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A GENERALIZATION OF A THEOREM OF DELAUNAY TO ROTATIONAL W-HYPERSURFACES OF σ_l-TYPE IN H^{n+1} AND S^{n+1}

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In 1841 Delaunay proved that if one rolls a conic section on a line in a plane and then rotates about that line the trace of a focus, one obtains a constant mean curvature surface of revolution in R^3. Conversely, all such surfaces, except spheres, are constructed in this way. In 1981, Hsiang and Yu generalized Delaunay's theorem to constant mean curvature rotation hypersurfaces in R^{n+1}. In 1982, Hsiang further generalized Delaunay's theorem to rotational W-hypersurfaces of σ_l-type in R^{n+1}. These are hypersurfaces such that the lth-basic symmetric polynomial of the principal curvatures $(k_i(x))$, namely,

$$\sigma_l(k_1,\dots,k_n) = \sum_{i_1<\cdots<i_l} k_{i_1}\cdots k_{i_l}, \qquad 1\le l\le n,$$

is constant.

Here we generalize Delaunay's theorem to rotational W-hypersurfaces of σ_l-type in hyperbolic $(n+1)$-space H^{n+1} and spherical $(n+1)$-space S^{n+1}. Specifically we generalize the "rolling construction" of Delaunay. Various geometrical properties of these surfaces and their generating curves have been studied by Hsiang.

1. The reduced ODE of rotational σ_l-W-hypersurfaces in R^{n+1}, H^{n+1}, or S^{n+1}. Following Hsiang [4], in this section we shall give a unified treatment of the reduced ODE of rotational σ_l-W-hypersurfaces in a space form of constant curvature. In order to do so, we shall first give a unified description of the orbital geometry of the $O(n)$ transformation on the simply connected $(n+1)$-dimensional space of constant curvature c, $M^{n+1}(c)$.

Orbital geometry of the $O(n)$-action on $M^{n+1}(c)$. Let $M^{n+1}(c)$ be the simply connected $(n+1)$-dimensional Riemannian space of constant sectional curvature c and $O(n)$ be an isometric transformation group on $M^{n+1}(c)$ fixing a given geodesic line, namely, a rotational transformation group of the usual type fixing a rotational axis M^1. Then $(O(n), M^{n+1}(c))$ consists of only two types of orbits, namely, fixed points and orbits of the type $S^{n-1} = O(n)/O(n-1)$. Let $O(n-1)$ be an arbitrarily chosen and then fixed principal isotropy subgroup of $(O(n), M^{n+1}(c))$, $Z_2 = N(O(n-1))/O(n-1)$ and $M^2(c) = F(O(n-1))$. Then it is easy to see that the upper half-plane $M^2_+(c)$ consists of a fundamental domain of $(O(n), M^{n+1}(c))$ which is perpendicular to all orbits. Hence

$$M^{n+1}(c)/O(n) = M^2(c)/Z_2 = M^2_+(c),$$

where $F(O(n)) = M^1 =$ the boundary of $M^2_+(c)$. We shall parametrize $M^2_+(c)$ by the following coordinate system:

Choose a base point $0 \in M^1$ and let x be the arc length on M^1 travelling in the positive orientation of $M^1 = \partial M^2_+(c)$. To each point $p \in M^2_+(c)$, let $\overline{pq}$ be a geodesic arc which realizes the shortest distance between p and M^1 (such a $\overline{pq}$ is unique except when p is the center of $M^2_+(c)$, $c > 0$). We shall assign to the point p the coordinate (x, y) where x is the coordinate of q in M^1 and $y =$ the length of $\overline{pq}$. It follows from the above definition that

$$\begin{cases} -\infty < x < +\infty, & 0 \le y \le +\infty \text{ if } c \le 0, \\ -\dfrac{\pi}{\sqrt{c}} \le x \le \dfrac{\pi}{\sqrt{c}}, & 0 \le y \le \dfrac{\pi}{\sqrt{c}} \text{ if } c > 0. \end{cases}$$

(In the case $c > 0$, the coordinate of the center of $M^2_+(c)$ is $(x, \pi/\sqrt{c})$, x arbitrary, and hence non-unique.)

