorphisms:

$$F(A) \xrightarrow{\hat{\phi}_{+*} - \hat{\phi}_{-*}} F(A_\Phi) \xrightarrow{\pi} F(\mathscr{K} \otimes B) \xrightarrow{e_*^{-1}} F(B),$$

where $\pi \colon F(A_\Phi) \to F(\mathscr{K} \otimes B)$ is a left inverse of $j_* \colon F(\mathscr{K} \otimes B) \to F(A_\Phi)$ (this exists by split exactness; also, since $\hat{\phi}_{+*} - \hat{\phi}_{-*}$ maps into the kernel of p_*, Ψ_* does not depend on the particular choice of π).

For example, consider the cycle $\langle 1, 0 \rangle$ of §2.8 whose class in $KK(A, A)$ is 1_A. The C^*-algebra A_Φ is $A \oplus \mathscr{K} \otimes A$, $\hat{\phi}_+(a) = a \oplus e(a)$, $\hat{\phi}_-(a) = a \oplus 0$, and $\pi \colon F(A_\Phi) \to F(\mathscr{K} \otimes A)$ may be chosen to be q_*, where $q(a \oplus x) = x$. It follows that $\langle 1, 0 \rangle_* = 1_{F(A)}$. This illustrates the following result.

3.5. Theorem. *The homomorphism* $\Phi_* \colon KK(A, A) \to KK(A, B)$ *maps* 1_A *to* $\{\Phi\}$.

Proof. By §2.8 the image of 1_A under $\hat{\phi}_{+*} - \hat{\phi}_{-*}$ is $\{\langle \hat{\phi}_+, \hat{\phi}_- \rangle\}$, and by Lemma 2.13 the image of this under π is $\{(r \circ e \circ \hat{\phi}_+, r \circ e \circ \hat{\phi}_-, 1)\}$, where $r \colon \mathscr{M}(\mathscr{K} \otimes A_\Phi) \to \mathscr{M}(\mathscr{K} \otimes \mathscr{K} \otimes B)$ is the canonical map. Now, this is equal to the class of $(r \circ e \circ \hat{\phi}_+, r \circ e \circ \hat{\phi}_-, e \otimes 1)$ and modulo interchanging the copies of $\mathscr{K}$ in $\mathscr{K} \otimes \mathscr{K}$ (a unitary equivalence) this is equal to the image of $(\phi_+, \phi_-, 1)$ under the map $\mathscr{M}(\mathscr{K} \otimes B) \simeq \mathscr{M}(\mathscr{K} \otimes \mathscr{K} \otimes B)$. But by Proposition 2.7 this last map induces $e_* \colon KK(A, B) \to KK(A, \mathscr{K} \otimes B)$, and so $e_*\{\Phi\} = \pi \circ (\hat{\phi}_{+*} - \hat{\phi}_{-*})(1_A)$; hence $\{\Phi\} = \Phi_*(1_A)$. □

3.6. Lemma. *Any* $KK(A, B)$-*cycle is equivalent to one of the form* $(\phi_+, \phi_-, 1)$, *which may also be chosen to be special. If two cycles for which* $U = 1$ *are equivalent there exist degenerate cycles and a homotopy implementing the equivalence for which also* $U = 1$.

Proof. By replacing an arbitrary $KK(A,B)$-cycle (ϕ_+, ϕ_-, U) with the equivalent cycle

$$\left(\begin{pmatrix}\phi_+ & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}\phi_- & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}U & 1-UU^*\\ 1-U^*U & (U^*U-2)U^*\end{pmatrix}\right)$$

we may assume that U is invertible (this trick is due to Connes [**8**]). Next, the path $U((1-t)+t|U|^{-1})$ $(t \in [0,1])$ deforms U to its unitary part in the polar decomposition, so we may assume that U is unitary. Finally, by replacing (ϕ_+, ϕ_-, U) with the unitarily equivalent cycle $(\mathrm{Ad}(U)\circ\phi_+, \phi_-, 1)$ we obtain the desired cycle (if we started the procedure with a special cycle clearly we would have finished with one). The second part of the lemma follows by applying the same procedure to the equivalence. □

3.7. THEOREM. *If* $x \in F(A)$ *then there exists a unique natural transformation* α: $KK(A,-) \to F$ *such that* $\alpha_A(1_A) = x$.

