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DERIVATIONS ON THE LINE AND FLOWS ALONG ORBITS

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The closure of the derivation λD: $C_c^1(\mathbb{R}) \to C_0(\mathbb{R})$ defined by $(\lambda D)(f) = \lambda f'$, where λ: $\mathbb{R} \to \mathbb{R}$ is continuous, generates a C_0-group on $C_0(\mathbb{R})$ (corresponding to a flow on $\mathbb{R}$) if and only if $1/\lambda$ is not locally integrable on either side of any zero of λ or at $\pm\infty$.

If S is a flow on a locally compact, Hausdorff, space X with fixed point set X_S^0, δ_S is the generator of the induced action on $C_0(X)$, λ: $X \setminus X_S^0 \to \mathbb{R}$ is continuous, and bounded on sets of low frequency under S, and $t \to \lambda(S_t\omega)^{-1}$ is not locally integrable on either side of any zero or at $\pm\infty$, then the flows along the orbits of S form a flow on X whose generator acts as $\lambda\delta_S$.

1. Introduction. Let S be a flow on a locally compact, Hausdorff, space X, and δ_S be the generator of the associated one-parameter group of *-automorphisms of $C_0(X)$, the commutative C^*-algebra of continuous complex-valued functions on X which vanish at infinity. Thus

$$\delta_S f = \lim_{t\to 0} t^{-1}(f \circ S_t - f)$$

whenever the limit exists (pointwise, and hence uniformly) and defines a function in $C_0(X)$. Let $\mathscr{D}_S^\infty = \bigcap_{n\geq 1}\mathscr{D}(\delta_S^n)$. Then $\mathscr{D}_S^\infty$ is a dense *-subalgebra of $C_0(X)$. If δ: $\mathscr{D}_S^\infty \to C_0(X)$ is a *-derivation, then there is a function λ: $X \to \mathbb{R}$ such that

$$\delta f = \lambda \delta_S f \qquad (f \in \mathscr{D}_S^\infty)$$

[**1**]. The function λ may be chosen arbitrarily on the fixed point set X_S^0:

$$X_S^0 = \{\omega \in X\colon S_t\omega = \omega \text{ for all } t\} = \{\omega \in X\colon \delta_S f(\omega) = 0 \text{ for all } f \text{ in } \mathscr{D}_S^\infty\},$$

and we shall always assume that $\lambda = 0$ on X_S^0. However, λ is uniquely determined and continuous on $X \setminus X_S^0$, and satisfies a bound of the form

$$(*) \qquad |\lambda(\omega)| \leq c(1 + \nu(\omega)^n) \qquad (\omega \in X \setminus X_S^0)$$

for some constant $c \geq 0$, and integer $n \geq 0$, where $\nu(\omega)$ is the frequency of ω, so

$$\nu(\omega)^{-1} = \inf\{t > 0\colon S_t\omega = \omega\}$$

($\nu(\omega) = 0$ if ω is aperiodic) (see [**4**]).

We shall therefore study the *-derivations $\lambda\delta_S$ defined by

$$(\lambda\delta_S)f = \begin{cases} \lambda\delta_S f & \text{on } X \setminus X_S^0 \\ 0 & \text{on } X_S^0 \end{cases}$$

whenever the right-hand side defines a function in $C_0(X)$. Here $\lambda\colon X \setminus X_S^0 \to \mathbb{R}$ is a continuous function. The domain $\mathscr{D}(\lambda\delta_S)$ contains $\mathscr{D}_S^\infty$ if and only if λ satisfies a bound of the form $(*)$, but this will not necessarily be assumed. Nevertheless, $\mathscr{D}(\lambda\delta_S)$ is always reasonably large. Indeed for any ω in $X \setminus X_S^0$, $\varepsilon > 0$ such that $2\varepsilon\nu(\omega) < 1$ and F in $C^\infty[-\varepsilon, \varepsilon]$, there exists f in $\mathscr{D}_S^\infty$ such that $f(S_t\omega) = F(t)$ ($|t| \le \varepsilon$), and $\operatorname{supp} f \subset X \setminus X_S^0$ [4]. In particular, $f \in \mathscr{D}(\lambda\delta_S)$.

The properties of interest are whether there is a flow T whose generator δ_T extends $\lambda\delta_S$, and if so whether T is unique and whether $\mathscr{D}(\lambda\delta_S)$ (or some smaller subalgebra) is a core for δ_T. Considering both functions which vary transversally and along the orbits of S, it is apparent that T should be a flow along the orbits of S whose speed is given at each point by the function λ. Thus

$$T_t S_s \omega = S_{\tau_\omega(s,t)}\omega$$

where τ_ω is a flow on $\mathbb{R}$ such that

$$\partial\tau_\omega/\partial t = \lambda_\omega \circ \tau_\omega$$

where $\lambda_\omega(s) = \lambda(S_s\omega)$.

The first stage (§2) therefore is to study flows T on $\mathbb{R}$ satisfying the differential equation

$$\partial T/\partial t = \lambda \circ T$$

where $\lambda\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. If $1/\lambda$ is not locally integrable on either side of any zero of λ or at $\pm\infty$, then there is a unique flow T of this type, each zero of λ is a fixed point of T, and $C_c^\infty(\mathbb{R})$ is a core for δ_T. Otherwise, there may be no flows or there may be many flows.

In §3, it is shown that if each λ_ω satisfies these conditions of reciprocal non-integrability, then the flows with speeds λ_ω along the orbits together define a flow on X whose generator extends $\lambda\delta_S$.

There is some overlap between §2 of this paper, a paper of de Laubenfels [6], which left several questions incompletely answered, and an unpublished manuscript of the author's [2] which has circulated and been cited quite widely. The results of §3 are more general than those obtained in [3, 7], where it was assumed that λ satisfies a Lipschitz condition

$$|\lambda(S_t\omega) - \lambda(\omega)| \le |t|\kappa(\nu)$$

whenever $\nu(\omega) \leq \nu$. Such a condition implies the reciprocal non-integrability conditions.

I am grateful to R. de Laubenfels for his helpful response to my queries concerning [6], and to D. W. Robinson for his encouragement in reviving this subject while I was visiting the Australian National University at his invitation.

2. The real line. Sakai [9] has raised the question of characterizing all flows T on $[0,1]$ whose generator extends λD, where $\lambda \in C[0,1]$ and D denotes differentiation defined on $C^1[0,1]$. The motivation for this was the fact that, for any flow T on $[0,1]$, there is a homeomorphism θ of $[0,1]$ such that $\delta_{\theta T\theta^{-1}}$ extends λD for some λ. Similar remarks apply to flows on $\mathbb{R}$, where D itself is the generator for the flow of translations, and we shall work on the whole line, at least initially.