Let $G(p(x, y))$ be the orbit of $p(x, y)$. Then it is not difficult to show that $G(p(x, y))$ is isometric to the $(n-1)$-sphere of radius $f(y)$ where

$$f(y) = \begin{cases} y & \text{if } c = 0 \text{ (euclidean case)}, \\ \dfrac{1}{\sqrt{c}} \sin \sqrt{c}\, y & \text{if } c > 0 \text{ (spherical case)}, \\ \dfrac{1}{\sqrt{-c}} \sinh \sqrt{-c}\, y & \text{if } c < 0 \text{ (hyperbolic case)}. \end{cases}$$

Moreover, the orbital distance metric on $M^{n+1}(c)/O(n)$ is the same as the restriction metric of $M^2_+(c)$ and hence it can be given in terms of the coordinates (x, y) as follows:

$$ds^2 = \begin{cases} dx^2 + dy^2 & \text{if } c = 0 \text{ (euclidean case)}, \\ \cos^2 \sqrt{c}\, y \cdot dx^2 + dy^2 & \text{if } c > 0 \text{ (spherical case)}, \\ \cosh^2 \sqrt{-c}\, y \cdot dx^2 + dy^2 & \text{if } c < 0 \text{ (hyperbolic case)}, \end{cases}$$

Reduced ODE. Let Σ^n be a given $O(n)$-invariant hypersurface in $M^{n+1}(c)$, $\gamma = \Sigma^n/O(n) \subset M^{n+1}(c)/O(n) = M^2_+(c)$ be the generating curve of Σ^n and II be the second fundamental form of Σ^n at $\gamma(x) = (x(s), y(s)) \neq 0$. Then it is easy to see that II is $O(n-1)$-invariant and hence it has only two distinct eigenvalues corresponding to the two non-conjugate $O(n-1)$-invariant subspaces of the tangent space of Σ^n of $\gamma(s)$.

PROPOSITION 1 [4]. *The principal curvatures of* Σ^n *at* $\gamma(s)$ *are given as follows*:

$$k_0 = \frac{d\sigma}{ds} - f''(y)\frac{dx}{ds}, \quad \textit{multiplicity } 1,$$
$$k_1 = -\cos\sigma\frac{f'(y)}{f(y)}, \quad \textit{multiplicity } (n-1),$$

where σ *is the angle between* $\partial/\partial x$ *and the tangent vector of* γ.

Suppose Σ^n *is a rotational* $\sigma_r W$*-hypersurface in* $M^{n+1}(c)$. *Then its generating curve* $\gamma = \Sigma^n/O(n)$ *is a curve in* $M_+^2(c)$ *which is characterized by the following ODE, namely*

$$(\mathrm{I}_l) \qquad \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}\left(-\cos\sigma\frac{f'(y)}{f(y)}\right)^{l-1}\cdot\left(\frac{d\sigma}{ds} - F''(y)\frac{dx}{ds}\right) + \begin{bmatrix} n-1 \\ l \end{bmatrix}\left(-\cos\sigma\frac{f'(y)}{f(y)}\right)^{l} = (-1)^l\begin{bmatrix} n-1 \\ l \end{bmatrix}h_l, \quad 1 \le l \le n-1,$$

$$(\mathrm{I}_n) \qquad \left(-\cos\sigma\frac{f'(y)}{f(y)}\right)^{n-1}\left(\frac{d\sigma}{ds} - f''(y)\frac{dx}{ds}\right) = (-1)^n h_n, \qquad l = n,$$

where h_l *is the normalized lth mean curvature of the* $\sigma_r W$*-hypersurface* Σ^n.

REMARK. In the case of l odd, the normalized lth mean curvature, h_l, changes its sign if one reverses the orientation of the hypersurface. Here we shall always choose the orientation so that $h_l \ge 0$.

2. The rolling construction and a generalization of Delaunay's theorem in H^{n+1} and S^{n+1}.

REMARK. The constructions, theorems, and proofs are similar for the hyperbolic and spherical cases. Hence we carry out the details only in the hyperbolic case. After the main theorem we will state the corresponding result for the spherical case.