Proof. If α: $KK(A,-) \to F$ is a natural transformation and Φ is a cycle as in 3.3 then it is easy to see that $\alpha_B \circ \Phi_* = \Phi_* \circ \alpha_A$. Hence $\alpha_B(\{\Phi\}) = \alpha_B(\Phi_*(1_A)) = \Phi_*(\alpha_A(1_A))$, and so $\alpha_A(1_A)$ determines $\alpha_B(\{\Phi\})$.

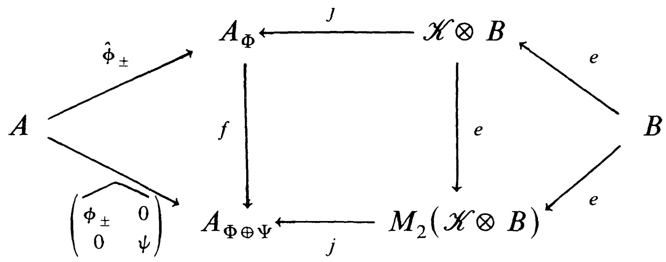
If $x \in F(A)$ define $\bar{\alpha}_B(\Phi) = \Phi_*(x)$. Let us show that $\bar{\alpha}_B(\Phi)$ depends only on $\{\Phi\}$. If $\Psi = (\phi, \phi, 1)$ is a degenerate cycle with $U = 1$ then bearing in mind that $A_{\Phi\oplus\Psi}$ is equal to

$$\left\{a \oplus (x_{ij}) \in A \oplus M_2(\mathcal{M}(\mathcal{K}\otimes B)): (x_{ij}) = \begin{pmatrix}\phi_+(a) & 0\\ 0 & \phi(a)\end{pmatrix}, \ \mathrm{mod}\ M_2(\mathcal{K}\otimes B)\right\},$$

we can define

$$f\colon A_\Phi \to A_{\Phi\oplus\Psi} \quad \text{by} \quad f(a\oplus x) = a \oplus \begin{pmatrix}x & 0\\ 0 & \phi(a)\end{pmatrix}.$$

As the diagram



commutes (for suitable choices of e), it follows that $(\Phi \oplus \Psi)_* = \Phi_*$. Also, if Φ_0 and Φ_1 are homotopic via a homotopy Φ with $u = 1$ then $\Phi_{i*} = \varepsilon_{i*} \circ \Phi_*$ where ε_i: $B \otimes C[0,1] \to B$ is evalution at $i = 0, 1$. Hence $\Phi_{0*} = \Phi_{1*}$ by homotopy invariance. So it follows from Lemma 3.6 that $\bar{\alpha}_B$ gives a map α_B: $KK(A, B) \to F(B)$. Notice that by the computation following Definition 3.4, $\alpha_A(1_A) = x$. Finally, if Φ is special then it is easy to check that $(g_\sharp \Phi)_* = g_* \circ \Phi_*$ if g: $B \to B'$; so α is a natural transformation. □

We end by noting that all these results have contravariant analogues. For example, corresponding to 3.5 there is the following result.

3.10. LEMMA. *If* Φ^*: $KK(B, B) \to KK(A, B)$ *is the composition*

$$KK(B,B) \xrightarrow{e^{*-1}} KK(\mathscr{K} \otimes B, B) \xrightarrow{\sigma} KK(A_\Phi, B) \xrightarrow{\hat{\phi}_+^* - \hat{\phi}_-^*} KK(A, B),$$

then $\Phi^*(1_B) = \{\Phi\}$. (σ *is right inverse to* j^*.)