In fact, one can, by choosing θ appropriately, arrange that $\theta T\theta^{-1}$ is one of the flows T_U^ε described in the following example [**10**, p. 26]. But this fact does not directly help to decide when λD extends to a generator, nor is it helpful in considering flows on general spaces.

EXAMPLE 2.1. For each open interval I in $\mathbb{R}$, define flows T_I on I as follows:

$$T_{(a,b)}(x,t) = \frac{b(x-a)e^{(b-a)t} + a(b-x)}{b-x+(x-a)e^{(b-a)t}},$$

$$T_{(a,\infty)}(x,t) = a+(x-a)e^{t},$$

$$T_{(-\infty,b)}(x,t) = b+(x-b)e^{-t},$$

$$T_{\mathbb{R}}(x,t) = x+t.$$

Now let U be an open subset of $\mathbb{R}$, $\mathscr{C}_U$ be the set of all connected components of U, and ε be a function of $\mathscr{C}_U$ into $\{-1,1\}$. Define

$$T_U^\varepsilon(x,t) = \begin{cases} T_I(x,\varepsilon(I)t) & (x \in I \in \mathscr{C}_U), \\ x & (x \in \mathbb{R}\setminus U). \end{cases}$$

Then T_U^ε is a flow on $\mathbb{R}$, and its generator is the closure of $\lambda_U^\varepsilon D \mid C_c^\infty(\mathbb{R})$, where

$$\lambda_U^\varepsilon(x) = \begin{cases} \varepsilon((a,b))(x-a)(b-x) & (x \in (a,b) \in \mathscr{C}_U), \\ \varepsilon((a,\infty))(x-a) & (x \in (a,\infty) \in \mathscr{C}_U), \\ \varepsilon((-\infty,b))(b-x) & (x \in (-\infty,b) \in \mathscr{C}_U), \\ \varepsilon(\mathbb{R}) & (\text{if } U = \mathbb{R}), \\ 0 & (x \in \mathbb{R}\setminus U). \end{cases}$$

Let $\lambda: \mathbb{R} \to \mathbb{R}$ be any continuous function, and put

$$Z(\lambda) = \{x \in \mathbb{R}: \lambda(x) = 0\},$$
$$U(\lambda) = \mathbb{R} \setminus Z(\lambda) = \{x: \lambda(x) \neq 0\}.$$

For x in $U(\lambda)$, let

$$\alpha_x = \sup\{y < x: \lambda(y) = 0\},$$
$$\beta_x = \inf\{y > x: \lambda(y) = 0\}$$

with the convention that the supremum of the empty set is $-\infty$, and the infimum is $+\infty$.

Let $A_l^+(\lambda)$ (respectively, $A_l^-(\lambda)$) be the set of all points x in $Z(\lambda) \cup \{\infty\}$ such that for some $y < x$, $\lambda \geq 0$ (respectively, $\lambda \leq 0$) in (y, x) and $1/\lambda$ is integrable over (y, x). Let $A_r^+(\lambda)$ (respectively, $A_r^-(\lambda)$) be the set of all x in $Z(\lambda) \cup \{-\infty\}$ such that for some $z > x$, $\lambda \geq 0$ (respectively, $\lambda \leq 0$) in (x, z) and $1/\lambda$ is integrable over (x, z). Let

$$A_l(\lambda) = A_l^+(\lambda) \cup A_l^-(\lambda), \qquad A_r(\lambda) = A_r^+(\lambda) \cup A_r^-(\lambda),$$
$$A(\lambda) = A_l(\lambda) \cup A_r(\lambda).$$

The first lemma specifies the properties which amount to a flow on $\mathbb{R}$ having speed λ. The proof is elementary and will be omitted.

LEMMA 2.2. *Let T be a flow on $\mathbb{R}$, and $\lambda: \mathbb{R} \to \mathbb{R}$ be continuous. The following are equivalent*:

(i) *T is differentiable with respect to t, and $\partial T/\partial t = \lambda \circ T$,*

(ii) *$C_c^\infty(\mathbb{R}) \subset \mathscr{D}(\delta_T)$ and δ_T extends $\lambda D \mid C_c^\infty(\mathbb{R})$,*

(iii) *$C_c^1(\mathbb{R}) \subset \mathscr{D}(\delta_T)$ and δ_T extends $\lambda D \mid C_c^1(\mathbb{R})$,*

(iv) *If $x \in U(\lambda)$ and $T_t x \in (\alpha_x, \beta_x)$, then*

$$\int_x^{T_t x} \frac{dy}{\lambda(y)} = t;$$

if $x \in \operatorname{int} Z(\lambda)$, then $T_t x = x$.

COROLLARY 2.3. *Let T be a flow with speed λ (so that T satisfies the conditions of Lemma 2.2) and $x \in U_\lambda$. The following are equivalent*:

(i) $\{T_t x: t \in \mathbb{R}\} \subset U(\lambda)$,

(ii) $\{T_t x: t \in \mathbb{R}\} = (\alpha_x, \beta_x)$,

(iii) $\alpha_x \notin A_r(\lambda)$ *and* $\beta_x \notin A_l(\lambda)$.

The following result (for $[0,1]$ rather than $\mathbb{R}$) was included in [**6**], but no proof was given of the core property. The construction of T appeared earlier in [**11**].

THEOREM 2.4. *Let* $\lambda\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function. The following are equivalent*:

(i) *There is a flow* T *such that* δ_T *is the closure of* $\lambda D \mid C_c^\infty(\mathbb{R})$,

(ii) $A(\lambda) = \varnothing$.

Proof. (i) $\Rightarrow$ (ii). For y in $Z(\lambda)$, $(\delta_T f)(y) = 0$ for all f in $C_c^\infty(\mathbb{R})$, and hence for all f in $\mathscr{D}(\delta_T)$. It follows that $T_t y = y$. Thus for x in $U(\lambda)$, $\{T_t x\} \subset U(\lambda)$, so, by Corollary 2.3, $\alpha_x \notin A_r(\lambda)$ and $\beta_x \notin A_l(\lambda)$. Now if there exists z in $A_l(\lambda)$, then there exists x in $U(\lambda)$ such that $x < z$ and $1/\lambda$ is integrable over (x, z) and therefore over (x, β_x). But then $\beta_x \in A_l(\lambda)$, which is a contradiction. Similarly, $A_r(\lambda)$ is empty.