REMARK. For simplicity and comparison we follow the format of Hsiang [2, §3].

Since all $M^{n+1}(c)$, $c < 0$, are obviously homothetically equivalent, it is easy to reduce our investigation to the special case of $c = -1$. Therefore, in this section, we shall always assume that $c = -1$ and denote $M^{n+1}(-1)$ simply by H^{n+1}.

Rolling construction. Suppose Γ is a curve in $M^2_+(-1)$ given as a geodesic polar coordinate graph of $r = r(\theta)$. If one rolls Γ along the x-axis, then the locus of the origin of the geodesic polar coordinate system attached to Γ plots another curve Ω. As indicated in Figure B, one has the following geometric relationship between Γ and Ω:

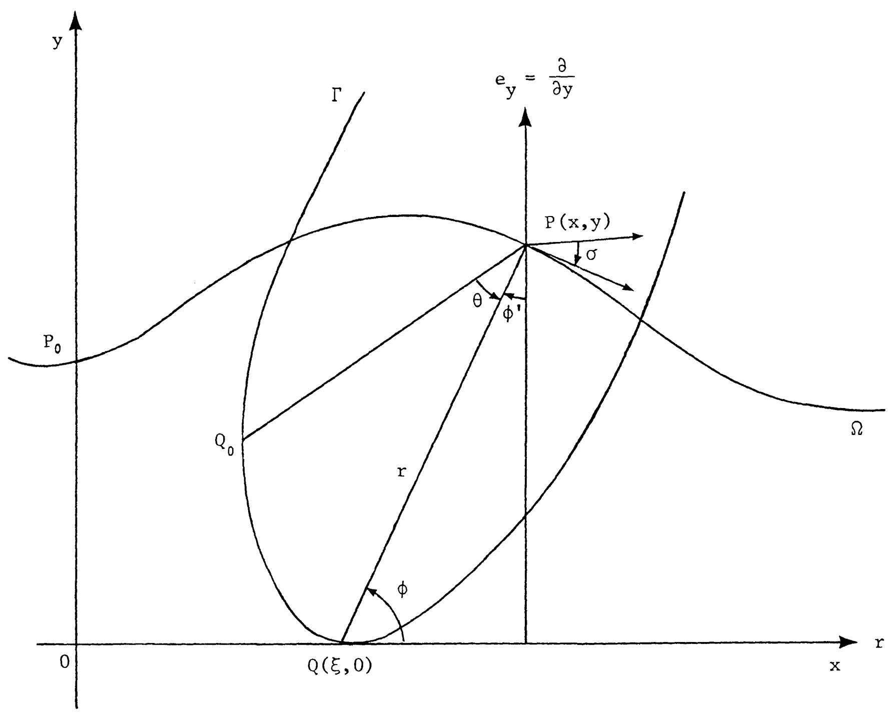

FIGURE B

Let s and ξ be the respective arc length parameters of Ω and Γ starting at a pair of corresponding points P_0 and Q_0. Let ϕ be the angle between $\overline{Ox}$ and $\overline{QP}$ and ϕ' be the angle between $-\partial/\partial y$ and $\overline{PQ}$. Then

$$\frac{dy}{ds} = \sin\sigma, \quad \cosh y\,\frac{dx}{ds} = \cos\sigma, \tag{2.0}$$

thus $dy/dx = \cosh y \cdot \tan\sigma$ and by hyperbolic trigonometry,

$$(2.1)\qquad \begin{array}{ll} \text{(A)} & \cos\phi = \tanh(x-\xi)\cdot\coth r \\ \text{(B)} & \sinh y = \sin\phi\cdot\sinh r \\ \text{(C)} & dr = -\cos\phi\, d\xi \\ \text{(D)} & \cosh r = \cot\phi\cdot\cot\phi' \\ \text{(E)} & \cos\phi' = \tanh y\cdot\coth r \\ \text{(F)} & \cos\phi = \sin\phi'\cdot\cosh y. \end{array}$$