Proof. Let $\pi \in KK(A_\Phi, \mathscr{K} \otimes B)$ be the element corresponding to the split exact sequence $0 \to \mathscr{K} \otimes B \to A_\Phi \overset{\hat{\phi}_+}{\leftrightarrows} A \to 0$. We may take σ to be Kasparov product with π; having done so, it is easy to check that $\Phi^*(1_B) = \phi_*(1_A)$ from which the lemma follows. □

CHARACTERIZATION OF KK

We begin by examining the product in the light of the above results.

4.1. THEOREM. *The Kasparov product is associative.*

Proof. Let $x \in KK(A, B)$ and $y \in KK(B, C)$. The two homomorphisms from $KK(C, D)$ to $KK(A, D)$ given by $z \mapsto (x \otimes_B y) \otimes_C z$ and $z \mapsto x \otimes_B (y \otimes_C z)$ are both natural in D and both map 1_C to $x \otimes_B y$. Therefore they are equal by Theorem 3.7. □

4.2. THEOREM. *If* Φ *is a cycle as in* 3.3 *then* (i) $x \otimes_A \{\Phi\} = \Phi_*(x)$, *and* (ii) $\{\Phi\} \otimes_B x = \Phi^*(x)$.

Proof. This follows from the functoriality of the product, 3.5 and 3.10. □

In particular, 4.2 shows that it is possible to recover the product from the homotopy invariance, stability, and split exactness of $KK(A, -)$.

4.3. THEOREM. *There is a unique bilinear pairing* γ: $KK(A,B) \times KK(B,C) \to KK(A,C)$ *which is functorial in* A *and* C *and which satisfies the normalization condition* $\gamma(1_B, 1_B) = 1_B$.

Proof. By functoriality and 4.2, $\gamma(\{\Phi\},\{\Psi\}) = \Phi^*\Psi_*(\gamma(1_B, 1_B)) = \Phi^*\Psi_*(1_B)$. □

4.4. The associativity of the product allows us to construct a category **K** from KK-theory. The objects of **K** are separable C^*-algebras, the set $\mathbf{K}(A, B)$ of morphisms from A and B is $KK(A, B)$; and the law of composition is the Kasparov product: $y \circ x = x \otimes_B y$ (the element $1_A \in KK(A, A)$ serves as the identity morphism). In fact **K** is an additive category, which means that each $\mathbf{K}(A, B)$ is abelian group and the composition law is bilinear. There is a canonical functor C from the category of separable C^*-algebras and $*$-homomorphisms (henceforth denoted **C*-Alg**) to **K**, namely $C(f) = f_*(1_A)$ if f: $A \to B$.

Now, let F: **C*-Alg** $\to$ **A** be any functor into an additive category **A** such that if X is any object of **A** then $\mathbf{A}(X, F(-))$ is a homotopy invariant, stable and split exact functor into abelian groups. Equivalently, suppose that F satisfies the following three properties:

(i) F: **C*-Alg** $\to$ **A** is a homotopy functor;

(ii) the morphism e_*: $F(B) \to F(\mathscr{K} \otimes B)$ is invertible; and

(iii) if $0 \to J \overset{j}{\to} D \underset{p}{\overset{s}{\leftrightarrows}} D/J \to 0$ is a split exact sequence then $F(D)$ is the direct sum (coproduct) of $F(J)$ and $F(D/J)$ via the maps j_* and s_*. (Compare [**16**], §I.18)

4.5. THEOREM. *There exists a unique functor* $\hat{F}$: **K** $\to$ **A** *such that* $\hat{F} \circ C = F$.

Proof. Obviously, on objects, $\hat{F}$ is given by $\hat{F}(A) = F(A)$. If $x \in KK(A, B)$ then define $\hat{F}(x)$: $F(A) \to F(B)$ to be the image of x under the natural transformation $KK(A, -) \to \mathbf{A}(F(A), F(-))$ which maps 1_A to $1_{F(A)}$. Using the description of the product in 4.2 it follows that $\hat{F}$ is a functor; uniqueness follows from uniqueness of the above transformation. □

Thus C: **C*-Alg** $\to$ **K** is characterized as the *universal* functor with properties (i), (ii) and (iii). This has an interesting consequence: by purely algebraic methods one may construct such a universal functor; this must of course be equal to C: **C*-Alg** $\to$ **K**, and hence it is possible to construct the Kasparov groups and the product purely formally—without any reference to functional analysis at all! We will discuss this, and related issues, in a future paper. We finish here with an application.