(ii) $\Rightarrow$ (i). For x in $U(\lambda)$, there is a (unique) function q such that $q(x) = 0$ and $q' = 1/\lambda$ in (α_x, β_x); q is injective, and, by assumption, q maps (α_x, β_x) onto $\mathbb{R}$. Define $T_t x = q^{-1}(t)$. For y in $Z(\lambda)$, define $T_t y = y$. It is easy to verify that T is a flow with speed λ.

The open set $U(\lambda)$ may be decomposed into a countable union of disjoint open intervals (a_i, b_i). Let $\mathscr{D}(\lambda)$ be the algebra of all functions f in $C_c^1(\mathbb{R})$ which are constant in some neighborhood of each a_i and in some neighborhood of each b_i. Since T fixes each a_i and each b_i, $\mathscr{D}(\lambda)$ is invariant under the dual action of T—the derivative of $f \circ T_t$ is $(\lambda \circ T_t)(f' \circ T_t)/\lambda$ on $U(\lambda)$. Since $\mathscr{D}(\lambda)$ is dense in $C_0(\mathbb{R})$, and contained in $\mathscr{D}(\delta_T)$, it follows that $\mathscr{D}(\lambda)$, and therefore $C_c^1(\mathbb{R})$, is a core for δ_T. Finally, given f in $C_c^1(\mathbb{R})$ with support in $[-N, N]$, there is a sequence f_n in $C_c^\infty(\mathbb{R})$ with support in $[-N, N]$ such that $\|f - f_n\| \to 0$, $\|f' - f_n'\| \to 0$. Then $\|\delta_T f_n - \delta_T f\| \to 0$. Thus $C_c^\infty(\mathbb{R})$ is a core for δ_T.

If $A(\lambda) \neq \varnothing$, there may or may not be a flow with speed λ, and any such flow may or may not be unique. Suppose for example that there exists x in $A_l^+(\lambda) \cap A_r^-(\lambda)$. Then any flow with speed λ would reach x from neighboring points on either side in a finite length of time, but would have no way of leaving x. So there is no flow with speed λ. On the other hand, if there are sufficiently many zeros of λ, a flow T may be delayed at the zeros. These delays are measured by μ where

$$\mu(I_T(x,t)) = |t| - \int_{I_T(x,t)} \frac{dy}{|\lambda(y)|} \tag{1}$$

for x in $U(\lambda)$, where $I_T(x, t)$ is the open interval between x and $T_t x$. Since the intervals $I_T(x, t)$ are disjoint from the fixed point space $\mathbb{R}_T^0$, there is no restriction on μ on $\mathbb{R}_T^0$, and, for standardisation, one may as well assume that $\mu(\mathbb{R}_T^0) = 0$. Thus a (positive) measure μ, defined on the

Borel subsets of $\mathbb{R}$, will be said to be a *delay measure* for T if (1) is satisfied and $\mu(\mathbb{R}_T^0) = 0$.

Conversely, it is possible to reconstruct T from μ by observing that $T_t x = y$ if $x < y$ and

$$\int_x^y \frac{dz}{\lambda(z)} + (\operatorname{sgn} t)\mu(x, y) = t.$$

This sets up a bijective correspondence between flows with speed λ and a certain class of measures, which have to be identified. A formal statement will be made in Theorem 2.5, for which the following notation and terminology is needed. As suggested above, finiteness of the delays and integrability of $1/\lambda$ on one side of a zero of λ has to be balanced on the other side with no change of sign of λ.

For a measure μ on $\mathbb{R}$, let $F_l(\mu)$ (respectively, $F_r(\mu)$) be the set of all x in $(-\infty, \infty]$ (respectively, $[-\infty, \infty)$) for which $\mu(y, x) < \infty$ for some $y < x$ (respectively, $\mu(x, z) < \infty$ for some $z > x$). Then μ will be said to be a *fluid measure* for λ if μ is non-atomic,

$$A_l^{\pm}(\lambda) \cap F_l(\mu) = A_r^{\pm}(\lambda) \cap F_r(\mu), \tag{2}$$

and μ is carried by $A_l(\lambda) \cap F_l(\mu)$ $(= A_r(\lambda) \cap F_r(\mu))$. Note that all these sets are Borel measurable, and that $A_l(\lambda) \setminus A_r(\lambda)$ etc. are countable and therefore null for measures μ which are non-atomic.

THEOREM 2.5. *Let λ: $\mathbb{R} \to \mathbb{R}$ be a continuous function. For any fluid measure μ for λ, there is a unique flow T on $\mathbb{R}$ with speed λ for which μ is a delay measure. Conversely, for any flow T with speed λ, there is a unique delay measure μ for T, and μ is a fluid measure for λ.*

Proof. For simplicity, we shall write A_l^+, F_l, etc. in place of $A_l^+(\lambda)$, $F_l(\lambda)$ etc., and put

$$V^+ = \{x: \lambda(x) \geq 0\}, \quad V^- = \{x: \lambda(x) \leq 0\},$$
$$U^+ = \{x: \lambda(x) > 0\}, \quad U^- = \{x: \lambda(x) < 0\}.$$

Let μ be a fluid measure. Define an equivalence relation on $\mathbb{R}$ by saying that points x and y with $x < y$ are equivalent if $\mu(x, y) < \infty$ and $1/\lambda$ is integrable over (x, y). Let C_x be the equivalence class of x; it is clear that C_x is some interval in $\mathbb{R}$. If C_x consists of the single point x, define $T_t x = x$. Otherwise, let a and b be the endpoints of C_x, so that $-\infty \leq a \leq x \leq b \leq \infty$. To define $T_t x$, the first stage is to show that C_x is contained in V^+ or in V^-. Suppose that there exist y^- in $C_x \cap U^-$ and y^+

in $C_x \cap U^+$, and suppose for the sake of argument that $y^- < y$. Let $y = \sup((y^-, y^+) \cap U^+)$, so that $y^- < y < y^+$. Then (y, y^+) is contained in V^+, and y is equivalent to y^+, so $y \in A_r^+ \cap F_r$. By (2), $y \in A_l^+$ which contradicts the fact that y is the limit of an increasing sequence in U^-.