Differentiating (A) and (B) gives

$$\text{(A}'\text{)}\qquad -\sin\phi\, d\phi = -\tanh(x-\xi)\cdot\operatorname{csch}^2 r\, dr + \operatorname{sech}^2(x-\xi) \cdot\coth r(dx - d\xi)$$

$$\text{(B}'\text{)}\qquad \cosh y\, dy = \sin\phi\cdot\cosh r\, dr + \cos\phi\cdot\sinh r\, d\phi.$$

Substituting (A) in (A′) and (B) in (B′) gives

$$\text{(A}''\text{)}\qquad -\sin\phi\, d\phi = -\cos\phi\cdot\operatorname{sech} r\cdot\operatorname{csch} r\, dr + (\coth r - \tanh r\cdot\cos^2)(dx - d\xi)$$

$$\text{(B}''\text{)}\qquad \cos\phi\, d\phi = \operatorname{csch} r\cdot\cosh y\, dy - \sin\phi\cdot\coth r\, dr.$$

(A″) and (B″) combined give

$$(*)\qquad \big(-\cos\phi\cdot\operatorname{sech} r\cdot\operatorname{csch} r\, dr + (\coth r - \tanh r\cdot\cos^2\phi)\cdot(dx - d\xi)\big)\cos\phi + (\operatorname{csch} r\cdot\cosh y\, dy - \sin\phi\cdot\coth r\, dr)\sin\phi = 0.$$

Using (C), the coefficient of dr in $(*)$ becomes

$$-\cos^2\phi\cdot\operatorname{sech} r\cdot\operatorname{csch} r + \coth r - \tanh r\cdot\cos^2\phi - \sin^2\phi\cdot\coth r$$

which by a simple computation is 0. Hence $(*)$ becomes

$$(*')\qquad (\coth r - \tanh r\cdot\cos^2\phi)\cos\phi\, dx + \operatorname{csch} r\cdot\cosh y\cdot\sin\phi\, dy = 0.$$

Using (2.0), another simple computation yields

$$(*'')\qquad \cosh r = -\cot\phi\cdot\cot\sigma.$$

This combined with (D) finally gives

$$(2.2)\qquad \sigma = -\phi'.$$

Geometrically this corresponds to the fact, as in the euclidean case, that the tangent vector to Ω is orthogonal to $\overline{PQ}$.

Hence

$$\tanh r = \frac{\tanh y}{\cos\phi'} = \frac{\tanh y}{\cos\sigma}$$

and

$$dr = -\sin\phi' \cdot \cosh y\, d\xi = \sin\sigma \cdot \cosh y\, d\xi.$$

By differentiating with respect to s, one obtains

$$\begin{aligned} dr &= \cosh^2 r \cdot \tan\sigma\left(\operatorname{sech}^2 y + \tanh r\,\frac{d\sigma}{ds}\right) ds, \\ d\xi &= \operatorname{sech} y \cdot \cosh^2 r \cdot \sec\sigma\left(\operatorname{sech}^2 y + \tanh r\,\frac{d\sigma}{ds}\right) ds, \qquad (2.3) \\ d\theta &= \frac{1}{\tanh r}\left(\operatorname{sech}^2 y + \tanh r\,\frac{d\sigma}{ds}\right) ds. \end{aligned}$$

PROPOSITION 2. *Suppose Ω is a C^2-curve given by $y = f(x) > 0$. If the center of curvature of Ω never lies on the x-axis, then there exists a unique geodesic polar coordinate graph Γ such that Ω is the trace of the "pole" by rolling Γ along the x-axis.*