4.7. *Tensor products.* By considering the (minimal) tensor product of C^*-algebras along with the tensor product of $*$-homomorphisms we obtain a functor $\otimes$: **C*-Alg** $\times$ **C*-Alg** $\to$ **C*-Alg**. By applying Theorem 4.5 we can carry this over to the category **K**. As a result, for example, we obtain the more general product

$$KK(A_1, B_1 \otimes D) \otimes KK(A_2 \otimes D, B_2) \to KK(A_1 \otimes A_2, B_1 \otimes B_2)$$

mentioned in the introduction.

4.8. THEOREM. *There exists a unique functor* $\boxtimes$: $\mathbf{K} \times \mathbf{K}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{C^*\text{-}Alg} \times \mathbf{C^*\text{-}Alg} & \overset{C\times C}{\to} & \mathbf{K} \times \mathbf{K} \\ \otimes \downarrow & & \downarrow \boxtimes \\ \mathbf{C^*\text{-}Alg} & \underset{C}{\to} & \mathbf{K} \end{array}$$

commutes.

Proof. Since the minimal tensor product preserves *split* exact sequences, the functors **C*-Alg** $\to$ **K** defined by

$$\left(A \xrightarrow{f} B\right) \mapsto (1 \otimes f)_*(1_{D\otimes A}) \in KK(D \otimes A, D \otimes B),$$

and

$$\left(A \xrightarrow{f} B\right) \mapsto (f \otimes 1)_*(1_{A\otimes E}) \in KK(A \otimes E, B \otimes E),$$

satisfy the conditions of 4.4, and so extend to functors $1_D \boxtimes \_$ and $\_\boxtimes 1_E$ from **K** to itself. Let $x \in KK(A_1, B_1)$ and let g: $B_2 \to B_2'$. Then it follows from Theorem 3.7 that $(1 \otimes g)^*(x \boxtimes 1_{B_2'}) = (1 \otimes g)_*(x \boxtimes 1_{B_2})$ since both of the functions

$$x \mapsto (1 \otimes g)^*(x \boxtimes 1_{B_2'}) \quad \text{and} \quad x \mapsto (1 \otimes g)_*(x \boxtimes 1_{B_2}) \qquad (x \in KK(A_1, B_1)),$$

are natural in B_1 and both send $x = 1_{A_1}$ to $(1 \otimes g)_*(1_{A_1\otimes B_2})$. It follows from this, and the functoriality of the Kasparov product that both of the maps

$$y \mapsto (1_{A_1} \boxtimes y) \otimes (x \boxtimes 1_{B_2}) \quad \text{and} \quad y \mapsto (x \boxtimes 1_{A_2}) \otimes (1_{B_1} \boxtimes y) \qquad (y \in KK(A_2, B_2))$$

(where $\otimes$ denotes Kasparov product), are natural in B_2. Since both map 1_{A_2} to $x_1 \boxtimes 1_{A_2}$ it follows from Theorem 3.7 again that they are equal. Hence:

$$(1_{A_1} \boxtimes y) \otimes (x \boxtimes 1_{B_2}) = (x \boxtimes 1_{A_2}) \otimes (1_{B_1} \boxtimes y)$$

and mapping (x, y) to either of these expressions gives a suitable functor. Uniqueness follows from the uniqueness of the functors $1 \boxtimes \_$ and $\_ \boxtimes 1$. □

APPENDIX

For the purpose of describing the product it is useful to work with a slightly different definition of $KK(A, B)$-cycle, using (if only in a minor way) the notion of a $\mathbf{Z}/2$-*grading* on a C^*-algebra E. This is a decomposition $E = E^{(0)} \oplus E^{(1)}$ of E into a direct sum of two closed, self-adjoint, complementary subspaces ("degree zero" and "degree one" elements) such that $E^{(i)} \cdot E^{(i)} \subset E^{(i+j)}$ with $(i + j)$ taken modulo 2.