Now suppose for the sake of argument that C_x is contained in V^+ (the other case is similar). If $a = x$, then $x \notin A_l^+ \cap F_l = A_r^+ \cap F_r$, so $b = x$. Thus we need only consider the case $a < x < b$. Define

$$\varphi(x') = \begin{cases} -\int_{x'}^{x} \frac{dy}{\lambda(y)} - \mu(x', x) & (a \leq x' \leq x), \\ \int_{x}^{x'} \frac{dy}{\lambda(y)} + \mu(x, x') & (x \leq x' \leq b). \end{cases}$$

By definition of the equivalence relation, and (2),

$$a \notin A_l \cap F_l \supset A_r^+ \cap F_r.$$

Since $(a, x) \subset V^+$, it follows that either $\mu(a, x) = \infty$ or $\int_a^x \lambda(y)^{-1}\,dy = \infty$, so $\varphi(a) = -\infty$. Similarly, $\varphi(b) = \infty$. In particular, neither a nor b is equivalent to x, so $C_x = (a, b)$.

Since μ is non-atomic, φ is continuous, and φ is clearly strictly increasing. Thus for each t in R, there is a unique point $T_t x$ in (a, b) such that $\varphi(T_t x) = t$, and $t \mapsto T_t x$ is a homeomorphism of $\mathbb{R}$ onto $(a, b) = C_x$. It is clear that $T_0 x = x$ and (1) holds.

If T is defined on $\mathbb{R} \times \mathbb{R}$ in this way, then for $s, t \geq 0$ and with the above notation and assumptions, using (1) with x replaced by $T_t x$,

$$\begin{aligned} \varphi(T_{s+t}x) = s + t &= \int_{T_t x}^{T_s T_t x} \frac{dy}{\lambda(y)} + \mu(T_t x, T_s T_t x) + t \\ &= \varphi(T_s T_t x) - \varphi(T_t x) + \varphi(T_t x) = \varphi(T_s T_t x), \end{aligned}$$

so $T_{s+t}x = T_s T_t x$. Dealing similarly with other cases, it follows that T satisfies the group property. Since T_t is an order-preserving homeomorphism of each C_x, it is a homeomorphism of $\mathbb{R}$. It is clear from the construction that $t \mapsto T_t x$ is continuous, so T is a continuous flow on $\mathbb{R}$. (For flows on $\mathbb{R}$, it is elementary to establish joint continuity from separate continuity, but flows on general spaces have the same property (see [**5**, Lemma 2.4]) for example).

For x in $A_l \cap F_l$, C_x is non-trivial, so x is not fixed by T. Thus $A_l \cap F_l$ is disjoint from $\mathbb{R}_T^0$ (actually $\mathbb{R}_T^0 = \mathbb{R} \setminus (A_l \cap F_l)$). Since μ is carried by $A_l \cap F_l$, μ is a delay measure. Since $\mu(U) = 0$, it follows from (1) and the construction that Lemma 2.2(iv) is satisfied, so that T has speed λ.

Let S be any flow with speed λ for which μ is a delay measure. For x in U^+, $S_t x$ increases with t for small t by Lemma 2.2(iv), and hence for all t (since $t \mapsto S_t x$ is either strictly monotone or constant by the group property). Now $S_t x$ is determined by (1). Similarly $S_t x$ is uniquely determined for x in U^-. Any interior point of Z is fixed under S. Thus $S_t x$ is uniquely determined for all x in a dense subset of $\mathbb{R}$, so by continuity S is unique.

Now let T be a flow with speed λ, let x be a point in $\mathbb{R} \setminus \mathbb{R}^0_T$ and C be the trajectory of x. Now $t \mapsto T_t x$ is injective, hence strictly monotone, and suppose for the sake of argument that it is increasing, so C is contained in V^+ by Lemma 2.2(i). If for some $\varepsilon > 0$ and $s_1 < s_2$, $\lambda(T_t x) < \varepsilon$ whenever $s_1 < t < s_2$, then by Lemma 2.2(iv),

$$T_{s_2} x - T_{s_1} x < \varepsilon(s_2 - s_1).$$

For $t_1 < t_2$, $\{ y \in (T_{t_1} x, T_{t_2} x) \colon \lambda(y) < \varepsilon\}$ is a countable union of disjoint intervals of the type $(T_{s_1} x, T_{s_2} x)$, so it follows that its Lebesgue measure is less than $\varepsilon(t_2 - t_1)$. Hence $Z \cap (T_{t_1} x, T_{t_2} x)$ is (Lebesgue) null.

If $\lambda(T_t x) > 0$ whenever $s_1' < t < s_2'$, then by Lemma 2.2(iv),

$$s_2' - s_1' = \int_{T_{s_1'} x}^{T_{s_2'} x} \frac{dy}{\lambda(y)}. \tag{3}$$

Now $U^+ \cap (T_{t_1} x, T_{t_2} x)$ is a countable union of disjoint intervals of the form $(T_{s_1'} x, T_{s_2'} x)$ and, taking the sum over these intervals and using the nullity of $Z \cap (T_{t_1} x, T_{t_2} x)$ gives

$$t_2 - t_1 \geq \int_{U \cap (T_{t_1} x, T_{t_2} x)} \frac{dy}{\lambda(y)} = \int_{T_{t_1} x}^{T_{t_2} x} \frac{dy}{\lambda(y)}. \tag{4}$$

Define a function F_C on C by

$$F_C(T_t x) = \begin{cases} t + \displaystyle\int_{T_t x}^{x} \frac{dy}{\lambda(y)} & (t \leq 0), \\ t - \displaystyle\int_{x}^{T_t x} \frac{dy}{\lambda(y)} & (t > 0). \end{cases}$$

Then F_C is continuous and (4) shows that F_C is increasing. So F_C determines a (positive) non-atomic Lebesgue-Stieltjes measure μ_C on C, and μ_C may be regarded as a measure on $\mathbb{R}$. Furthermore μ_C is independent of the choice of x in C, since replacing x by $T_t x$ alters F_C only by a constant. For $t > 0$ it is immediate that

$$\int_{x}^{T_t x} \frac{dy}{\lambda(y)} + \mu_C(x, T_t x) = t. \tag{5}$$

Also (3) shows that any compact subinterval of the open set $C \cap U^+$, and hence $C \cap U^+$ itself, is μ_C-null, so μ_C is carried by $C \cap Z$.

Similarly for a non-trivial trajectory C contained in V^-, one may construct a non-atomic measure μ_C, carried by $C \cap Z$, such that

$$(6) \qquad \int_x^{T_t x} \frac{dy}{\lambda(y)} - \mu_C(x, T_t x) = t \qquad (t < 0).$$

There are only countably many non-trivial trajectories C; let μ be the sum of all the corresponding measures μ_C. It is clear that $\mu(\mathbb{R}_T^0) = 0$, and (5) and (6) show that (1) also holds, so μ is a delay measure for T.