Proof. Under the assumption of the proposition, one always has $\operatorname{sech}^2 y + \tanh r\, d\sigma/ds \neq 0$. Therefore it never changes its sign. Choose a starting point (x_0, y_0) and assign the corresponding values of $s = 0$, $\theta = 0$, $r = r_0 = \operatorname{arctanh}(\tanh y_0/\cos\sigma_0)$, then

$$\theta = \theta(s) = \int_0^s \frac{1}{\tanh r}\left(\operatorname{sech}^2 y + \tanh r\,\frac{d\sigma}{ds}\right) ds$$

is clearly a strictly monotonic function of s. Hence one may solve for s in terms of θ and substitute it into $r = \operatorname{arctanh}(\tanh y/\cos\sigma) = r(s)$. It is rather straightforward to verify that if one rolls the curve Γ (defined by the above geodesic polar coordinate graph $r = r(\theta)$) on the x-axis, then the trace of the origin of its attached geodesic polar coordinate system is exactly the given Ω. $\square$

Rolling construction of solution curves of (I_l). Let γ be the generating curve of an $O(n)$-invariant hypersurface Σ^n in H^{n+1} satisfying the W-condition $\sigma_l(k_1, \ldots, k_n) = h$. Then γ is a solution of (I_l), namely

$$(\mathrm{I}_l) \quad \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}\left(-\cos\sigma\,\frac{\cosh y}{\sinh y}\right)^{l-1}\left(\frac{d\sigma}{ds} - \sinh y\,\frac{\cos\sigma}{\cosh y}\right) + \begin{bmatrix} n-1 \\ l \end{bmatrix}\left(-\cos\sigma\,\frac{\cosh y}{\sinh y}\right)^{l} = (-1)^l \begin{bmatrix} n-1 \\ l \end{bmatrix} h_l, \quad 1 \le l \le n-1,$$

$$(\mathrm{I}_n) \quad \left(-\cos\sigma\,\frac{\cosh y}{\sinh y}\right)^{n-1}\left(\frac{d\sigma}{ds} - \sinh y\,\frac{\cos\sigma}{\cosh y}\right) = (-1)^n h_n, \qquad l = n.$$

Note that (I_l) (resp. (I_n)) has the "circular" solution $\tanh r = ((n/(n-l))h_l)^{1/l}$ (resp. $\tanh r = (h_n)^{1/n}$).

LEMMA 1. *Let γ be a solution curve of (I_l). If there exists a point, $\gamma(s_0)$, on γ whose center of curvature lies on the x-axis, then γ is a circular solution.*

Proof. Suppose $\gamma(s_0) = (x_0, y_0)$ is such a point of γ. Let $Q_0 = (\xi, 0)$ be the center of curvature and R_0 be the radius of curvature of γ at (x_0, y_0). Then it follows easily from equation (I_l) that $R_0 = \operatorname{arctanh}((h_n)^{1/n})$ if $l = n$ and $R_0 = \operatorname{arctanh}(((n/(n-1))h_l)^{1/l})$ if $l \leq n-1$. Therefore, the circular solution of radius R_0 and center at Q_0 is a solution curve of (I_l) which tangents γ at (x_0, y_0). Hence it follows from the uniqueness of (I_l) that γ must coincide with the above circular solution. □

THEOREM 1. *Suppose γ is a non-circular solution curve of (I_l). Then γ can be obtained by the above rolling construction with respect to the geodesic polar coordinate graph Γ of $r = r(\theta)$, where $1/\tanh r$ is the inverse function of the following integral, namely*

$$\theta = \pm \int b_l(w)^{-1/2}\, dw, \qquad w = 1/\tanh r,$$

where

$$b_l(w) = \begin{cases} \alpha \left| nw^l - (n-l)h_l \right|^{2/n} - (w^2 - 1), & 1 \leq l \leq n-1, \\ \alpha \left| w^n - 1 \right|^{2/n} - (w^2 - 1), & l = n. \end{cases}$$

Proof. By Proposition 3 and Lemma 1, there exists a unique geodesic polar coordinate graph Γ of a suitable function $r = r(\theta)$ such that γ can be obtained by the rolling construction of Γ. It follows from (2.2) and (2.3) that

$$(2.4)\qquad \begin{cases} \tanh r = \dfrac{\tanh y}{\cos \sigma} \\ \dfrac{dr}{d\theta} = \sinh r \cdot \cosh r \cdot \tan \sigma, \quad \text{or} \quad \dfrac{d}{d\theta}(\ln \tanh r) = \tan \sigma \\ \dfrac{d\sigma}{d\theta} = \dfrac{\tanh r\, d\sigma/ds}{1 + \tanh r\{ d\sigma/ds + \sinh y \cos \sigma / \cosh y \}}. \end{cases}$$