Examples are: (1) the trivial grading, that is: $E^{(0)} = E$ and $E^{(1)} = 0$; and (2) the grading included by a symmetry $X \in E$ (a self-adjoint unitary), for which the degree zero elements are those which commute with X and the degree one elements are those which anticommute with it.

A1. DEFINITION. A $KK(A, B)$-cycle is a triple (X, ϕ, F) where X is a symmetry in $\mathcal{M}(\mathcal{K})$, ϕ: $A \to \mathcal{M}(\mathcal{K} \otimes B)$ is $*$-homomorphism which is grading preserving with respect to the trivial grading on A and the grading on $\mathcal{M}(\mathcal{K} \otimes B)$ induced by $X \otimes 1$, and F is a degree one element of $\mathcal{M}(\mathcal{K} \otimes B)$ such that $[\phi(a), F]$, $\phi(a)(F^2 - 1)$, and $\phi(a)(F - F^*)$ are elements of $\mathcal{K} \otimes B$, if $a \in A$.

Let us relate this to Definition 2.1. To obtain from a cycle (X, ϕ, F) as above a cycle of the type considered there, we can proceed as follows. Let $P = (2X \otimes 1) - 1$, the projection onto the $+1$ eigenspace of $X \otimes 1$. Because ϕ is grading preserving, the symmetry $X \otimes 1$, and so also the projection P, commutes with every $\phi(a)$ $(a \in A)$. Therefore P reduces ϕ and we can write $\phi = \phi_P + \phi_{1-P}$. On the other hand, F anticommutes with $X \otimes 1$ and therefore with respect to the decomposition $1 = P + (1 - P)$ it is a matrix of the form $\left(\begin{smallmatrix} 0 & V \\ U & 0 \end{smallmatrix}\right)$. We obtain the triple

$$\left(\phi_P, \phi_{1-P}, \begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}\right),$$

which is a cycle in the sense of Definition 2.1. (Note that from the relation $\phi(a)(F - F^*) \in \mathcal{K} \otimes B$, it follows that $U^* = V$ modulo $\mathcal{K} \otimes B$ and multiplication by $\phi_p(a)$ and $\phi_{(1-p)}(a)$.)

Addition of cycles is given by

$$\left(\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}, \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}\right)$$

and exactly as in §2 we obtain the group $KK(A, B)$ from the equivalence relation generated by addition of degenerate cycles (cycles for which $[\phi(a), F]$, etc., equal zero) and homotopy (which is given by $KK(A, B \otimes C[0,1])$-cycles); it is the same group, of course. Functoriality is obtained via special cycles as in §2.

We should also compare our definition with that of Kasparov ([**15**], Definition 4.1). This is given in terms of Hilbert modules, and is related to ours by the fact that the algebra of adjoinable operators on the standard Hilbert module H_B is ($*$-isomorphic to) $\mathcal{M}(\mathcal{K} \otimes B)$. (For information on Hilbert modules see [**14**] and [**15**].) When written in terms of algebras rather than modules, Kasparov's definition is the same as ours, except that the element of $\mathcal{M}(\mathcal{K} \otimes B)$ which determines the grading—the symmetry $X \otimes 1$ in our scheme—is allowed to be any symmetry at all. However, by virtue of the stabilization theorem ([**14**], Theorem 3.2) the same group $KK(A, B)$ is obtained.