Suppose x is a point in Z with non-trivial trajectory C. Assuming that C is contained in V^+, (5) gives

$$\int_{T_{-1}x}^{T_1 x} \frac{dy}{\lambda(y)} + \mu_C(T_{-1}x, T_1 x) = 2,$$

so $x \in A_l^+ \cap F_l \cap A_r^+ \cap F_r$.

Now consider a point in $\mathbb{R}_T^0 \cap A_l^+$. For all sufficiently large $x' < x$, (x', x) is contained in V^+ and $1/\lambda$ is integrable over (x', x). Let x'' be any point of $U^+ \cap (x', x)$. The trajectory C of x'' is contained in $(-\infty, x)$, so

$$\mu(x', x) \geq \mu_C(x'', x) \geq \lim_{t \to \infty} \mu_C(x'', T_t x'')$$
$$= \lim_{t \to \infty} \left\{ t - \int_{x''}^{T_t x''} \frac{dy}{\lambda(y)} \right\} = \infty,$$

using (5) in the penultimate step. Thus $x \notin F_l$.

These and similar arguments show that

$$(A_l \cap F_l) \cup (A_r \cap F_r)$$
$$\subset Z \setminus \mathbb{R}_T^0 \subset \left[(A_l^+ \cap A_r^+) \cup (A_l^- \cap A_r^-) \right] \cap F_l \cap F_r.$$

Thus μ is a fluid measure.

Finally, let μ' be any delay measure for T. Then (1) shows that μ' is uniquely determined on any open subinterval of a non-trivial trajectory, and is σ-finite on the trajectory. Hence μ' is uniquely determined on each non-trivial trajectory. Since μ' is carried by the union of the countable set of non-trivial trajectories, it follows that μ' is unique. This completes the proof of Theorem 2.5.

From Theorem 2.5, it is a routine matter of measure theory to determine those λ for which there is a (unique) flow with speed λ.

COROLLARY 2.6. *There is at least one flow on* $\mathbb{R}$ *with speed* λ *if and only if* $(x, y) \cap Z(\lambda)$ *is uncountable whenever* $-\infty \leq x < y \leq \infty$ *and either* $x \in (A_r^+(\lambda) \setminus A_l^+(\lambda)) \cup (A_r^-(\lambda) \setminus A_l^-(\lambda))$ *or* $y \in (A_l^+(\lambda) \setminus A_r^-(\lambda)) \cup (A_l^-(\lambda) \setminus A_r^-(\lambda))$. *The flow is unique if and only if* $A_l^+(\lambda) = A_r^+(\lambda)$, $A_l^-(\lambda) = A_r^-(\lambda)$ *and* $A(\lambda)$ *is countable. If there are two distinct flows with speed* λ, *then there are uncountably many.*

If $\overline{\lambda D \mid C_c^\infty(\mathbb{R})}$ generates a C_0-semigroup τ, then the derivation law implies that τ_t is an endomorphism of $C_0(\mathbb{R})$. Since all C_0-groups of *-automorphisms arise from flows, Theorem 2.5 covers all cases when $\overline{\lambda D \mid C_c^\infty(\mathbb{R})}$ generates a C_0-group. A C_0-semigroup of endomorphisms corresponds to a half-flow T on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ which fixes $\pm\infty$, that is, a continuous mapping $T: \overline{\mathbb{R}} \times [0, \infty) \to \overline{\mathbb{R}}$ such that

$$T_0 x = x, \quad T_s T_t = T_{s+t}, \quad T_t \infty = \infty, \quad T_t(-\infty) = -\infty.$$

The analogue of Theorem 2.4 follows.

PROPOSITION 2.7. *Let* $\lambda: \mathbb{R} \to \mathbb{R}$ *be continuous. The following are equivalent*:

(i) $\overline{\lambda D \mid C_c^\infty(\mathbb{R})}$ *generates a* C_0*-semigroup on* $C_0(\mathbb{R})$,

(ii) $A_r^+(\lambda) = A_l^-(\lambda) = \varnothing$.

The C_0-semigroup in Proposition 2.7 arises from a half-flow on $\mathbb{R}$ (as opposed to $\overline{\mathbb{R}}$) if and only if $-\infty \notin A_r^-(\lambda)$ and $\infty \notin A_l^+(\lambda)$, that is, $1/\lambda$ is not integrable at $\pm\infty$.

All the results of this section have analogues for $\mathbb{T}$ $(= \mathbb{R}/\mathbb{Z})$ and $[0, 1]$, provided that $A_l^+(\lambda)$ etc. are interpreted correctly. For $\mathbb{T}$, regard λ: $\mathbb{T} \to \mathbb{R}$ as a periodic function on $\mathbb{R}$ and let $A_l^+(\lambda)$ consist of those x in $Z(\lambda)$ such that for some $y < x$, $\lambda \geq 0$ in (y, x) and $1/\lambda$ is integrable over (y, x), etc. The statements of Theorems 2.4 and 2.5 and Proposition 2.7 are almost unchanged. For $[0, 1]$, let $A_l^+(\lambda)$ consist of those $x \neq 0$ in $Z(\lambda)$ such that, for some $0 < y < x$, $\lambda \geq 0$ in (y, x) and $1/\lambda$ is integrable over (y, x); let $A_r^+(\lambda)$ consist of those $x \neq 1$ in $Z(\lambda)$ such that for some $x < z < 1$, $\lambda \leq 0$ in (x, z) and $1/\lambda$ is integrable over (x, z), etc. The statements of Theorem 2.4 become:

(i) There is a flow T on $[0, 1]$ such that $\delta_T = \overline{\lambda D}$,

(ii) $A(\lambda) = \varnothing$; $\lambda(0) = \lambda(1) = 0$.

Theorem 2.5 is valid, but only for functions satisfying $\lambda(0) = \lambda(1) = 0$. The conditions of Proposition 2.7 are:

(i) $\overline{\lambda D}$ generates a C_0-semigroup on $C[0, 1]$,

(ii) $A_r^+(\lambda) = A_l^-(\lambda) = \varnothing$; $\lambda(0) \geq 0$, $\lambda(1) \leq 0$.

This answers a question raised in [**6**]. In particular, Theorem 4 of [**6**] remains valid if the assumption that the derivation is well-behaved is dropped, provided that the assertion that $p(0) = p(1) = 0$ is replaced by the conditions $p(0) \geq 0$, $p(1) \leq 0$. Some of the claims made in [**6**] about the example on p. 77 are incorrect, and the true position is set out below. (In comparing this paper with [**6**], the reader should bear in mind that there is a difference in sign conventions in defining generators.)