Since $\cos\sigma = \cos\phi' = \tanh y \cdot \coth r$, (I_l), $l \leq n-1$, becomes

$$(\mathrm{I}_l) \qquad \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}\left(\frac{1}{\tanh r}\right)^{l-1}\left\{\frac{d\sigma}{ds} - \sinh y \frac{\cos\sigma}{\cosh y}\right\} + \begin{bmatrix} n-1 \\ l \end{bmatrix}\left(\frac{-1}{\tanh r}\right)^{l} = (-1)^{l}\begin{bmatrix} n-1 \\ l \end{bmatrix} h_l$$

or

$$\frac{-l}{(n-l)} \tanh r\left\{\frac{d\sigma}{ds} - \sinh y \cdot \frac{\cos\sigma}{\cosh y}\right\} + 1 = \tanh^l r \cdot h_l$$

or

$$\tanh r\left\{\frac{d\sigma}{ds} - \sinh y \frac{\cos\sigma}{\cosh y}\right\} = \frac{n-l}{l}\left(1 - \tanh^l r \cdot h_l\right).$$

Also:

$$\tanh r \frac{d\sigma}{ds} = \frac{n-l}{l}\left(1 - \tanh^l r \cdot h_l\right) + \tanh^2 r \cdot \cos^2\sigma.$$

Combining all the above relations, one obtains the following corresponding ODE of $r = r(\theta)$ by differentiation and substitution.

$$(\mathrm{II}_l) \quad \frac{d^2}{d\theta^2}(ln \tanh r) = (1 + \tan^2\sigma)\frac{d\sigma}{d\theta} = \left\{1 + \left(\frac{d\sigma}{d\theta} ln \tanh r\right)^2\right\} \times \frac{((n-l)/l)\left(1 - \tanh^l r \cdot h_l\right) + \tanh^2 r \cdot \cos^2\sigma}{1 + ((n-l)/l)\left(1 - \tanh^l r \cdot h_l\right)}.$$

Next, let us proceed to integrate the above ODE explicitly by a suitable substitution of variables. Set

$$(2.5) \qquad u = ln \tanh r, \qquad v = \frac{du}{d\theta} = \frac{1}{\tanh r}\frac{dr}{d\theta}.$$

Then

$$\frac{d^2}{d\theta^2} ln \tanh r = \frac{dv}{d\theta} = \frac{du}{d\theta} \cdot \frac{dv}{du} = v\frac{dv}{du}$$

and hence (II_1) becomes

$$v\frac{dv}{du} = (1 + v^2)\frac{(n-l)\left(1 - e^{lu} \cdot h_l\right) + e^{2u}/(1+v^2)}{n - (n-l)e^{lu} \cdot h_l}$$

or

$$v\frac{dv}{du} = 1 + v^2 + \frac{le^{2u}}{(n-l)(1-e^{lu}\cdot h_l)}\,\frac{(n-l)(1-e^{lu}\cdot h_l)}{n-(n-l)e^{lu}\cdot h_l}$$

or

$$\frac{d}{du}ln(1+v^2-e^{2u}) = \frac{2(n-l)(1-e^{lu}\cdot h_l)}{n-(n-l)e^{lu}\cdot h_l}. \tag{2.6}$$

Integrating both sides of (2.6), one gets

$$ln(1+v^2-e^{2u}) = 2\left\{\frac{n-l}{n}u + \frac{1}{n}ln|n-(n-l)e^{lu}\cdot h_l\right\} + \alpha_1. \tag{2.7}$$

That is,

$$1+v^2-e^{2u} = \alpha e^{2u}\left\{e^{-lu}|n-(n-l)e^{lu}\cdot h_l|\right\}^{2/n}, \qquad \alpha = e^{\alpha_1} > 0. \tag{2.8}$$