A2. Definition. Let (X_1, ϕ, F) be a special $KK(A, B)$-cycle and let (X_2, ψ, G) be a $KK(B, C)$-cycle. Define the $*$-homomorphism $\bar{\psi}\colon \mathcal{M}(\mathcal{K} \otimes \tilde{B}) \to \mathcal{M}(\mathcal{K} \otimes \mathcal{K} \otimes C)$ to be the composition

$$\mathcal{M}(\mathcal{K} \otimes \tilde{B}) \xrightarrow{(1 \otimes \tilde{\psi})_*} \mathcal{M}(\mathcal{K} \otimes \mathcal{M}(\mathcal{K} \otimes C)) \to \mathcal{M}(\mathcal{K} \otimes \mathcal{K} \otimes C),$$

and let $\bar{F} = \bar{\psi}(F)$, $\bar{G} = X_1 \otimes G \in \mathcal{M}(\mathcal{K} \otimes \mathcal{K} \otimes C)$. A *Kasparov product* of (X_1, ϕ, F) and (X_2, ψ, G) is a $KK(A, C)$-cycle of the form $(X_1 \otimes X_2, \bar{\psi} \circ \phi, M^{1/2}\bar{F} + N^{1/2}\bar{G})$ where M and N are positive, degree zero elements of $\mathcal{M}(\mathcal{K} \otimes \mathcal{K} \otimes C)$ such that $M + N = 1$, and:

(i) M and N commute, modulo $\mathcal{K} \otimes \mathcal{K} \otimes C$, with all elements of the subspace $\mathcal{F} \subset \mathcal{M}(\mathcal{K} \otimes \mathcal{K} \otimes C)$ generated by $\bar{\psi}(\phi(a))$ (for all $a \in A$), $\bar{F}$ and $\bar{G}$;

(ii) $M \cdot E_1 \subset \mathcal{K} \otimes \mathcal{K} \otimes C$, where E_1 is the C^*-subalgebra of $\mathcal{M}(\mathcal{K} \otimes \mathcal{K} \otimes C)$ generated by $[\bar{\psi}(\phi(a)), \bar{F}]$, $\bar{\psi}(\phi(a))(\bar{F}^2 - 1)$ and $\bar{\psi}(\phi(a))(\bar{F} - \bar{F}^*)$ $(a \in A)$; and

(iii) $N \cdot E_2 \subset \mathcal{K} \otimes \mathcal{K} \otimes C$, where E_2 is the C^*-subalgebra of $\mathcal{M}(\mathcal{K} \otimes \mathcal{K} \otimes C)$ generated by $[\bar{\psi}(\phi(a)), \bar{G}]$, $\bar{\psi}(\phi(a))(\bar{G}^2 - 1)$, $\bar{\psi}(\phi(a))(\bar{G} - \bar{G}^*)$ $(a \in A)$, and $\bar{F}\bar{G} + \bar{G}\bar{F}$.

Some discussion of this definition is perhaps in order. To begin with, suppose given two degree one symmetries, F_1 and F_2, on graded Hilbert spaces (X_1, H_1) and (X_2, H_2) (the $X_i \in \mathscr{B}(H_i)$ are grading operators). Then $\alpha^{1/2}F_1 \otimes 1 + \beta^{1/2}X_1 \otimes F_2$ is a degree one symmetry on $(X_1 \otimes X_2, H_1 \otimes H_2)$, if α and β are non-negative numbers such that $\alpha + \beta = 1$: multiplying out we get

$$\begin{aligned}&\left(\alpha^{1/2}F_1 \otimes 1 + \beta^{1/2}X_1 \otimes F_2\right)^2\\ &\qquad = \alpha + \alpha^{1/2}\beta^{1/2}(F_1X_1 \otimes F_2 + X_1F_1 \otimes F_2) + \beta,\end{aligned}$$

and since F_1 anticommutes with X_1, the two middle terms cancel out. This helps motivate the choice of the operator $M^{1/2}\overline{F} + N^{1/2}\overline{G}$ in the definition of the product. Of course, $\overline{F}$ and $\overline{G}$ only "approximately" anticommute (except in special circumstances: for example, if G commutes with $\psi(B)$); however this problem is overcome by the particular choice of the operators M and N in the definition.