EXAMPLE 2.8 [**6**, p. 77]. Consider $\lambda\colon [0,1] \to \mathbb{R}$ defined by $\lambda(x) = -2x^{1/2}$. Then

$$A_r^-(\lambda) = \{0\}, \quad A_r^+(\lambda) = A_l^-(\lambda) = A_l^+(\lambda) = \varnothing.$$

Thus condition (ii) is satisfied, and $\overline{\lambda D}$ is the generator of the half-flow T^-, where

$$T_t^- x = \left(\max(x^{1/2} - t, 0)\right)^2.$$

On the other hand, $-\lambda$ does not satisfy (ii) because $-\lambda(1) < 0$ and $0 \in A_r^+(-\lambda)$. The half-flow T^+ defined by

$$T_t^+ x = \left(\min(x^{1/2} + t, 1)\right)^2$$

satisfies

$$\delta_{T^+} f(x) = -\lambda(x) f'(x)$$

for $0 < x < 1$, but behaves differently at both endpoints.

3. General spaces. Let S be a flow on a locally compact Hausdorff space X, with fixed point set X_S^0, and let $\lambda\colon X \setminus X_S^0 \to \mathbb{R}$ be a continuous function. The problem now is to determine conditions under which there is a flow with "speed λ relative to S", and how such flows behave at the points of X_S^0. The first result interprets the relative speed in two different, but equivalent, ways.

PROPOSITION 3.1. *Let T be a flow on X, and $\lambda\colon X \setminus X_S^0 \to \mathbb{R}$ be a continuous function. The following are equivalent*:

(i) *For $\omega \in \operatorname{int} X_S^0$, $T_t\omega = \omega$; for $\omega \in X \setminus X_S^0$, there is a function τ_ω: $\mathbb{R} \to \mathbb{R}$ such that $T_t\omega = S_{\tau_\omega(t)}\omega$ ($t \in \mathbb{R}$) and $\tau'_\omega(0) = \lambda(\omega)$,*

(ii) *If $f \in \mathcal{D}(\delta_S)$ and $g \in C_0(X)$ are such that*

$$g = \begin{cases} \lambda\delta_S f & \text{on } X \setminus X_S^0 \\ 0 & \text{on } X_S^0, \end{cases}$$

then $f \in \mathcal{D}(\delta_T)$ and $\delta_T f = g$.

Proof. (i) ⇒ (ii). This is a standard argument, but the details are included for completeness. For ω in $X \setminus X_S^0$,

$$\lim_{t\to 0} \frac{f(T_t\omega) - f(\omega)}{t} = \frac{d}{dt}\Big(f\big(S_{\tau_\omega(t)}\omega\big)\Big)\Big|_{t=0}$$
$$= \tau_\omega'(0)\delta_S f(\omega) = g(\omega).$$

Replacing ω by $T_s\omega$, it follows that

$$\left|\frac{f(T_t\omega) - f(\omega)}{t} - g(\omega)\right| = \left|\frac{1}{t}\int_0^t \left\{\frac{d}{ds}(f(T_s\omega)) - g(\omega)\right\} ds\right|$$
$$\leq \frac{1}{|t|}\int_0^t |g(T_s\omega) - g(\omega)|\, ds \leq \sup_{|s|\leq|t|} \|g \circ T_s - g\|.$$

By continuity, this estimate remains valid for ω in $\overline{X \setminus X_S^0}$, while it is trivially valid for ω in $\operatorname{int} X_S^0$. Thus

$$\|t^{-1}(f \circ T_t - f) - g\| \leq \sup_{|s|\leq|t|} \|g \circ T_s - g\| \to 0 \quad \text{as } t \to 0.$$

Thus $f \in \mathscr{D}(\delta_T)$, and $\delta_T f = g$.

(ii) ⇒ (i). Firstly, consider ω in $X \setminus X_S^0$. The argument used in [3] to show that $\{T_t\omega\} \subset \{S_s\omega\}$ is still valid, so there is a function τ_ω such that $T_t = S_{\tau_\omega(t)}\omega$. Furthermore, τ_ω is uniquely determined modulo the S-period of ω, and one may (uniquely) arrange that τ_ω is continuous and $\tau_\omega(0) = 0$. It was shown in [4, Theorem 2.1] that there exists f in $\mathscr{D}(\delta_S)$ such that $f(S_s\omega) = s$ for all small $|s|$, and $\operatorname{supp} f \subset X \setminus X_S^0$. It follows from (ii) that $f \in \mathscr{D}(\delta_T)$ and

$$\lambda(\omega) = (\delta_T f)(\omega) = \lim_{t\to 0} \frac{\tau_\omega(t)}{t} = \tau_\omega'(0).$$

Next, for any function h in $C_0(X)$ with $\operatorname{supp} h$ contained in $\operatorname{int} X_S^0$, it follows from (ii) that $h \in \mathscr{D}(\delta_T)$ and $\delta_T h = 0$. The local nature of δ_T ensures that each point of $\operatorname{int} X_S^0$ is fixed by T.

REMARK. The class $\mathscr{D}$ of functions f which satisfy condition (ii) of Proposition 3.1 is a *-subalgebra of $\mathscr{D}(\delta_S)$, but it may not separate the points of X_S^0. Furthermore the flow T may not fix every point of X_S^0 (so that T may not be a "fluctuation" of S in the sense of [2]). For example, let $X = \mathbb{R}^2$, $S_t(x, y) = (x + ty, y)$, $T_t(x, y) = (x + t, y)$. Here $X_S^0 = \mathbb{R} \times \{0\}$ and $\lambda(x, y) = 1/y$ $(y \neq 0)$, while $\mathscr{D}$ fails to separate any points of X_S^0.

A sufficient condition that T fixes each point of X_S^0 is condition (i) in Theorem 3.2 below (see [3] and the proof of Theorem 3.2). Sufficient conditions that $\mathscr{D}$ is a core for δ_T (in particular, $\mathscr{D}$ separates the points of X, and T fixes X_S^0) were given in [3, 7, 8].

THEOREM 3.2. *Let* $\lambda\colon X \setminus X_S^0$ *be a continuous function, and suppose that*

(i) *For any compact set* $K \subset X$, *there exists* $\varepsilon > 0$ *such that* λ *is bounded on* $\{\omega \in K \setminus X_S^0\colon \nu(\omega) < \varepsilon\}$,

(ii) *If* $\lambda(\omega) = 0$ *for some* ω *in* $X \setminus X_S^0$, *then* $t \mapsto \lambda(S_t\omega)^{-1}$ *is not integrable over* $(0, a)$ *or over* $(-a, 0)$ *for any* $a > 0$,

(iii) *For any* ω *in* $X \setminus X_S^0$, $t \mapsto \lambda(S_t\omega)^{-1}$ *is not integrable over* $(0, \infty)$ *or over* $(-\infty, 0)$.