Solving for

$$v = \frac{1}{\tanh r}\frac{d\tanh r}{d\theta},$$

$$\begin{aligned}\frac{1}{\tanh r}\frac{d\tanh r}{d\theta} &= v = \pm\left\{\alpha e^{2u}\left(e^{-lu}|n-(n-l)e^{lu}\cdot h_l|\right)^{2/n} - (1-e^{2u})\right\}^{1/2}\\ &= \pm\Big\{\alpha\tanh^2 r\left(\tanh^{-l}r|n-(n-l)\tan^l r\cdot h_l|\right)^{2/n}\\ &\qquad -(1-\tanh^2 r)\Big\}^{1/2}.\end{aligned}$$

Set $w = 1/\tanh r$. Then

$$\begin{aligned}\frac{dw}{d\theta} &= \frac{d(1/\tanh r)}{d\theta} = -\frac{1}{\tanh^2 r}\frac{d\tanh r}{d\theta}\\ &= \mp\left\{\alpha|nw-(n-l)h_l|^{2/n} - (w^2-1)\right\}^{1/2}.\end{aligned} \tag{2.9}$$

Therefore, $\theta = \mu\int b_l(w)^{-1/2}\,dw$, where $b_l(w) = \alpha|nw-(n-l)h_l|^{2/n} - (w^2-1)$. The case $l = n$ is similar. □

The spherical case. Similarly, in the spherical case we may assume $c = 1$, and we have

THEOREM 1′. *Suppose γ is a non-circular solution curve of* (I_l), *in* $M_+^2(1)$. *Then γ can be obtained by a spherical rolling construction with respect to a spherical geodesic polar coordinate graph Γ of $r = r(\theta)$, where $1/\tan r$ is the inverse function of the following integral, namely,* $\theta = \pm\int b_l(w)^{-1/2}\,dw$, $w = 1/\tan r$, *where*

$$b_l(w) = \begin{cases} \alpha\left|nw^l - (n-l)h_l\right|^{2/n} - (w^2+1), & 1 \le l \le n-1, \\ \alpha\left|w^n - 1\right|^{2/n} - (w^2+1), & l = n. \end{cases}$$

Proof. Similar to that of Theorem 1 and hence omitted.

REMARK. For the case $n = 2$, $l = 1$, i.e. constant mean curvature hypersurfaces in H^3 we have

$$b_1(w) = (2w - h_1) - w^2 + 1 \quad \text{or} \quad (h_1 - 2w) - w^2 + 1$$

which can easily be integrated to obtain the equation for the rolling curve

$$\frac{1}{\tanh r} = w = a + b\cos(\theta + c), \qquad a,\ b,\ c \text{ constants},$$

i.e.

$$\tanh r = \frac{1}{a + b\cos(\theta + c)}.$$

Similarly for S^3 we have

$$\tan r = \frac{1}{a + b\cos(\theta + c)}.$$

Recall in R^3, in the classical Delaunay theorem, the rolling curve is a conic, which in polar coordinates is

$$r = \frac{1}{a + b\cos(\theta + c)}.$$

REFERENCES

[1] C. Delaunay, *Sur la surface de revolution dont la courbure mayenne est constante*, J. Math. Pures Appl., Ser. 1, **6** (1841), 309–320.

[2] W. Y. Hsiang, *On W-hypersurfaces of generalized rotational type in* E^{n+1}, I, (to appear).

[3] ______, *On generalization of theorems of A. D. Alexandrov and C. Delaunay on hypersurfaces of constant mean curvature*, Duke J. Math., **49**, No. 3 (1982), 485–496.

[4] ______, *On rotational W-hypersurfaces in spaces of constant curvature and generalized laws of sine and cosine*, Bull. Inst. Math. Acad. Sinica, **11**, no. 3 (Sept. 1983), 349–373.

[5] W. Y. Hsiang and W. Yu, *A generalization of a theorem of Delaunay*, J. Differential Geom., **16** (1981), 161–177.

[6] I. Sterling, *New examples of imbedded spherical soap bubbles in* $S^n(1)$ *and a generalization of Delaunay's theorem*, Thesis (1985), University of California, Berkeley.

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