Let us comment on the use of $\mathbf{Z}/2$-gradings in the product. Put in terms of Definition 2.1, the product is a cycle of the form

$$\left(\begin{pmatrix} \overline{\psi_+}\phi_+ & 0 \\ 0 & \overline{\psi_-}\phi_- \end{pmatrix}, \begin{pmatrix} \overline{\psi_+}\phi_- & 0 \\ 0 & \overline{\psi_-}\phi_+ \end{pmatrix}, \begin{pmatrix} M^{1/2}\overline{\psi_+}(U) & -N^{1/2}(1 \otimes V^*) \\ N^{1/2}(1 \otimes V) & M^{1/2}\overline{\psi_-}(U^*) \end{pmatrix}\right).$$

It is evident then that $\mathbf{Z}/2$-gradings provide a very concise and natural method of bookkeeping. The rather strange formula for the operator in the above cycle is quite familiar from K-theory: for example given two essentially unitary operators U and V, the operator

$$U\#V = \begin{pmatrix} M^{1/2}(U \otimes 1) & -N^{1/2}(1 \otimes V^*) \\ N^{1/2}(1 \otimes V) & M^{1/2}(U^* \otimes 1) \end{pmatrix}$$

for suitable M and N Is the natural operator obtained from U and V for which

$$\operatorname{Index}(U\#V) = \operatorname{Index}(U) \cdot \operatorname{Index}(V).$$

In fact, the space of all Fredholm operators forms a classifying space for K-theory and the product in K-theory may be obtained from a construction such as this. (Of course, we are just describing a special case of the Kasparov product.)

At a more basic level, conditions (i), (ii) and (iii) above on M and N are simply the obvious sufficient ones to make a Kasparov product $(X_1 \otimes X_2, \overline{\psi} \circ \phi, M^{1/2}\overline{F} + N^{1/2}\overline{G})$ into a $KK(A, C)$-cycle (as checking

Definition A1 immediately reveals). Within these constraints, the choice of (M, N) is unimportant, for if (M', N') is another admissible pair then the path of pairs, $(tM + (1-t)M', tN + (1-t)N')$ $(t \in [0,1])$ links the corresponding Kasparov products by a homotopy, and so they determine the same element of $KK(A, C)$.

The existence of products follows from the following separation result (see [**15**] or [**12**]). The *graded commutator* of elements in a graded algebra is $[x, y]_{gr} = xy - (-1)^{\deg(x)\deg(y)}yx$ (see [**15**]).

A3. THEOREM. *Let D be a separable graded C^*-algebra, let $\mathscr{A}$ and $\mathscr{B}$ be separable graded C^*-subalgebras of $\mathscr{M}(D)$, and let $\mathscr{F}$ be a separable graded subspace. If $[\mathscr{F}, \mathscr{A}]_{gr} \subset \mathscr{A}$ and $\mathscr{A} \cdot \mathscr{B} \subset D$ then there exist positive degree zero elements $M, N \in \mathscr{M}(D)$ such that $M + N = 1$, $[M, \mathscr{F}] \subset D$, $M \cdot \mathscr{A} \subset D$ and $N \cdot \mathscr{B} \subset D$. Furthermore, if D is an essential ideal in D' and $\mathscr{A}, \mathscr{B}$, and $\mathscr{F}$ are contained in the subalgebra $\mathscr{M}(D')$ of $\mathscr{M}(D)$ then M and N may be chosen in $\mathscr{M}(D')$.* □

We take $D = \mathscr{K} \otimes \mathscr{K} \otimes C$, $\mathscr{A} = \overline{\bar{\psi}(\mathscr{K} \otimes B) + D}$ (noting that $E_1 \subset \bar{\psi}(\mathscr{K} \otimes B)$), $\mathscr{B} = E_2$, and $\mathscr{F} = \mathscr{F}$; the hypothesis of Theorem A3 are easily verified.

We will close by proving Theorem 2.9, considering first the functoriality of the product.

A4. THEOREM. *The product gives a well-defined bilinear mapping $KK(A, B) \times KK(B, C) \to KK(A, C)$ which is natural in A, B and C in the sense of Theorem* 2.9.