Then there is a unique flow T *on* X *with speed* λ *relative to* S (*so that the conditions of Proposition* 3.1 *are valid*).

Proof. For ω in $X \setminus X_S^0$, let $\lambda_\omega(t) = \lambda(S_t\omega)$. It follows from assumptions (ii) and (iii) and Theorem 2.5 that there is a unique flow θ_ω on $\mathbb{R}$ with speed λ_ω. This flow is characterised by the properties:

$$x \text{ is a fixed point of } \theta_\omega \Leftrightarrow \lambda(S_x\omega) = 0,$$

$$\int_x^{\theta_\omega(x,t)} \frac{ds}{\lambda(S_s\omega)} = t \quad \text{if } \lambda(S_x\omega) \neq 0.$$

The uniqueness of the flows, together with the relation

$$\lambda_{S_t\omega}(x) = \lambda_\omega(x+t),$$

ensures that the flows θ_ω are coherent in the sense that

$$\theta_{S_t\omega}(x, s) + t = \theta_\omega(x+t, s).$$

Let $\tau_\omega(t) = \theta_\omega(0, t)$ and

$$T_t\omega = \begin{cases} S_{\tau_\omega(t)}\omega & (\omega \in X \setminus X_S^0), \\ \omega & (\omega \in X_S^0). \end{cases}$$

Then T satisfies the group property $T_sT_t = T_{s+t}$.

In order to show that T is a flow, it remains to show that $(\omega, t) \mapsto T_t\omega$ is jointly continuous. Let (ω_α) and (t_α) be nets such that $\omega_\alpha \to \omega$, $t_\alpha \to t$. By passing to subnets and replacing λ by $-\lambda$, it suffices to assume that $t_\alpha \geq 0$ and to consider six cases:

1. $\omega_\alpha \in X_S^0$;
2. $\omega_\alpha \in X \setminus X_S^0$, $\lambda(\omega_\alpha) = 0$;

3. $\omega_\alpha \in X \setminus X_S^0$, $\omega \in X \setminus X_S^0$, $\lambda(\omega_\alpha) > 0$, $\lambda(\omega) > 0$, $\tau_{\omega_\alpha}(t_\alpha) \to \tau$, where $0 \leq \tau \leq \infty$;

4. $\omega_\alpha \in X \setminus X_S^0$, $\omega \in X \setminus X_S^0$, $\lambda(\omega_\alpha) > 0$, $\lambda(\omega) = 0$;

5. $\omega_\alpha \in X \setminus X_S^0$, $\omega \in X_S^0$, $\lambda(\omega_\alpha) > 0$, $\nu(\omega_\alpha) > \nu$, where $\nu > 0$;

6. $\omega_\alpha \in X \setminus X_S^0$, $\omega \in X_S^0$, $\lambda(\omega_\alpha) > 0$, $\nu(\omega_\alpha) \to 0$.

Cases 1 *and* 2. Since X_S^0 is closed and λ is continuous, either $\omega \in X_S^0$ or $\lambda(\omega) = 0$. Thus

$$T_{t_\alpha}\omega_\alpha = \omega_\alpha \to \omega = T_t\omega.$$

Case 3. Firstly, suppose that $\tau > \tau_\omega(t)$. Then, by construction of τ_ω, there exists θ such that $\tau_\omega(t) < \theta < \tau$, $\lambda(S_s\omega) > 0$ for $0 \leq s \leq \theta$. Since S is jointly continuous, $\lambda(S_s\omega_\alpha)^{-1} \to \lambda(S_s\omega)^{-1}$ as $\alpha \to \infty$ uniformly for $0 \leq s \leq \theta$, and therefore

$$\int_0^\theta \frac{ds}{\lambda(S_s\omega_\alpha)} \to \int_0^\theta \frac{ds}{\lambda(S_s\omega)}.$$

But for large α, $\tau_\omega(t) < \theta < \tau_{\omega_\alpha}(t_\alpha)$, so

$$t_\alpha > \int_0^\theta \frac{ds}{\lambda(S_s\omega_\alpha)} \to \int_0^\theta \frac{ds}{\lambda(S_s\omega)} > t.$$

This is a contradiction, so it follows that $\tau \leq \tau_\omega(t)$. For all sufficiently small $\theta' > \tau$, $\lambda(S_s\omega) > 0$ for $0 \leq s \leq \theta'$, and the same argument as above shows that

$$t_\alpha \leq \int_0^{\theta'} \frac{ds}{\lambda(S_\alpha\omega_\alpha)} \to \int_0^{\theta'} \frac{ds}{\lambda(S_s\omega)}.$$

Hence $\theta' \geq \tau_\omega(t)$. Since $\theta' > \tau$ is arbitrarily small, it follows that $\tau \geq \tau_\omega(t)$. Thus $\tau = \tau_\omega(t)$ and

$$T_{t_\alpha}\omega_\alpha = S_{\tau_{\omega_\alpha}(t_\alpha)}\omega_\alpha \to S_\tau\omega = S_{\tau_\omega(t)}\omega = T_t\omega.$$

Case 4. By assumption (ii), for any $\eta > 0$, $\int_0^\eta |\lambda(S_s\omega)|^{-1}\,ds = \infty$, and therefore

$$\lim_{\varepsilon \to 0+} \int_0^\eta \frac{ds}{|\lambda(S_s\omega)| + \varepsilon} = \infty.$$

Since $(|\lambda(S_s\omega_\alpha)| + \varepsilon)^{-1} \to (|\lambda(S_s\omega)| + \varepsilon)^{-1}$ uniformly on $(0, \eta)$, it follows that

$$\lim_{\varepsilon \to 0+} \lim_{\alpha \to \infty} \int_0^\eta \frac{ds}{|\lambda(S_s\omega_\alpha)| + \varepsilon} = \infty.$$

It follows that

$$\lim_{\alpha\to\infty}\int_0^\eta \frac{ds}{|\lambda(S_s\omega_\alpha)|} = \infty$$

and therefore $\tau_{\omega_\alpha}(t_\alpha) < \eta$ for large α. Thus $\tau_{\omega_\alpha}(t_\alpha) \to 0$, so

$$T_{t_\alpha}\omega_\alpha = S_{\tau_{\omega_\alpha}(t_\alpha)}\omega_\alpha \to \omega = T_t\omega.$$

Case 5. For each α,

$$\tau_{\omega_\alpha}(t_\alpha) = m_\alpha \nu(\omega_\alpha)^{-1} + \theta_\alpha$$

where m_α is an integer, $0 \le \theta_\alpha < \nu(\omega_\alpha)^{-1} \le \nu^{-1}$. Passing to a subnet, one may assume that $\theta_\alpha \to \theta$. Then

$$T_{t_\alpha}\omega_\alpha = S_{\tau_{\omega_\alpha}(t_\alpha)}\omega_\alpha = S_{\theta_\alpha}\omega_\alpha \to S_\theta\omega = \omega.$$

Case 6. Let K be any compact neighbourhood of ω, and let

$$\tau_\alpha = \inf\{t > 0\colon S_t\omega_\alpha \notin K\}.$$

Supose that $\tau_\alpha \to \tau < \infty$. Then $S_{\tau_\alpha}\omega_\alpha \to S_\tau\omega = \omega$, so $\omega \in \overline{X \setminus K}$. This is a contradiction. It follows (on passing to subnets) that $\tau_\alpha \to \infty$.