Proof. Let (X_1, ϕ, F) be a special $KK(A, B)$-cycle and let (X_2, ψ, G) be a $KK(B, C)$-cycle. If (X_1, ϕ, F) is degenerate we may take $M = 1$, $N = 0$, and then the product is degenerate. Similarly, if (X_2, ψ, G) is degenerate we may take $M = 0$, $N = 1$ and the product is again degenerate. By taking the product of (X_1, ϕ, F) with a homotopy (a $KK(B, C \otimes C[0,1])$-cycle), we see that the product only depends on the homotopy class of (X_2, ψ, G); by taking the product of a homotopy with the $KK(B \otimes C[0,1], C \otimes C[0,1])$-cycle $(X_2, \psi \otimes 1, G \otimes 1)$ we see that the product also only depends on the homotopy class of (X_1, ϕ, F). Therefore, since it is clearly additive, the product passes to the KK-groups. Functoriality in A and B is obvious; to obtain functoriality in C, we need only consider the case where (X_2, ϕ, G) is special. Then by the last part of Theorem A4 we may choose M and N from the subalgebra $\mathcal{M}(\mathcal{K} \otimes \mathcal{K} \otimes \tilde{C})$ of $\mathcal{M}(\mathcal{K} \otimes \mathcal{K} \otimes C)$, from which the result follows. □

We pass on to the normalization condition. Note that the "identity" element $1_A \in KK(A, A)$ is given by the cycle $(1, e, 0)$, where e: $A \to \mathcal{K} \otimes A$, $e(a) = e \otimes a$.

A5. Theorem. *If* $x \in KK(A, B)$ *then* $1_A \otimes_A x = x = x \otimes_B 1_B$.

Proof. Let (X, ϕ, F) be a special $KK(A, B)$-cycle, and let $x = \{(X, \phi, F)\}$. Let us consider $1_A \otimes_A x$ first. We may take say $M = (1 - e) \otimes 1$ in the construction of the product, and then the cycle so obtained is $(1 \otimes X, \bar{\phi} \circ e, (e \otimes 1)\phi(F))$. Since the map $\bar{\phi} \circ e$ is equal to

$$e \otimes \phi \colon a \mapsto e \otimes \phi(a) \in \mathcal{M}(\mathcal{K}) \otimes \mathcal{M}(\mathcal{K} \otimes B) \subset \mathcal{M}(\mathcal{K} \otimes \mathcal{K} \otimes B),$$

we may write the product as $(1 \otimes X, e \otimes \phi, e \otimes F)$, which is equal to (X, ϕ, F) plus a zero cycle. Hence $1_A \otimes_A x = x$ by the definition of equivalence in $KK(A, B)$. In the construction of $x \otimes_B 1_B$ we may take $M = 1$, which gives the cycle $(X \otimes 1, \bar{e} \circ \phi, \bar{e}(F))$. The projection $P = 1 \otimes e \otimes 1$ reduces $\bar{e}$: $\mathcal{M}(\mathcal{K} \otimes B) \to \mathcal{M}(\mathcal{K} \otimes \mathcal{K} \otimes B)$, (so $\bar{e} = \bar{e}_M + \bar{e}_{(1-M)}$) and therefore we may write

$$\begin{aligned}(X \otimes 1, \bar{e} \circ \phi, \bar{e}(F)) = (X \otimes 1, (\bar{e} \circ \phi)_P, P\bar{e}(F)) \\ + (X \otimes 1, (\bar{e} \circ \phi)_{(1-P)}, (1 - P)\bar{e}(f)).\end{aligned}$$

Since $\mathcal{K} \otimes B$ is contained in the kernel of $\bar{e}_{(1-M)}$, the second cycle is degenerate. Thus the product is equivalent to $(X \otimes 1, (\bar{e} \circ \phi)_P, P\bar{e}(F))$, which, after interchanging the copies of $\mathcal{K}$ in $\mathcal{K} \otimes \mathcal{K}$, is $(1 \otimes X, e \otimes \phi, e \otimes F)$. This is equivalent to $(e \otimes X, e \otimes \phi, e \otimes F)$ which is equivalent to the original cycle (X, ϕ, F). □

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DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA, CANADA B3H 4H8