By assumption (i), there is a constant c such that $|\lambda(S_s\omega_\alpha)| \le c$ whenever $0 \le s \le \tau_\alpha$, so that, for any $\eta > 0$,

$$\int_0^\eta \frac{ds}{|\lambda(S_s\omega_\alpha)|} \ge \frac{\eta}{c}$$

for all sufficiently large α. In particular, $\tau_{\omega_\alpha}(t_\alpha) \le ct_\alpha$. Passing to a subnet, one may assume that $\tau_{\omega_\alpha}(t_\alpha) \to \tau < \infty$. Then

$$T_{t_\alpha}\omega_\alpha \to S_\tau\omega = \omega = T_t\omega.$$

It is clear that T satisfies condition (i) of Proposition 3.1, and it remains only to establish uniqueness. If $\tilde{T}$ is any flow with relative speed λ, then for ω in $X \setminus X_S^0$, there is a unique continuous function $\tilde{\tau}_\omega\colon \mathbb{R} \to \mathbb{R}$ such that $\tilde{\tau}_\omega(0) = 0$ and $T_t\omega = S_{\tilde{\tau}_\omega(t)}\omega$. Furthermore $\tau_\omega'(0) = \lambda(\omega)$. The uniqueness ensures that

$$\tilde{\tau}_\omega(s+t) = \tilde{\tau}_\omega(s) + \tilde{\tau}_{S_{\tilde{\tau}_\omega(s)}\omega}(t)$$

and therefore there is a flow $\tilde{\theta}_\omega$ on $\mathbb{R}$ given by

$$\tilde{\theta}_\omega(x,t) = \tilde{\tau}_{S_x\omega}(t) + x.$$

Now $\tilde{\theta}_\omega$ has speed λ_ω, and it follows from the uniqueness of flows with speed λ_ω that $\tilde{\theta}_\omega = \theta_\omega$. In particular

$$\tilde{\tau}_\omega(t) = \tilde{\theta}_\omega(0, t) = \theta_\omega(0, t) = \tau_\omega(t),$$

so $\tilde{T}_t\omega = T_t\omega$ ($\omega \in X \setminus X_S^0$).

For $\omega \in \operatorname{int} X_S^0$, $\tilde{T}_t\omega = \omega = T_t\omega$. Thus $\tilde{T}_t$ and T_t coincide on a dense subset of X, and therefore $\tilde{T} = T$.

REMARK. Under the assumptions of Theorem 3.2, the algebra $\mathscr{D}$ considered in the remark following Proposition 3.1 equals $\mathscr{D}(\delta_S) \cap \mathscr{D}(\delta_T)$, but it is still unclear whether it is automatically a core for δ_T. Let

$$\mathscr{D}_0 = \{ f \in \mathscr{D} \colon f(S.\omega) \in \mathscr{D}(\lambda_\omega) \text{ for all } \omega \in X \setminus X_S^0, \quad f \text{ has compact support} \},$$

where $\mathscr{D}(\lambda_\omega)$ is as defined in the proof of Theorem 2.4. Then $\mathscr{D}_0$ is a T-invariant *-subalgebra of $\mathscr{D}$, but it is not clear that $\mathscr{D}_0$ separates the points of X. If so, then $\mathscr{D}$ is a core for δ_T.

EXAMPLE 3.3. In Theorem 3.2, it is not possible to replace (ii) and (iii) by the weaker assumption

(iii)′ For each ω in $X \setminus X_S^0$, there is a unique flow on $\mathbb{R}$ with speed λ_ω (where $\lambda_\omega(t) = \lambda(S_t\omega)$),

even if (i) is replaced by the stronger assumption that λ is bounded. For example, let

$$X = \mathbb{R} \times [0,1], \qquad S_t(x, y) = (x + t, y)$$

$$\lambda(x, y) = \begin{cases} \dfrac{|x|^{1/2}}{1 + (1/y + 1)(1 - |x|)^{1/y}|x|^{1/2}} & (|x| \le 1, y \neq 0), \\ |x|^{1/2} & (|x| \le 1,\ y = 0), \\ 1 & (|x| \ge 1). \end{cases}$$

Then

$$\int_0^2 \frac{dx}{\lambda(x,0)} = 3 = \int_0^1 \frac{dx}{\lambda(x,y)} \qquad (y \neq 0).$$

Since $Z(\lambda_{(0,y)}) = A_l^+(\lambda_{(0,y)}) = A_r^+(\lambda_{(0,y)}) = \{0\}$ and $A_l^-(\lambda_{(0,y)}) = A_r^-(\lambda_{(0,y)}) = \varnothing$, there is a unique measure μ satisfying the conditions of Theorem 2.5 for $\lambda = \lambda_{(0,y)}$, namely $\mu = 0$. The corresponding flow θ_y on $\mathbb{R}$ satisfies

$$\theta_y(s, t) = s + \tau_y(t) \quad \text{where } \int_0^{\tau_y(t)} \frac{dx}{\lambda(x, y)} = t.$$

If T is any flow on $\mathbb{R}$ satisfying the conditions of Proposition 3.1, then T induces flows $\tilde{\theta}_y$ on $\mathbb{R}$ such that

$$T_t(x, y) = \left(\tilde{\theta}_y(x, t), y\right),$$

and $\tilde{\theta}_y$ has speed $\lambda_{(0,y)}$. Hence $\tilde{\theta}_y = \theta_y$, so

$$T_3(0, y) = \left(\tau_y(3), y\right) = (1, y) \qquad (y \neq 0)$$
$$T_3(0,0) = \left(\tau_0(3), 0\right) = (2,0).$$

This contradicts the continuity of T.

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